CHAPTER 4
Paraboloids

4.1 *Circular and Elliptic Paraboloids.*

Imagine the parabola

$$x^2 = 4qz \quad , \qquad 4.1.1$$

which is a parabola whose semi latus rectum is of length 2*q*.

I have drawn it in figure IV.1 for $4q = 1$. The distance between vertex and focus is *q*, and the length of the latus rectum is 4*q*.

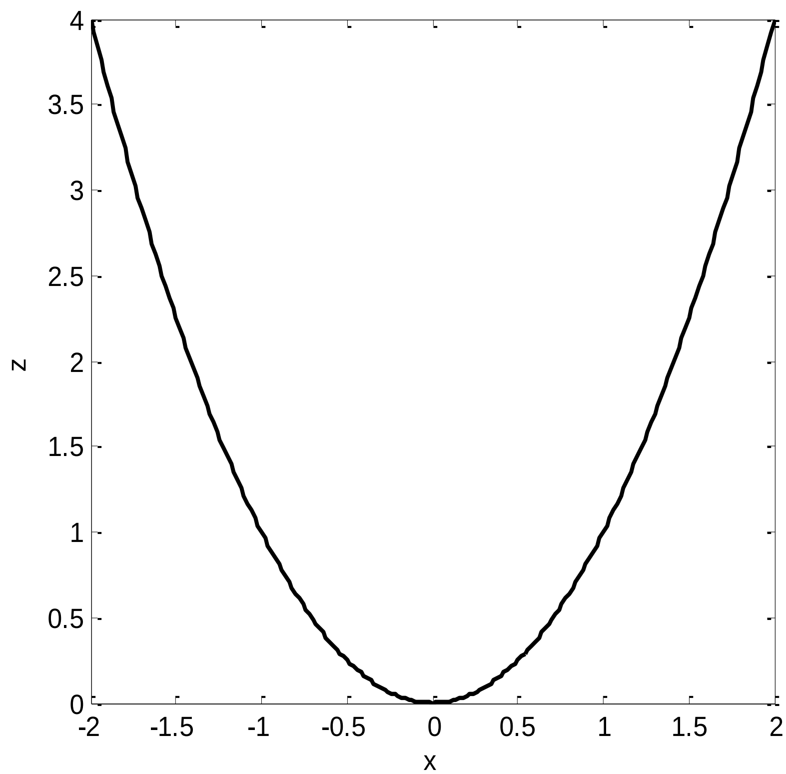

FIGURE IV.1

Rotate this parabola about the vertical axis. You obtain a *paraboloid* of circular cross-section, or a *circular paraboloid* - like a telescope mirror (not Ritchey-Chrétien!), or a stirred cup of coffee.

Its equation is

$$x^2 + y^2 = 4qz . \qquad 4.1.2$$

If we introduce two lengths a and h by

$$q = \frac{a^2}{2h}, \qquad 4.1.3$$

the equation to the circular paraboloid becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{2z}{h}. \qquad 4.1.5$$

Of course a paraboloid need not be circular in cross-section, and the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{h} \qquad 4.1.6$$

represents an *elliptic paraboloid*. It cannot be obtained simply by rotation of a parabola.

If we were to translate the origin of the coordinate axes (without rotation), we would introduce terms in x, y and z as well as a constant term into the equation. If, further, we were to rotate the coordinate axes about the z- axis, we would introduce a term in xy.

Thus an equation of the form

$$ax^2 + 2hxy + by^2 + 2ux + 2vy + 2wz + d = 0 \qquad 4.1.7$$

(with no terms in z^2, yz or zx) represents a paraboloid in which the z axis is parallel to the symmetry axis of the paraboloid. This will be useful to recall in Section 7.1 of Chapter 7.

Neither a parabola nor a paraboloid has a *centre* of symmetry. Equation 4.1.6 contains an odd power of z. It is not unchanged if you substitute $-z$ for z.

4.2 *Hyperbolic Paraboloid*

The elliptical paraboloid described by equation 4.1.6 is easy to visualize. Slightly less easy (but by no means unreasonably difficult) to visualize is a *hyperbolic paraboloid*, described by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{h} \qquad 4.2.1$$

This is saddle-shaped.

In the plane $y = 0$, the cross section is a nose-down parabola similar to figure IV.1, with semi latus rectum of length $\frac{a^2}{h}$.

In the plane $x = 0$, the cross-section is a nose-up parabola with semi latus rectum of length $\frac{b^2}{h}$.

In the plane $z = 0$, the cross-section is two straight lines, $y = \pm \frac{b}{a} x$, which are the asymptotes to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

In the plane $z = \frac{1}{2} h$, the cross-section is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

In the plane $z = -\frac{1}{2} h$, the cross-section is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$. This is the conjugate hyperbola to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Like the circular and elliptical paraboloids, the hyperbolic paraboloid is not a central quadric.