Chapter 7
Equations that contain powers of derivatives.

By this I mean equations that contain expressions such as $\left(\frac{dy}{dx}\right)^2$, which of course is not the same thing as $\frac{d^2y}{dx^2}$.

In this chapter I shall often use the symbol p to mean y' or $\frac{dy}{dx}$.

Also, I shall maintain the convention that $\sqrt{x}$ means $|\,x^{1/2}\,|$. That is, while $4^{1/2} = \pm 2$, $\sqrt{4} = 2$.

An example would be the not-too-difficult equation

$$(2 - y)p^2 = y.$$

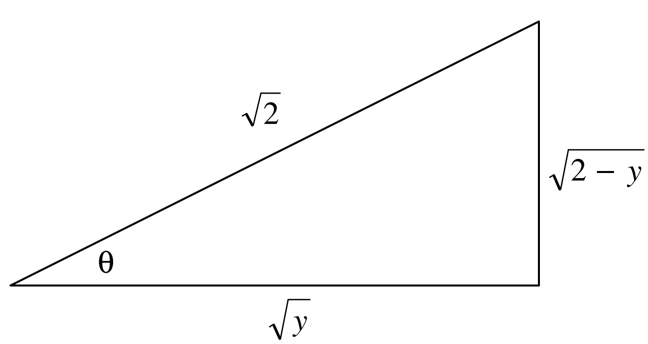
The obvious and perfectly legitimate way to do this is to write it as

$$p = \frac{dy}{dx} = \pm\sqrt{\frac{y}{2-y}}$$

or

$$x = \pm\int\sqrt{\frac{2-y}{y}}\,dy.$$

The "differential equation" part of the problem is now over. All we have to do now is an integration, which can be done by means of the Brilliant Substitution (I shan't say how many attempts I made) let $\tan\theta = \sqrt{\frac{2-y}{y}}$, or, equivalently, $y = 2\cos^2\theta$.

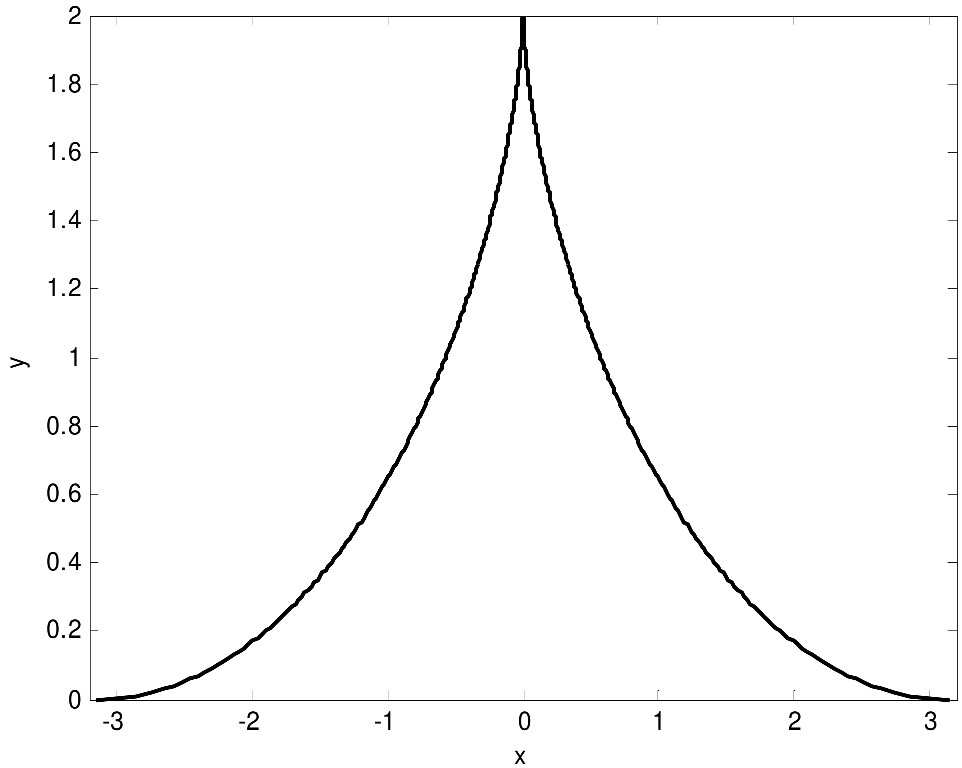


You will soon arrive at

$$\underline{\underline{x = \pm 2(\sin\theta\cos\theta - \theta) + C}},$$

where sin θ, cosθ and θ are given as functions of y by the above triangle drawing.

Here is it, with $C = 0$:



Here is another way of doing it. The original equation was:

$$(2 - y)p^2 = y.$$

Re-write this as:

$$y = \frac{2p^2}{1 + p^2}.$$

Differentiate this with respect to x. (An unexpected move, since we are trying to integrate the equation, not differentiate it!) The derivative of the left hand side with respect to x is, of course, just p.

$$p = \frac{4p}{(1+p^2)^2}\frac{dp}{dx}.$$

We now have a differential equation in *p* and *x* rather than in *y* and *x*. We just have to do a simple integration:

$$x = 4\int\frac{dp}{(1+p^2)^2}.$$

This is easy (make the substitution $p = \tan\phi$), and we arrive at

$$\underline{\underline{x = 2(\sin\phi\cos\phi + \phi) + C}}.$$

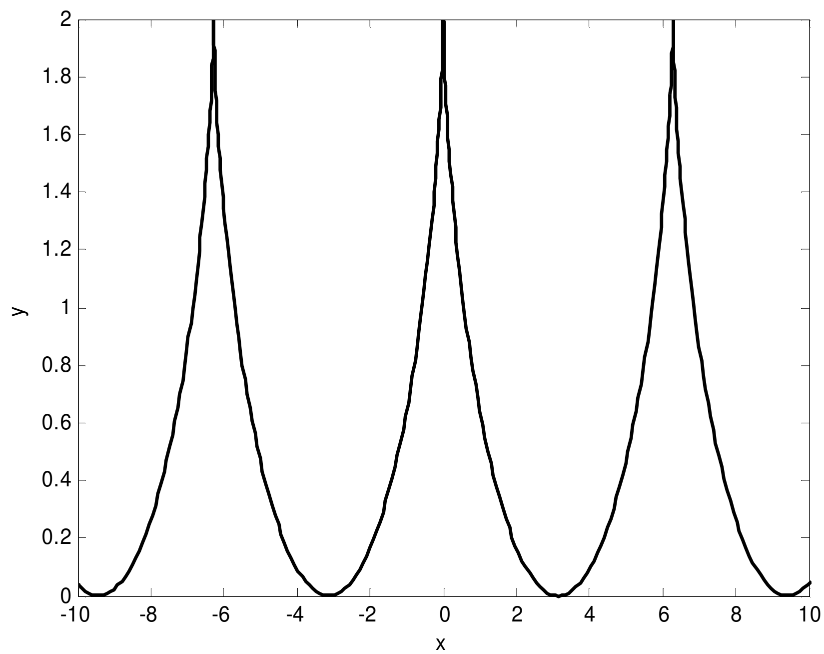
(Don't worry about the plus sign. ϕ is not the same as θ.)

As for *y*, recall that $y = \frac{2p^2}{1+p^2}$ and that $p = \tan\phi$, from which we arrive at

$$\underline{\underline{y = 2\sin^2\phi}}$$

You could try and eliminate ϕ to obtain an explicit relation between *x* and *y* (good luck!), but it is more satisfactory to leave the equations in parametric form.

Here it is, with *C* chosen to be π. You will no doubt be relieved to see that, for real *x*, *y* is never greater than 2.



Example

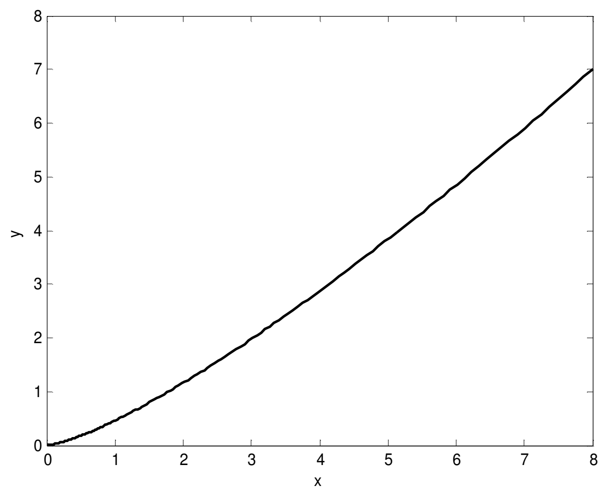
$$x = 3p^2 + 5p^4$$

This will be considered solved if you can find *y* as a function of *p*. You then have the solution in parametric form, and can easily draw a graph of *y* versus *x*.

Are you completely stuck? It's actually very, very easy. Just do the unexpected thing that you did in the last example - except that this time differentiate both sides with respect to *y*. What is the derivative of the left hand side with respect to *y*? You should very soon arrive at

$$y = 2p^3 + 4p^5 + C$$

Here it is with $C = 0$, calculated from $p = 0$ to 1.



Clairaut's Equation

These are equations of the form

$$y = xf_1(p) + f_2(p),$$

They can be quite exciting.

Let's try a simple example first, in which $f_1(p) = p$ and $f_2(p) = p^3$:

$$y = xp - p^3$$

Differentiate with respect to *x*, as we did in the previous example.

$$p = p + xp' - 3p^2 p',$$

in which $p' = \dfrac{dp}{dx}$.

That is, $$p'(x - 3p^2) = 0.$$

This is exciting because there are *two solutions*, one of which is satisfied by $p' = 0$, and the other is satisfied by $x = 3p^2$. We'll see shortly that there is a nice geometric relation between the two solutions.

The solution of $p' = 0$ is $p = C$. Substitution of this into the original equation $y = xp - p^3$ gives us

$$\underline{\underline{y = Cx - C^3}},$$

which is a *family of straight lines*, each member of which has its own value of *C*. For reasons that are not quite clear to me, this solution, representing a family of curves each with its own value of the parameter *C* (constant of integration) is called the *complete primitive.*

What about the other solution - i.e. the solution of $\underline{x = 3p^2}$? Substitution of this into the original equation $y = xp - p^3$ gives us $\underline{y = 2p^2}$. The two underlined equations represents the second solution in parametric form, and we can easily eliminate *p* to find the (x, y) equation of the second solution, namely

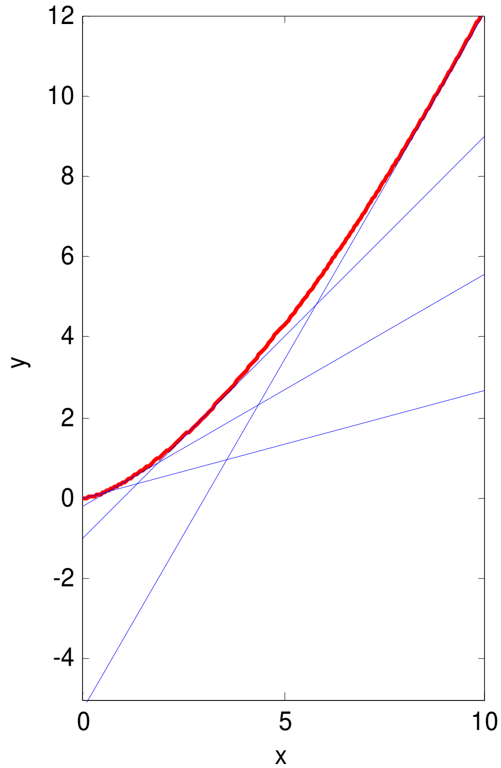
$$\underline{\underline{\frac{x^3}{y^2} = \frac{27}{4}}}.$$

As we have pointed out, the first solution is a family of straight lines, each with its own value of *C*. The second equation has no arbitrary constant, and is the *singular solution*.

Below I have drawn the family $y = Cx - C^3$ (the complete primitive) in **blue** for values of

C = tan (15, 30, 45, 60) degrees, and the singular solution in **red.**

The singular solution is the *envelope* of the complete primitive. The complete primitive is tangent to the singular solution.



Now let's try a Clairaut equation $y = xf_1(p) + f_2(p)$ with $f_1(p) = p$ and $f_2(p) = 1/p$:

$$y = xp + \frac{1}{p}.$$

Carry through exactly the same procedure, and you should find that the complete primitive is the family of straight lines

$$y = mx + \frac{1}{m},$$

where I have chosen to use the symbol m to represent the arbitrary constant of integration, rather than C

The lover of conic sections may recognize this family.
[See www.astro.uvic.ca/~tatum/celmechs/celm2.pdf - especially Section 2,4, equation 2.4.6 and figure II.22]. Indeed the lover of conic sections may already guess what the singular solution is, but you should in any case work it through to show that the singular solution is

$$y^2 = 4x,$$

which is the parabolic envelope of the family of straight lines given by the complete primitive $y = mx + \frac{1}{m}$. FigureII.22 in the above link shows you this.

As long as $f_1(p)$ is just *p*, we'll always find a family or curves (the Complete primitive) and its envelope (the singular solution). But not if $f_1(p)$ is not just *p*.

Now an example in which $f_1(p)$ is not just *p*, but something a little (but not too much) more complicated, say $f_1(p)$. And we'll take $f_2(p) = p^3$, so the Clairaut equation to be solved is

$$y = xp^2 + p^3 \tag{7.1}$$

As before, we'll make the unexpected move of differentiating with respect to *x* (although by this time it is not so unexpected):

$$p = p^2 + 2xpp' + 3p^2 p'.$$

There is an obvious singular solution, which you may regard as "trivial" but is nevertheless a valid solution, namely $p = 0$, hence (from equation 1) $y = 0$ (i.e. the *x*-axis)..

Otherwise $$p' = \frac{1-p}{2x+3p}$$

or $$\frac{dx}{dp} = \frac{2x+3p}{1-p}$$

Just as a temporary measure, which some readers may find helpful (others may not), I'll change the notation so that $x \equiv Y$ and $p \equiv X$: Then

$$\frac{dY}{dX} - \left(\frac{2}{1-X}\right)Y = \frac{3X}{1-X}$$

To solve this differential equation, instantly and without hesitation we multiply the equation through by $(1-X)^2$. [Just in case you didn't do this instantly and without hesitation, go back to Chapter 2, the section on Equations that Require an Integrating Factor.]

$$(1-X)^2\frac{dY}{dX} - 2(1-X)Y = \frac{d}{dX}[(1-X)^2 Y] = 3X(1-X)$$

Integrate: $$(1-X)^2 Y = \tfrac{3}{2}X^2 - X^3 + C$$

It is time to return to the (x, p) notation:

$$(1-p)^2 x = \tfrac{3}{2}p^2 - p^3 + C \tag{7.2}$$

The solution is, then, found by eliminating p between this and the original equation $y = xp^2 + p^3$. This will be a family of curves, each with its own value of C. Eliminating p algebraically is not easy, but the curves can be calculated and drawn by varying p to find first x and then y. Here they are calculated from $p = \tan 30°$ to $\tan -75°$ for

$C = -2$

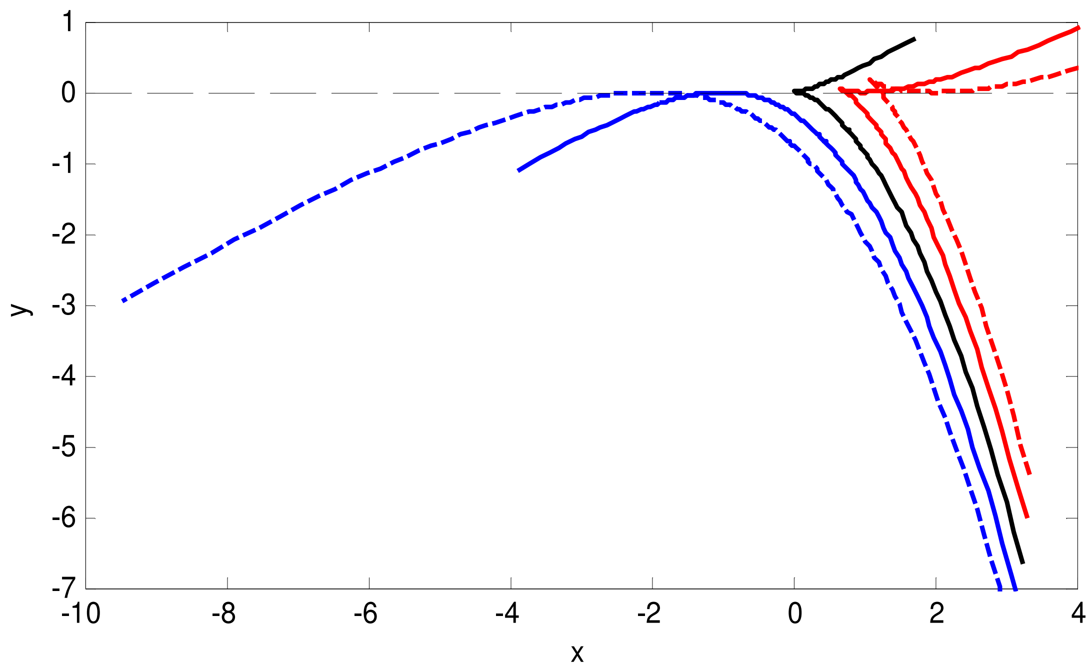
$C = -1$

$C = 0$

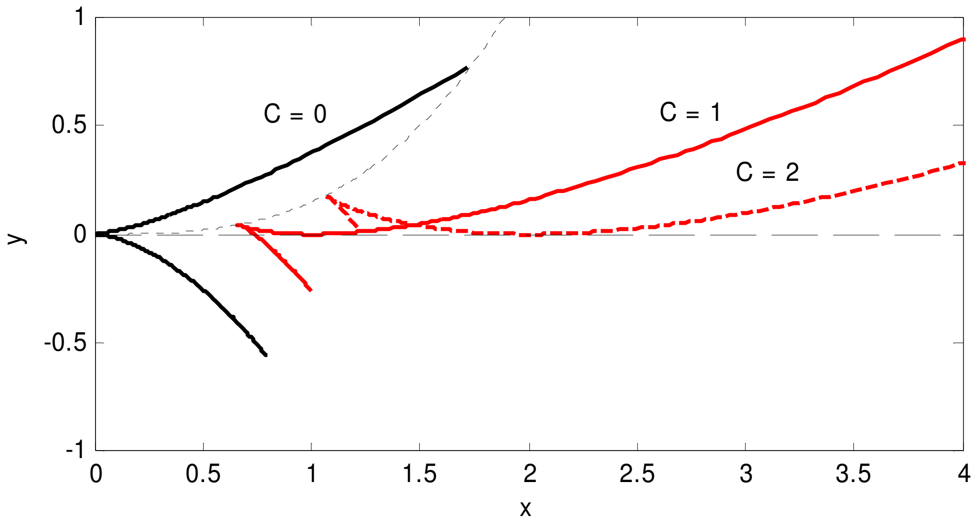
$C = 1$

$C = 2$

Also shown in the graphs, as a thin horizontal black dashed line – – – – – – , is the singular solution, $y = 0$,which is a common tangent to all the curves.



Below is a close-up view near the three cusps. The dotted line shows the locus of the cusps.



Having arrived at the solution, namely that there is a family of curves given by the equations $(1-p)^2 x = \frac{3}{2}p^2 - p^3 + C$ and $y = xp^2 + p^3$, together with the singular solution $y = 0$, we have essentially finished the problem as an exercise in solving a differential equation. However, it might be of interest just to look a little at the geometry of the solutions. I give here the results that I have found from various algebraic manipulations, the details of which I do not give here. I leave it to the reader to derive them if s/he wishes.

Each curve comes to a maximum or a minimum at the point $(C, 0)$. It is a maximum if $C < 0$ and a minimum if $C > 0$. If $C = 0$ there is a maximum *and* a minimum - you will see what I mean if you look at the graphs.

If $C \geq 0$ there is a cusp. The cusps lie on the locus $y = \frac{4}{27}x^3$ (derivation not given here), which is shown by a dotted line in the drawing above.

After some algebraic manipulation (which I do not give here), I find that, for a given curve with parameter *C*, the values of *p*, the slope angle θ, *x* and *y* can be calculated by application, in order, of the following equations:

$$p^3 - 3p^2 + 3p = -2C$$

$$\theta = \tan^{-1} p$$
$$x = -\tfrac{2}{3} p$$
$$y = \tfrac{4}{27} x^3$$

Thus for the curve with $C = 2$ (the dashed red curve in the figure),

$$p = -0.709976$$
$$\theta = 144^{\circ}\ 37'.$$
$$x = 1.064964$$
$$y = 0.178937$$