Chapter 1
Integrals

How would you do the following integral?

$$\int \frac{dx}{\sqrt{2x^2 - 3x + 7}}.$$

Well of course today you would use one of the modern computer packages, such as *Mathematica* or *Maple* or *Wolfram*, that do calculus and algebraic manipulations for you. Some of these packages are quite astonishing in what they can do, and they usually do give a correct answer. (I have never known them not to do so!) I say "a" correct answer, because sometimes the expressions they give, while formally correct, are rather complicated and unreduced, or not always in the most convenient form. For example I once saw one of these mathematics packages give, as the answer to a problem, the expression $\cosh(\sqrt{-x})$. This happened to be formally correct, but if you have not been handling hyperbolic functions recently (and most of us don't see them all that often), it may be incomprehensible. Quick – calculate $\cosh(\sqrt{-2})$. Be honest now – did you find the answer (0.15594) in less than five minutes? Does your table of hyperbolic cosines (you haven't got one) have an entry for the square root of minus two? (No it doesn't) Actually $\cosh(\sqrt{-x})$ is the same thing as $\cos\sqrt{x}$, so that $\cosh(\sqrt{-2}) = \cos(\sqrt{2}) = 0.15594$.

These on-line integration programs have, in recent years, improved enormously, and the problem of unduly complicated or unreduced or incomprehensible solutions appears only occasionally - though you do sometimes get one. One could argue that there is no need, these days, to learn the art of integration, or to waste one's time on it. We are in the 21st century and we must use the tools available. To insist on working out the integrals oneself rather than by using a computer is rather like insisting on doing numerical calculations by long multiplication and division with pencil and paper rather than by just pressing the × or ÷ sign on one's calculator. What, then, is the use of preparing a file like this one that shows us how to carry out integrals ourselves?

I shan't argue with that. If we come across an integral that we need in the course of our scientific work, by all means look it up in *Wolfram* and get on with it. However, many of us enjoy, for relaxation, doing the little puzzles that appear in the daily newspapers, such as sudokus or crossword puzzles. An integral such as the one above is probably slightly more difficult than a newspaper sudoku (although some sudoku devotees have concocted some very difficult ones). Its difficulty level may be comparable to a killer sudoku (for those who haven't heard of a killer sudoku, it's a special variety with slightly different rules), and not nearly as difficult as a cryptic crossword in a British newspaper. I sometimes speculate, wouldn't it be nice if our daily newspapers were to give, in addition to a sudoku and a crossword puzzle, an integral for us to do?

I don't know if that is ever likely to happen, but I suspect that many of us enjoy or get a bit of personal satisfactions from doing little puzzles, and this file may help with doing integrations.

By the way, I have just worked out $\int \frac{dx}{\sqrt{2x^2 - 3x + 7}}$ myself the "old way", and I get

$$\frac{1}{\sqrt{2}} \ln\left[k\left(\sqrt{8(2x^2 - 3x + 7)} + 4x - 3\right)\right]$$

Now let me try *Wolfram* and see what it gives me. Here it is, *Wolfram*'s solution:

$$\frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{4x - 3}{\sqrt{47}}\right) + C$$

They don't look at all alike, do they? I'll let you decide if they are the same. And, if they are, which do you prefer? *Wolfram* is certainly shorter and more compact - but do you have a $\sinh^{-1}$ button on your calculator, or do you know how to program it on a computer? And by the way, if you have concluded that they are not the same and one of them must be wrong, think again. One way of testing would be to choose some number for x and see if both expressions give the same numerical answer. This won't work - because both expressions include an arbitrary constant of integration.

I now give a short list, for reference, of the integrals of the commonest simple mathematical functions, but, beyond that I am not giving a long table of integrals. Rather, I am giving a few hints as to how to start. Indeed it is usually starting that is the most difficult part. One often has to seek a Brilliant Substitution, and, once one hits upon a suitable substitution, the rest is straightforward. Of course, finding the best Brilliant Substitution is something that comes partly with experience. But is also comes in part from realizing that not all successful substitutions are necessarily "brilliant" – there are some that should be routine.

After the table of common integrals, I'll give a number of examples, with hints on how to start. I am assuming that the viewer does know the basics of integration, such as making Brilliant Substitutions, and how to integrate by parts. If not, you are probably not quite ready for this file, which is not for absolute beginners.

Do let me know (jtatum at uvic dot ca) if you find any mistakes anywhere. To err is human, but one of the advantages of Web publishing is that mistakes can be corrected.

I'm dealing only with *analytical* integration in these notes. For *numerical* integration, see Section 1.2 of http://orca.phys.uvic.ca/~tatum/celmechs/celm1.pdf

INTEGRALS OF THE COMMON SIMPLE FUNCTIONS

$f(x)$	$\int f(x)dx$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$
$1/x$	$\ln\lvert x\rvert + C$
$\ln x$	$x(\ln x - 1) + C$
e^x	$e^x + C$
a^x	$\dfrac{a^x}{\ln a} + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\tan x$	$\ln \sec x + C$
$\sec x$	$\ln(\sec x + \tan x) + C$
$\csc x$	$\ln(\csc x - \cot x) + C$
$\cot x$	$\ln \sin x + C$
$\sin^{-1} x$	$x\sin^{-1} x - \sqrt{1-x^2} + C$
$\cos^{-1} x$	$x\cos^{-1} x - \sqrt{1-x^2} + C$
$\tan^{-1} x$	$x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$
$\sec^{-1} x$	$x\sec^{-1} x - \ln\left(x + \sqrt{x^2-1}\right) + C$
$\csc^{-1} x$	$x\csc^{-1} x - \ln\left(x - \sqrt{x^2-1}\right) + C$
$\cot^{-1} x$	$x\cot^{-1} x - \ln\left(\sqrt{1+x^2} - x\right) + C$
$\sinh x$	$\cosh x + C$

$\cosh x$	$\sinh x + C$
$\tanh x$	$\ln(\cosh x) + C$
$\operatorname{sech} x$	$2\tan^{-1} e^{x} + C$
$\operatorname{csch} x$	$\ln k\left(\dfrac{e^{x} - 1}{e^{x} + 1}\right)$
$\coth x$	$\ln(k \sinh x)$
$\sinh^{-1} x$	$x \sinh^{-1} x - \sqrt{x^2 + 1} + C$
$\cosh^{-1} x$	$x \cosh^{-1} x - \sqrt{x^2 - 1} + C$
$\tanh^{-1} x$	$x \tanh^{-1} x + \frac{1}{2}\ln(x^2 + 1) + C$
$\operatorname{sech}^{-1} x$	$x \operatorname{sech}^{-1} x + \sin^{-1} x + C$
$\operatorname{csch}^{-1} x$	$x \operatorname{csch}^{-1} x + \sinh^{-1} x + C$
$\coth^{-1} x$	$x \coth^{-1} x + \frac{1}{2}\ln(x^2 - 1) + C$

I included the inverse hyperbolic functions, for "completeness" rather than for their importance. They won't mean very much unless you are aware of the following identities.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$\tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right)$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right)$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1 + \sqrt{1 + x^2}}{x}\right)$$

$$\coth^{-1} x = \tfrac{1}{2}\ln\left(\frac{x + 1}{x - 1}\right)$$

$\int \frac{dx}{x-1}$ is of some interest and needs some care. Many of us might, in a moment of haste, write $\ln(x-1) + C$. What about $\int \frac{dx}{1-x}$? We might, with similar haste, write $-\ln(1-x)+C$, which can also be written $\ln\frac{1}{1-x}+C$ or $\ln\frac{k}{1-x}$ No problem so far. But suppose we write $\frac{1}{x-1}$ as $-\frac{1}{1-x}$ and then integrate it. We would find

$$\int \frac{dx}{x-1} = -\int \frac{dx}{1-x} = -[-\ln(1-x)] + C = \ln(1-x) + C.$$

So, what is $\int \frac{dx}{x-1}$? Is it $\ln(x-1) + C$, or is it $\ln(1-x) + C$?

It seems to depend on whether $x > 1$ or $x < 1$. You can't take the logarithm of a negative number. (Well, you can't if all you have heard of is real numbers. Those of you who are familiar with complex numbers will know that, for example, $\ln(-\tfrac{1}{2}) = -0.69 + 3.1i$ and you'll know where to find it on the Argand diagram. Those who are not familiar with them should pretend you've never seen this paragraph.)

The truth is, of course, that $\int \frac{dx}{x-1} = \ln|x-1|$, and it doesn't matter what the value of x is. It might be interesting to convince yourself that

$\int_{0.2}^{0.6} \frac{dx}{x-1} = -0.69$ and $\int_{0.5}^{1.5} \frac{dx}{x-1} = 0$ and similarly with other limits, following them by looking at the corresponding areas under a graph of $y = \frac{1}{x-1}$.

I now have a look at several sorts of integrals that you might encounter, with some suggestions as to how to deal with them. Following that, some integrals for yourself to try.

1. **(a)** $\displaystyle\int \frac{dx}{x^2 + bx + c}$ **(b)** $\displaystyle\int \frac{dx}{\sqrt{x^2 + bx + c}}$ **(c)** $\displaystyle\int \sqrt{x^2 + bx + c}\,dx$

There are three cases to consider: **i.** $b^2 = 4c$ **ii.** $b^2 > 4c$ **iii.** $b^2 < 4c$

<u>Case i.</u> $b^2 = 4c$. In this case, the expression $x^2 + bx + c$ can be written as a perfect square, of the form $(x + \alpha)^2$, after which the integrals are easy.

<u>Case ii.</u> $b^2 > 4c$. In this case, the expression $x^2 + bx + c$ can be written as the product of two real linear terms, in the form $(x + \alpha)(x + \beta)$.

For the integral **(a)**¸ split the integrand into partial fractions:

$$\frac{1}{(x+\alpha)(x+\beta)} = \frac{1}{\beta - \alpha}\left(\frac{1}{(x+\alpha)} - \frac{1}{(x+\beta)}\right),$$ after which the integral is easy.

For the integrals **(b)** and **(c)**, let $y = x + \alpha$, and we then have to deal with integrals of the form

(b) $\displaystyle\int \frac{dy}{\sqrt{y(y+h)}}$ or **(c)** $\displaystyle\int \sqrt{y(y+h)}\,dy$, where $h = \beta - \alpha$.

Then let $y = h\tan^2\theta$. The rest should be straightforward.

<u>Case iii.</u> $b^2 < 4c$. In this case, add and subtract $\frac{1}{4}b^2$ ("half the coefficient of x¸ squared") to the expression $x^2 + bx + c$, which becomes

$$x^2 + bx + \tfrac{1}{4}b^2 + c - \tfrac{1}{4}b^2 = (x + \tfrac{1}{2}b)^2 + h^2, \text{ where } h^2 = c - \tfrac{1}{4}b^2.$$

Then let $x + \frac{1}{2}b = h\tan\theta$. From this, $dx = h\sec^2\theta\, d\theta$ and $(x + \frac{1}{2}b)^2 + h^2 = h^2\sec^2\theta$.

The three integrals then become:

(a) $\dfrac{1}{h}\int d\theta.$

(b) $\int \sec\theta d\theta.$

(c) $h^2\int \sec^3\theta d\theta.$ See example **5** below for this one.

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2. $\displaystyle\int \frac{dx}{x\sqrt{1+x}}.$ Try letting $x = \tan^2\theta.$

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3. $\displaystyle\int \frac{dx}{\sqrt{1+e^x}}.$ Try letting $e^x = \tan^2\theta.$

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4. $\displaystyle\int \frac{d\theta}{\sqrt{1+\cos\theta}}.$ Write $1+\cos\theta = 2\cos^2\tfrac{1}{2}\theta.$

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5. $\int \sec^n\theta d\theta.$

(a) If n is *even*.

$$\int \sec^2\theta d\theta = \tan\theta + C$$
$$\int \sec^4\theta d\theta = \int \sec^2\theta(1+\tan^2\theta)d\theta = \int \sec^2\theta d\theta + \int \tan^2\theta d\tan\theta.$$

$$\int \sec^6\theta d\theta = \int \sec^4\theta(1+\tan^2\theta)d\theta = \int \sec^4\theta d\theta + \int \sec^2\theta\tan^2\theta\sec^2\theta d\theta$$

$$= \int \sec^4\theta + \int (1+\tan^2\theta)\tan^2\theta d\tan\theta.$$

...and so on for higher even powers.

(b) If *n* is *odd*.

$$\int \sec\theta d\theta = \ln(\sec\theta + \tan\theta) + C.$$

$$\begin{aligned} I_3 &= \int \sec^3\theta d\theta = \int \sec\theta d\tan\theta = \sec\theta\tan\theta - \int \tan\theta d\sec\theta \\ &= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta \\ &= \sec\theta\tan\theta - I_3 + \int \sec\theta d\theta. \end{aligned}$$

...and so on for higher odd powers.

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6. $\int \sin^m\theta\cos^n\theta d\theta$ and $\int x^m(1 - x^2)^{n/2}dx.$

The second integral is the same as the first if you let $x = \sin\theta$, so we deal only with the first.

If one or both of *m* and *n* are *odd*: For example:
$\int \sin^3\theta\cos^5\theta d\theta = \int \sin^3\theta\cos^4\theta\cos\theta d\theta.$

Let $s = \sin\theta$, $\cos^2\theta = 1 - s^2$, $ds = \cos\theta d\theta$, and so we have $\int s^3(1 - s^2)^2 ds.$

If both *m* and *n* are *even* it's not quite so simple. For example

$$\int \sin^2\theta\cos^4\theta d\theta = \int (\cos^4\theta - \cos^6\theta)d\theta.$$

Let's just deal with $\int \cos^6\theta d\theta$, because, if you can deal with that, you can probably also deal with $\int \cos^4\theta d\theta.$
The most straightforward way is to use the identity

$$\cos^6\theta = \tfrac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$

In the very unlikely event that you did not know that

$$\cos^6\theta = \tfrac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10),$$

you'd need to be able to find it quickly. I can think of <u>two</u> quick methods. With practice, it might be possible to derive the identity in your head, though I haven't tried it myself. All you need is de Moivre's Theorem, which is the only theorem you need in trigonometry, because all trigonometric identities can be derived quickly from it. De Moivre's Theorem is: $e^{im\theta} = e^{mi\theta}$.

Thus: Let $z = e^{i\theta} = \cos\theta + i\sin\theta$, so that $1/z = \cos\theta - i\sin\theta$ and hence $z + \frac{1}{z} = 2\cos\theta$.

The binomial coefficients for $(a + b)^6$ (which you can get from Pascal's pyramid) are

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

so that $$\left(z + \frac{1}{z}\right)^6 = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20.$$

That is $$2^6\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20,$$

or $$\cos^6\theta = \tfrac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$

The other quick way to find this and similar identities is to look it up in the table that you will find in Section 3.8 of http://orca.phys.uvic.ca/~tatum/celmechs/celm3.pdf

The definite integrals $\int_0^{\pi/2}\sin^m\theta\cos^n\theta d\theta$ and $\int_0^1 x^m(1-x^2)^{n/2}dx$ can be evaluated from a simple formula, which is not difficult to derive, namely:

$$\int_0^{\pi/2}\sin^m\theta\cos^n\theta d\theta = \frac{(m-1)!!(n-1)!!}{(m+n)!!}f(m,n).$$

Here 13!! means, for example, $13 \times 11 \times 9 \times 7 \times 5 \times 3 \times 1$, and $1!! = 0!! = 1$. The function $f(m,n)$ equals $\pi/2$ if m and n are both even, and $f(m,n) = 1$ otherwise.

For example:

$$\int_0^{\pi/2}\sin^6\theta\cos^8\theta d\theta = \frac{5\times3\times7\times5\times3}{14\times12\times8\times6\times4\times2}\times\frac{\pi}{2} = \frac{5\pi}{4096} = \underline{\underline{0.00383495}}$$

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7. $$\int\frac{dx}{x^3 + Bx^2 + Cx + D}.$$

The cubic expression can always be expressed in the form $(x + \alpha)(x^2 + bx + c)$, and sometimes even in the form $(x + \alpha)(x + \beta)(x + \gamma)$. It may be easy to do so. For example:

$(x^3 + 1) = (x + 1)(x^2 - x + 1)$. Or it may be less easy, for example

$x^3 + 2x^2 + 5x - 11 = (x + \alpha)(x^2 + bx + c)$, where $\alpha = 1.227\,461\,483$, $b = 3.227\,461\,483$, $c = 8.961584\,658$.

In any case, you can split the integrand into partial fractions:

$$\frac{1}{(x+\alpha)(x^2+bx+c)} = \frac{P}{x+\alpha} + \frac{Qx+R}{x^2+bx+c},$$

where (I think – but you'd better check it)

$$P = \frac{1}{c+\alpha(b+\alpha)}, \quad Q = -P \quad \text{and} \quad R = -\frac{b-\alpha}{c+\alpha(b+\alpha)}.$$

$\int \frac{Pdx}{x+\alpha}$ and $\int \frac{Rdx}{x^2+bx+1}$ will cause no difficulty, but what about $\int \frac{xdx}{x^2+bx+c}$?

Try something like this:

$$\frac{x}{x^2+bx+c} = \frac{1}{2}\left(\frac{2x}{x^2+bx+c}\right) = \frac{1}{2}\left(\frac{2x+b}{x^2+bx+c} - \frac{b}{x^2+bx+c}\right).$$

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8. $\int x^n e^{\pm ax^2} dx.$

If n is *odd*, there is no difficulty. Let $y = ax^2$ and integrate by parts.

Thus $\int x^5 e^{x^2} dx = e^{x^2}(\frac{1}{2}x^4 - x^2 + 1) + C.$

If n is *even*, the integral cannot be expressed in terms of the simple elementary functions. If it has to be integrated between finite definite limits, it has to be evaluated numerically. However, the function (of t) $\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2} dx$ is called the *error function* erf t, and it is supported by many computer packages. For more on the error function see Section 4.0 of http://orca.phys.uvic.ca/~tatum/thermod.html

The definite integral $\int_0^\infty e^{-x^2} dx$ has the value $\sqrt{\pi}/2$. This can be derived as follows:

$\left(\int_0^\infty e^{-x^2} dx\right)^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dydx.$ Let $y = tx$, so that the inner integral becomes $\int_0^\infty xe^{-x^2(1+t^2)} dt$, in which, as far as this inner integral is concerned, x is constant.

Thus $\left(\int_0^\infty e^{-x^2}dx\right)^2 = \int_0^\infty\int_0^\infty xe^{-x^2(1+t^2)}dtdx$. Reverse the order of integration (we can always do this with a well-behaved function – when calculating an area, it doesn't matter whether we take elemental strips parallel to the *y*-axis and integrate them with respect to *x*, or do it the other way round.)

Thus $\left(\int_0^\infty e^{-x^2}dx\right)^2 = \int_0^\infty\int_0^\infty xe^{-x^2(1+t^2)}dxdt$.

The inner integral now is $\int_0^\infty xe^{-(1+t^2)x^2}dx$, in which *t* is constant. It is easily found (e.g. let $s = x^2$) that this integral comes to $1/(1+t^2)$, and therefore $\left(\int_0^\infty e^{-x^2}dx\right)^2 = \int_0^\infty \frac{dt}{2(1+t^2)}$. This integral is elementary (e.g. let $t = \tan\theta$) and comes to $\pi/4$.

Therefore $$\int_0^\infty e^{-x^2}dx = \tfrac{1}{2}\sqrt{\pi}.$$

We can go further. By substitution of $\sqrt{a}x$ for *x*, we easily see that

$$\int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Now each side of this equation is a function of *a*, not of *x*. If we now differentiate both sides again and again with respect to *a*, we obtain progressively

$$\int_0^\infty x^2e^{-ax^2}dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}},$$

$$\int_0^\infty x^4e^{-ax^2}dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}},$$

and so on.

We can also do this with the odd powers. It is easy to obtain

$$\int_0^\infty xe^{-ax^2}dx = \frac{1}{2a},$$

and by repeated differentiation with respect to *a* we obtain

$$\int_0^\infty x^3e^{-ax^2}dx = \frac{1}{2a^2},$$

$$\int_0^\infty x^5e^{-ax^2}dx = \frac{1}{a^3},$$

and so on.

Of course, $\int_{-\infty}^{\infty} \; = \; 2\int_0^{\infty}$ if n is even, and zero if n is odd.

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9. $\int e^{-(ax^2 + bx)}dx = e^{b^2/(4a)}\int e^{-a(x + b/(2a))^2}$, etc.

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10. $\int \cos bx . e^{-ax^2} dx = \mathrm{Re}\int e^{-ax^2 + bx} dx$

$\int \sin bx . e^{-ax^2} dx = \mathrm{Im}\int e^{-ax^2 + bx} dx$

In case the newspapers don't publish a daily integral in addition to their sudoku and crossword puzzle, here are a few for fun, chosen at random. As mentioned earlier, I estimate that on average each is about as difficult as a killer sudoku puzzle, but not as difficult as a cryptic crossword from a British newspaper. From time to time as the spirit moves me, I may add a few more. Following the list, I give a few hints. And after the hints, in case you are absolutely stuck, I give some worked solutions.

1. $\int \frac{dx}{x^2 - x}$ **2.** $\int \frac{dx}{x^3 - x}$ **3.** $\int \frac{x^2 dx}{x^2 + 1}$ **4.** $\int \left(\frac{x+1}{x-1}\right) dx$ **5.** $\int \frac{dx}{1 + \cos x}$

6. $\int \frac{(x+2)dx}{x^2 - x + 1}$ **7.** $\int \frac{(x^2 - 3x + 1)dx}{x^3 - 3x^2 + 2x}$ **8.** $\int \frac{dx}{\sqrt{x^2 - \frac{1}{4}} - x + \frac{1}{2}}$

9. $\int \frac{dx}{\sqrt{x^2 + \frac{1}{4}}}$ **10.** $\int \frac{(2x^2 + x + 1)dx}{x^3 - 1}$ **11.** $\int \frac{x^2 dx}{x^3 + 5x^2 + 8x + 4}$

12. $\int \frac{(x+1)^2 dx}{x^3 - x^2 + x - 1}$ **13.** $\int \frac{(x^3 + 1)dx}{x - 1}$ **14.** $\int \frac{(3x+1)dx}{7x^2 + x + 3}$

15. $\int \tan^2 x dx$ **16.** $\int \sin x \tan^2 x dx$ **17.** $\int \cos^3 x dx$ **18.** $\int \tan^3 x dx$

19. $\int \frac{dx}{(x^2+4)^2}$ **20.** $\int \frac{xdx}{x-1}$ **21.** $\int (1+x)^2 e^x dx$

22. $\int \sqrt{x^2-4}dx$ **23.** $\int \frac{x^2dx}{\sqrt{2-x}}$ **24.** $\int \sqrt{\frac{1+x}{1-x}}dx$ **25.** $\int x \ln x dx$

26. $\int e^{3x} \sin 2x dx$ **27.** $\int \frac{x^2dx}{(x^2+9)(x^2+4)^2}$ **28.** $\int \cos^{\frac{3}{2}} 2x \sin x dx$

29. $\int \frac{dx}{x^3+1}$ **30.** $\int_0^1 \frac{x^3dx}{\sqrt{1-x}}$ **31.** $\int \sqrt{\frac{3-x}{x}}dx$ **32.** $\int \frac{dx}{1+e^x}$

33. $\int \frac{x^3}{x-1}dx$ **34.** $\int e^{2x} \sin^2 3x dx$ **35.** $\int_0^{\pi/4} \ln(1+\tan\theta)d\theta$

36. $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$ **37.** $\int \frac{d\theta}{1+\varepsilon\cos\theta}$ **38.** $\int x\sqrt{1-x}\, dx$

39. $\int \frac{dx}{1+a\cos x}$ **40.** $\int \frac{dx}{x^4+1}$

HINTS

Here are some hints for the integrals above. I give worked solutions after the hints, but, when you have what you believe to be the answer, you can always differentiate your answer to see if you arrive at the original integrand.

Don't forget to add a constant to all of them.

1. $\frac{1}{x^2-x} = \frac{1}{x(x-1)}$ Split into partial fractions.

2. $x^3 - x = x(x-1)(x+1)$.

3. $\dfrac{x^2}{x^2+1} = \dfrac{x^2+1}{x^2+1} - \dfrac{1}{x^2+1}$

4. $\dfrac{x+1}{x-1} = \dfrac{x-1}{x-1} + \dfrac{2}{x-1}$

5. $\dfrac{1}{1+\cos x} = \dfrac{1-\cos x}{1-\cos^2 x} = \dfrac{1-\cos x}{\sin^2 x} = \csc^2 x - \csc x \cot x$

Or you could try something that I often use as a last resort when dealing with trigonometric functions, let $t = \tan\frac{1}{2}x$. It's often useful.

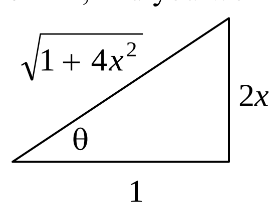
6. $\dfrac{x+2}{x^2-x+1}$

The denominator is a quadratic expression. You must look to see whether $b^2 > = < 4ac$. In this case $b^2 < 4ac$. You must "complete the square" in the denominator by writing it as $(x-\frac{1}{2})^2 + \frac{3}{4}$. Then, a substitution $u = x - \frac{1}{2}$ may help.

7. The denominator is $x(x-1)(x-2)$. Split the integrand into partial fractions.

8. If you had $\sqrt{\frac{1}{4}-x^2}$ you'd probably want to try $x = \frac{1}{2}\cos\theta$. But you have $\sqrt{x^2-\frac{1}{4}}$, so try $x = \frac{1}{2}\cosh\phi$ instead. This may result in something awful such as $\dfrac{\sinh\phi\, d\phi}{\sinh\phi - \cosh\phi + 1}$. But then remember that $\sinh\phi = \frac{1}{2}(e^{\phi} - e^{-\phi})$ and $\cosh\phi = \frac{1}{2}(e^{\phi} + e^{-\phi})$, and you'll probably get something that you can handle.

9. Try $x = \frac{1}{2}\tan\theta$. You may get an answer with $\tan\theta$ and $\sec\theta$ in it, and you won't know what to do with $\sec\theta$. Pythagoras might help you out.



10. $x^3 - 1 = (x-1)(x^2+x+1)$ Split the integrand into partial fractions.

11. The denominator is $(x+1)(x+2)^2$. Split the integrand into partial fractions.

12. $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$. Split the integrand into partial fractions.

13. Let $u = x - 1$.

14. I had to work a little with this one. The first thing I did was to try to make the numerator $3x+1$ look a bit like the derivative of the denominator $14x+1$.

Thus $\quad 3x+1 = \tfrac{3}{14}(14x+1) + \tfrac{11}{14}$

We now have two integrals:

$$\int \frac{(3x+1)dx}{7x^2+x+3} = \tfrac{3}{14}\int \frac{(14x+1)dx}{7x^2+x+3} + \tfrac{11}{14}\int \frac{dx}{7x^2+x+3} = I_1 + I_2.$$

I_1 is easy. Write I_2 as $\quad I_2 = \tfrac{11}{98}\displaystyle\int \frac{dx}{x^2+\tfrac{1}{7}x+\tfrac{3}{7}}$

In the denominator, $b^2 < 4ac$, so we complete the square by writing the denominator as $(x+\tfrac{1}{14})^2 + \tfrac{83}{196}$, followed by a substitution such as let $\quad x+\tfrac{1}{14} = \dfrac{\sqrt{83}}{14}$. I make the final answer

$$\int \frac{(3x+1)dx}{7x^2+x+3} = \tfrac{3}{14}\ln(7x^2+x+3) + \tfrac{11}{7\sqrt{83}}\tan^{-1}\left(\frac{14x+1}{\sqrt{83}}\right) + C$$

15. $\tan^2 x = \sec^2 x - 1$

16. Integrate by parts.

17. Write either $\cos^3 x = \tfrac{1}{4}(\cos 3x + 3\cos x)$

or $\qquad \cos^3 x = \cos x(1-\sin^2 x)$

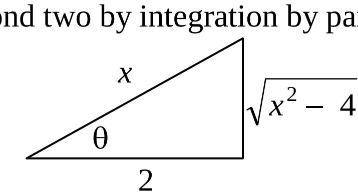
18. $\tan^3 x = \tan x(\sec^2 x - 1)$

19. It's nearly always a good idea, when you see $x^2 + a^2$, to let $x = a\tan\theta$.

20. $\dfrac{x}{x-1} = \dfrac{x-1}{x-1} + \dfrac{1}{x-1}$

21. $(1+x)^2e^x = e^x + 2xe^x + x^2e^x$, and do the second two by integration by parts.

22. Not particularly easy. Let $x = 2\sec\theta$



Then $\sqrt{x^2 - 1}\,dx = 4\sec\theta\tan^2\theta d\theta = 4\sec^3\theta - 4\sec\theta d\theta.$

The integral of $\sec^n\theta d\theta$ is dealt with on page 6 of this file.

The substitution $x = 2\cosh\phi$ will also work, though I expect many will find that $x = 2\sec\theta$ is easier.

I make the answer $\frac{1}{2}x\sqrt{x^2 - 4} - 2\ln\left[\frac{1}{2}\left(x + \sqrt{x^2 - 4}\right)\right] + C$

23. $u = 2 - x.$

24. Let $x = \sin\theta$. After some manipulation of trigonometric identities, you should arrive at $\int(1 + \sin\theta)d\theta.$

25. Integrate by parts. Either way will do.

26. There's probably more than one way, but you might try $\mathrm{Im}\int e^{3x}e^{2ix}dx.$

I make it $\frac{1}{13}(3\sin 2x - 2\cos 2x)e^{3x} + C$. This looks unlikely, but try differentiating it and see what you get. You never know - it might be right.

27. This looks like nothing more than hard work. If my algebra is right,

$$\frac{x^2}{(x^2 + 9)(x^2 + 4)} = \frac{1}{25}\left[-\frac{9}{x^2 + 9} + \frac{9}{x^2 + 4} - \frac{20}{(x^2 + 4)^2}\right]$$

Then a few deft substitutions, such as $x = 3\tan\theta$ and $x = 2\tan\phi$ should help.

28. Try any of $y = \cos x$, $y = \cos 2x$ or $y^2 = \cos 2x$. Surely one of them will put it in a form that you can cope with.

29. $x^3 + 1 = (x + 1)(x^2 - x + 1)$. Split the integrand into partial fractions.

30. This looks bad enough even as an indefinite integral. As a definite integral it looks even worse, because, at the upper limit, the integrand becomes infinite. Try $x = \sin^2\theta$

31. One suggestion: Try $x = 3\cos^2\theta$. If you do this, what is $\tan\theta$?

32. You could try the Brilliant Substitution $y = 1 + e^x$, or, alternatively, you could write the numerator as $1 + e^x - e^x$.

33. There's probably a better way, but all I can immediately think of is the Brilliant Substitution $y = x - 1$.

Indeed, after I wrote the above, Stuart McAlpine came up with another, very nice, solution. He suggests dividing x^3 by $x - 1$ by long division. Try it – it works!

34.
I think the first thing I'd do would be to write $\sin^2 3x$ as $\frac{1}{2} - \frac{1}{2}\cos 6x$.

Then
$$\int e^{2x} \sin^2 3x dx = \frac{1}{2}\int e^{2x} dx - \frac{1}{2}\int e^{2x} \cos 6x dx = \frac{1}{4}e^{2x} - \frac{1}{2}I,$$

where $I = \int e^{2x} \cos 6x dx$

35. I have not been able to find a simple analytical solution of the indefinite integral. As a definite integral, it can be integrated numerically (e.g. by Simpson's rule), and it is found to be 0.272198261... However, remarkably, an analytical solution for the definite integral can be found by making the extraordinarily simple substitution Let $\theta = \frac{\pi}{4} - \phi$.

36. This is a definite integral. You might suspect (you would be right!) that, like number **35**, I have been unable to find a simple analytical solution of the indefinite integral, but, by means of a simple substitution, it is possible to find an analytical solution for the definite integral. The expression $1 + x^2$ in the denominator suggest that the substitution Let $x = \tan\theta$ might help.

37. Unlike most of the other examples here, this isn't a "made-up" integral - I really came across it while I was doing some astronomical orbital calculations. This is not surprising, since $r = \dfrac{l}{1 + \varepsilon\cos\theta}$ is the equation, in polar coordinates, to a conic section of eccentricity ε. (The usual symbol for eccentricity is e, but I use ε here since we frequently use e for something else in these notes.) For such a simple-looking integral, it is surprisingly awkward. Usually I make the Brilliant (i.e. routine) Substitution Let $t = \tan\frac{1}{2}\theta$ only as a last resort, but it does work here. You will find, as you go, that there are three cases to consider: $\varepsilon < 1$, $\varepsilon = 1$, $\varepsilon > 1$, corresponding to elliptical, parabolic and hyperbolic orbits.

38. Let $y = \sqrt{1 - x}$

39. If $a = 0$, $+1$ or -1 the integral is easy. Indeed the case with $a = +1$ is example **5** of this group. For other values of a, try making the substitution $t = \tan\frac{1}{2}x$. This is often useful in trigonometry problems, but often only as a last resort when you can't think of anything else. With this substitution,

$$\tan x = \frac{2t}{1-t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}.$$

If a is between -1 and $+1$ the integral is slightly difficult. If $|a| > 1$ the function is not "well-behaved" - it has some infinities. You can get expressions for the indefinite integral, but you have to take care if you are doing a numerical integration between two limits that you are not going through one of the infinities.

40. $\displaystyle\int \frac{dx}{x^4+1}$.

(Before starting it might be worth mentioning that, if you are given $\displaystyle\int \frac{dx}{x^4+a^4}$, just substitute $x = ay$, and all will be well.)

$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$. Split the integrand into partial fractions. You'll need to know how to do this. That's algebra, not calculus.

SOLUTIONS

1.

$$\int \frac{dx}{x^2 - x} = \int \frac{dx}{x-1} - \int \frac{dx}{x} = \ln|x-1| - \ln|x| + C = \underline{\underline{\ln\left|\frac{k(x-1)}{x}\right|}}$$

2. $x^3 - x = x(x-1)(x+1)$

$$\int \frac{1}{x(x-1)(x+1)}dx = \tfrac{1}{2}\int \frac{1}{x-1}dx + \tfrac{1}{2}\int \frac{1}{x+1}dx - \int \frac{dx}{x} = \tfrac{1}{2}\ln|x-1| + \tfrac{1}{2}\ln|x+1| - \ln x + C$$

$$= \underline{\underline{\ln\left|\frac{k\sqrt{x^2-1}}{x}\right|}}$$

3. $\displaystyle\int \frac{x^2dx}{x^2+1} = \int \frac{x^2+1}{x^2+1}dx - \int \frac{1}{x^2+1}dx = \underline{\underline{x - \tan^{-1}x + C}}$

4. $\displaystyle\int \frac{x+1}{x-1}dx = \int \frac{x-1}{x-1}dx + \int \frac{2}{x-1}dx = \underline{\underline{x + 2\ln|x-1| + C}}$

5.

$$\int \frac{dx}{1+\cos x} = \int \frac{(1-\cos x)dx}{1-\cos^2 x} = \int \frac{(1-\cos x)dx}{\sin^2 x} = \int (\csc^2 x - \csc x \cot x)dx = \underline{\underline{-\cot x + \csc x + C}}$$

$\underline{\underline{\tan(x/2) + C.}}$

Alternatively, if you let $t = \tan\frac{1}{2}x$, then $dx = \dfrac{2dt}{1+t^2}$, and

$1 + \cos x = 2\cos^2\frac{1}{2}x = \dfrac{2}{(1+t^2)}$ and the integral becomes just $\int dt$.

6. $\displaystyle\int \frac{(x+2)dx}{x^2 - x + 1}$

The denominator is a quadratic expression. We start by looking at $b^2 - 4ac$. If it is ≥ 0, it will factorize into two real linear terms. In this case, it is less than zero, so it won't factorize. We "complete the square" by writing it as $(x - \frac{1}{2})^2 + \frac{3}{4}$. And if we now let $u = x - \frac{1}{2}$, the integral becomes (or, rather, the integrals become)

$$\int \frac{udu}{u^2 + \frac{3}{4}} + \frac{5}{2}\int \frac{du}{u^2 + \frac{3}{4}} = \frac{1}{2}\ln(u^2 + \tfrac{3}{4}) + \frac{5}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2u}{\sqrt{3}} + C$$

$$= \underline{\underline{\tfrac{1}{2}\ln(x^2 - x + 1) + \tfrac{5}{\sqrt{3}}\tan^{-1}\tfrac{2x-1}{\sqrt{3}} + C}}$$

7.

$$\int \frac{(x^2-3x+1)dx}{x^3-3x^2+2x} = \int \frac{(x^2-3x+1)dx}{x(x-1)(x-2)} = \frac{1}{2}\int\left(\frac{1}{x}+\frac{2}{x-1}-\frac{1}{x-2}\right)dx = \underline{\underline{\ln\left[k(x-1)\sqrt{\frac{x}{x-2}}\right]}}$$

$$= \underline{\underline{\tfrac{1}{2}\ln|x| + \ln|x-1| - \tfrac{1}{2}\ln|x-2| + C}}\ .$$

If $x>2$, you'd be safe in writing this as $\ln\left[k(x-1)\sqrt{\frac{x}{x-2}}\right]$, but if $x<2$ I wouldn't risk it.

8. You need to be pretty familiar with hyperbolic functions for this one.

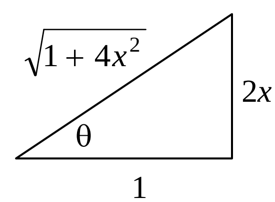
$$I = \int \frac{dx}{\sqrt{x^2-\frac{1}{4}} - x + \frac{1}{2}} \qquad \text{Let } x = \tfrac{1}{2}\cosh\phi. \text{ Then:}$$

$$I = \int \frac{\sinh\phi\, d\phi}{\sinh\phi - \cosh\phi + 1} = \tfrac{1}{2}\int(1+e^{\phi})d\phi = \tfrac{1}{2}(\phi + e^{\phi}) + C$$

$$= \underline{\underline{\tfrac{1}{2}\left[\cosh^{-1}(2x) + 2x + \sqrt{4x^2-1}\right] + C}}$$

9. $I = \int \frac{dx}{\sqrt{x^2+\frac{1}{4}}}$ Try $x = \frac{1}{2}\tan\theta$. Then

$$I = \int \sec\theta\, d\theta = \ln k(\sec\theta + \tan\theta) = \underline{\underline{\ln k\left(\sqrt{1+4x^2} + 2x\right)}}$$



10. $\int \frac{(2x^2+x+1)dx}{x^3-1}$ It is always useful is these situations to see if the denominator factorizes; and, if it does, split the expressions into partial fractions. In this case the denominator factorizes into $(x-1)(x^2+x+1)$ and the expression to be integrated the splits into $\frac{1}{3}\left[\frac{4}{x-1} + \frac{2x+1}{x^2+x+1}\right]$.

Thus:

$$\int \frac{(2x^2+x+1)dx}{x^3-1} = \tfrac{1}{3}\left[4\int\frac{dx}{x-1} + \int\frac{(2x+1)dx}{x^2+x+1}\right] = \tfrac{1}{3}\left[4\ln|x-1| + \ln(x^2+x+1) + \ln k\right]$$

$$= \tfrac{1}{3}\ln[k(x-1)^4(x^2+x+1)] = \underline{\underline{\tfrac{1}{3}\ln[k(x-1)^3(x^3-1)]}}.$$

This last form is OK whether x is less than or greater than 1.

11.

$$\int \frac{x^2dx}{x^3+5x^2+8x+4} = \int \frac{x^2dx}{(x+1)(x+2)^2} = \int\left[\frac{1}{x+1} - \frac{4}{(x+2)^2}\right]dx = \underline{\underline{\ln|x+1| + \frac{4}{x+2} + C}}$$

12. $\displaystyle\int \frac{(x+1)^2dx}{x^3-x^2+x-1} = \int \frac{(x+1)^2dx}{(x-1)(x^2+1)}$

$$= \int\left[\frac{2}{x-1} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right]dx = 2\ln|x-1| - \tfrac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$$

$$= \underline{\underline{\ln\left(\frac{k(x-1)^2}{\sqrt{x^2+1}}\right) + \tan^{-1}x}}$$

13.

$\displaystyle\int \frac{(x^3+1)dx}{x-1}$ Let $u = x-1$.

$$\int \frac{(x^3+1)dx}{x-1} = \int(u^2+3u+3+2/u)du = \tfrac{1}{3}u^3 + \tfrac{3}{2}u^2 + 3u + 2\ln|u| + \text{constant}$$

$$= \underline{\underline{\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + x + \ln[k(x-1)^2]}}$$

14. $\displaystyle\int \frac{(3x+1)dx}{7x^2+x+3}$ The denominator does not factorize into real factors. I'm going to manipulate the numerator to try to make it look like the derivative of the denominator.

$3x+1 = \frac{3}{14}(14x + \frac{14}{3}) = \frac{3}{14}(14x + 1 + \frac{11}{3}) = \frac{3}{14}(14x+1) + \frac{11}{14}$. So now we have

$$\int \frac{(3x+1)dx}{7x^2+x+3} = \frac{3}{14}\int\frac{(14x+1)dx}{7x^2+x+3} + \frac{11}{14}\int\frac{dx}{7x^2+x+3} = I_1 + I_2.$$

I_1 is easy. It is just $\frac{3}{14}\ln(7x^2+x+3) + \text{constant}$.

$$I_2 = \frac{11}{98}\int\frac{dx}{x^2+\frac{1}{7}x+\frac{3}{7}} = \frac{11}{98}\int\frac{dx}{x^2+\frac{1}{7}x+\frac{1}{196}+\frac{83}{196}} = \frac{11}{98}\int\frac{dx}{(x+\frac{1}{14})^2+\frac{83}{196}}$$

Let $u = x + \frac{1}{14}$. Then $I_2 = \frac{11}{98}\int \frac{du}{u^2 + \frac{83}{196}} = \frac{11}{7\sqrt{83}}\tan^{-1}\left(\frac{14u}{\sqrt{83}}\right) = \frac{11}{7\sqrt{83}}\tan^{-1}\left(\frac{14x+1}{\sqrt{83}}\right)$ plus a constant. Hence $\int \frac{(3x+1)dx}{7x^2+x+3} = \frac{3}{14}\ln(7x^2+x+3) + \frac{11}{7\sqrt{83}}\tan^{-1}\left(\frac{14x+1}{\sqrt{83}}\right) + C.$

15. $\int \tan^2 x dx = \int \sec^2 x dx - \int dx = \underline{\underline{\tan x - x + C}}$

16. **(a)**

$\int \sin x \tan^2 x dx = -\int \tan^2 x \, d\cos x = -\cos x \tan^2 x + \int \cos x d \tan^2 x$

$= -\sin x \tan x + 2\int \sec x \tan x dx = -\sin x \tan x + 2\sec x + C = \underline{\underline{\cos x + \sec x + C}}$

(b)

$\int \sin x \tan^2 x dx = \int \sin x \sec^2 x \, dx - \int \sin x dx = \int \sin x d \tan x - \int \sin x dx$

$= \sin x \tan x - \int \tan x d \sin x - \int \sin x dx = \sin x \tan x + \cos x + \cos x + C$

$= \underline{\underline{\sec x + \cos x + C}}$

17. **(a)**

$\int \cos^3 x dx = \frac{1}{4}\int(\cos 3x + 3\cos x)dx = \underline{\underline{\frac{1}{12}\sin 3x + \frac{3}{4}\sin x + C}}$

(b)

$\int \cos^3 x dx = \int \cos x dx - \int \cos x \sin^2 dx = \underline{\underline{\sin x - \frac{1}{3}\sin^3 x + C}}$

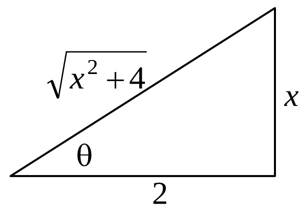
The two solutions, **(a)** and **(b)**, are trigonometric identities.

18. $\int \tan^3 x dx = \int \tan x \sec^2 x dx - \int \tan x dx = \underline{\underline{\frac{1}{2}\tan^2 x - \ln \sec x + C}}$

Wolfram gives $\frac{1}{2}\sec^2 x + \ln \cos x + C.$ Is that the same?

19. $\int \frac{dx}{(x^2+4)^2}$ It's nearly always a good idea, when you see $x^2 + a^2$, to let $x = a\tan\theta$. In this case, $a = 2$ and we obtain

$$\tfrac{1}{8}\int \frac{\sec^2\theta d\theta}{\sec^4\theta} = \tfrac{1}{8}\int \cos^2\theta d\theta = \tfrac{1}{16}\int(\cos 2\theta + 1)d\theta = \tfrac{1}{32}\sin 2\theta + \tfrac{1}{16}\theta + C$$



$$= \tfrac{1}{16}(\sin\theta\cos\theta + \theta) + C = \frac{1}{16}\left(\frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}} + \tan^{-1}\tfrac{1}{2}x\right) + C$$

$$= \underline{\underline{\frac{1}{16}\left(\frac{2x}{x^2+4} + \tan^{-1}\tfrac{1}{2}x\right) + C}}$$

20. $\displaystyle\int\frac{xdx}{x-1} = \int\frac{(x-1)dx}{x-1} + \int\frac{dx}{x-1} = \underline{\underline{x + \ln|x-1| + C}}$

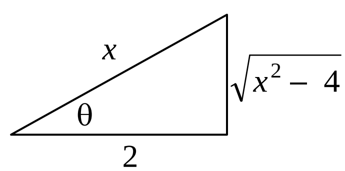
21.

$$\int(1+x)^2e^xdx = \int e^xdx + 2\int xe^xdx + \int x^2e^xdx$$

$$= e^x + 2xe^x - 2e^x + x^2e^x - 2xe^x + 2e^x + C$$

$$= \underline{\underline{(1+x^2)e^x + C}}$$

22. $\displaystyle\int\sqrt{x^2-4}dx.$ Let $x = 2\sec\theta.$



Then $\displaystyle\int\sqrt{x^2-4}dx = 4\int\sec\theta\tan^2\theta d\theta$

$$= 4\int\sec^3\theta d\theta - 4\int\sec\theta d\theta = 4I_1 - 4I_2.$$

$$I_1 = \int \sec^3\theta d\theta = \int \sec\theta(1+\tan^2\theta)d\theta = \int \sec\theta d\theta + \int \tan\theta d\sec\theta$$
$$= \int \sec\theta d\theta + \sec\theta\tan\theta - \int \sec\theta d\tan\theta$$
$$= \int \sec\theta d\theta + \sec\theta\tan\theta - \int \sec^3\theta d\theta$$
$$= \ln(\sec\theta+\tan\theta) + \sec\theta\tan\theta - I_1 + \text{constant}$$
$$\therefore I_1 = \tfrac{1}{2}\ln(\sec\theta+\tan\theta) + \tfrac{1}{2}\sec\theta\tan\theta + \text{constant}$$

$$I_2 = \ln(\sec\theta+\tan\theta) + \text{constant}$$

Therefore

$$\int\sqrt{x^2-4}dx = 2\sec\theta\tan\theta - 2[\ln(\sec\theta+\tan\theta] + \text{constant}$$
$$= \underline{\underline{\tfrac{1}{2}x\sqrt{x^2-4} - 2\ln\left[\tfrac{1}{2}\left(x+\sqrt{x^2-4}\right)\right] + C}} \qquad (x \geq 2)$$

Alternatively:

Let $x = 2\cosh\phi$, $\sqrt{x^2-4} = 2\sinh\phi$, $dx = 2\sinh\phi d\phi$.

Then

$$\int\sqrt{x^2-4}\,dx = 4\int\sinh^2\phi d\phi = \int\left(e^{\phi} - e^{-\phi}\right)^2 d\phi = \int\left(e^{2\phi} + e^{-2\phi} - 2\right)d\phi$$

$$= 2\int(\cosh 2\phi - 1)d\phi = \sinh 2\phi - 2\phi + C = 2\sinh\phi\cosh\phi - 2\phi + C$$

$$= \tfrac{1}{2}x\sqrt{x^2-4} - 2\cosh^{-1}\tfrac{1}{2}x + C$$

$$= \tfrac{1}{2}x\sqrt{x^2-4} - 2\ln\left[\tfrac{1}{2}\left(x + \sqrt{x^2-4}\right)\right] + C \qquad (x \geq 2)$$

23. $\displaystyle\int\frac{x^2dx}{\sqrt{2-x}}$

Let $u = 2 - x$

$$\int\frac{x^2dx}{\sqrt{2-x}} = -\int\frac{(2-u)^2du}{u} = -u^{1/2}(8 - \tfrac{8}{3}u + \tfrac{2}{5}u^2) + C$$
$$= \underline{\underline{-\tfrac{2}{15}\sqrt{2-x}(32+8x+3x^2) + C}}$$

24. $\int \sqrt{\dfrac{1+x}{1-x}}\,dx$ Let $x = \sin\theta$, then

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}\cos\theta d\theta$$

Multiply top and bottom by $\sqrt{1+\sin\theta}$.

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}\cos\theta d\theta = \int \frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta = \int(1+\sin\theta)d\theta$$

$$= \theta - \cos\theta + C = \underline{\underline{\sin^{-1}x - \sqrt{1-x^2} + C}}$$

Incidentally I tried $\int \sqrt{\dfrac{1+x}{1-x}}\,dx$ and $\int \sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}\cos\theta d\theta$ on *Wolfram* recently, and in both cases it returned impossibly complicated answers! I think they were actually correct, though it was very hard work to simplify them. Such cases are mercifully rare these days, and usually the answer is returned in a convenient form.

25. $\int x\ln x dx$

One way:

$$I = \int x\ln x dx = \int x d[x(\ln x - 1)] = x^2(\ln x - 1) - \int x(\ln x - 1)dx = x^2(\ln x - 1) - I + \int x dx$$

$$2I = x^2(\ln x - 1) + \tfrac{1}{2}x^2 + 2C = x^2(\ln x - \tfrac{1}{2}) + 2C$$

$$\underline{\underline{I = \tfrac{1}{2}x^2(\ln x - \tfrac{1}{2}) + C}}$$

The other way:

$$\int x\ln x dx = \int \ln x d(\tfrac{1}{2}x^2) = \tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\int x^2 d\ln x = \tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\int x dx = \underline{\underline{\tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + C}}$$

26.

$$\int e^{3x}\sin 2x dx = \operatorname{Im}\int e^{3x}e^{2ix}dx$$

$$\int e^{3x}e^{2ix}dx = \int e^{(3+2i)x}dx = \frac{1}{(3+2i)}e^{(3+2i)x} + C$$

$$= \frac{3-2i}{13}e^{(3+2i)x} + C = \tfrac{1}{13}e^{3x}(3-2i)(\cos 2x + i\sin 2x) + C$$

The imaginary part is

$$\underline{\underline{\tfrac{1}{13}e^{3x}(3\sin 2x - 2\cos 2x) + C}}$$

You could also try integrating by parts (either way) - but that's not so interesting.

27.

$$\int \frac{x^2dx}{(x^2+9)(x^2+4)^2} = \frac{1}{25}\int\left[-\frac{9}{x^2+9} + \frac{9}{x^2+4} - \frac{20}{(x^2+4)^2}\right]$$

$$= \tfrac{1}{25}\left[-3\tan^{-1}\tfrac{1}{3}x + \frac{9}{2}\tan^{-1}\tfrac{1}{2}x - \frac{5}{4}\left(\frac{2x}{x^2+4} + \tan^{-1}\tfrac{1}{2}x\right)\right] + C$$

$$= \underline{\underline{\tfrac{1}{100}\left[-12\tan^{-1}\tfrac{1}{3}x + 13\tan^{-1}\tfrac{1}{2}x - \frac{10x}{(x^2+4)}\right] + C}}$$

28. $\int \cos^{\frac{3}{2}}2x\sin x dx$

Let's try the first of my suggestions. First,
$y = \cos x, \quad dy = -\sin x dx \qquad \cos 2x = 2\cos^2 x - 1 = 2y^2 - 1$

$$\int \cos^{\frac{3}{2}}2x\sin x dx = -\int (2y^2-1)^{\frac{3}{2}}dy$$

Then maybe try letting $2y^2 = \sec^2\theta$. That is, $y = \frac{1}{\sqrt{2}}\sec\theta$. This looks promising, so let's go back to the beginning and make the truly Brilliant Substitution
$\cos x = \frac{1}{\sqrt{2}}\sec\theta, \ \sin x dx = -\frac{1}{\sqrt{2}}\sec\theta\tan\theta d\theta, \ \cos 2x = 2\cos^2 x - 1 = \sec^2\theta - 1 = \tan^2\theta.$
Then:

$$\int \cos^{\frac{3}{2}}2x\sin x dx = -\frac{1}{\sqrt{2}}\int \sec\theta\tan^4\theta d\theta.$$

We already know, from pages 6 and 7, how to integrate $\sec^n\theta$, so we'll use $\tan^2\theta = \sec^2\theta - 1$ and hence write the integral as

$$\int \cos^{\frac{3}{2}} 2x \sin x dx = -\frac{1}{\sqrt{2}} \int (\sec^5\theta - 2\sec^3\theta + \sec\theta) d\theta$$

On carrying out the procedures suggested on pages 6 and 7, I obtain

$$\int \sec\theta = \ln(\sec\theta + \tan\theta) + C_1$$
$$\int \sec^3\theta = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln(\sec\theta + \tan\theta) + C_3$$
$$\int \sec^5\theta = \tfrac{1}{4}\sec^3\theta\tan\theta + \tfrac{3}{8}\sec\theta\tan\theta + \tfrac{3}{8}\ln(\sec\theta + \tan\theta) + C_5$$

Thus the integral becomes, after a little tidying up,

$$\int \cos^{\frac{3}{2}} 2x \sin x dx = \tfrac{\sqrt{2}}{16}[\sec\theta\tan\theta(5 - 2\sec^2\theta) - 3\ln(\sec\theta + \tan\theta)] + C.$$

We have to get back to x, remembering that $\sec\theta = \sqrt{2}\cos x$, from which $\tan\theta = \sqrt{2\cos^2 x - 1} = \sqrt{\cos 2x}$. Thus

$$\int \cos^{\frac{3}{2}} 2x \sin x\, dx = \tfrac{\sqrt{2}}{16}\left[\sqrt{2\cos 2x}\cos x\,(5 - 4\cos^2 x) - 3\ln(\sqrt{2}\cos x + \sqrt{\cos 2x})\right] + C.$$

The second suggestion was let $y = \cos 2x$.

This leads to $\int \cos^{\frac{3}{2}} 2x \sin x dx = \int \sqrt{\frac{y^3}{8(1+y)}} dy.$

The third suggestion was let $y^2 = \cos 2x$.

This leads to $\int \cos^{\frac{3}{2}} 2x \sin x dx = \int \frac{y^4}{\sqrt{2(1+y^2)}} dy.$

You could try either of these and see where they lead. For *numerical* integration (see http://orca.phys.uvic.ca/~tatum/celmechs/celm1.pdf) these are far faster than the original expression.

29.

$$\int \frac{1}{x^3+1}dx \;=\; \int \frac{1}{(x+1)(x^2-x+1)}dx \;=\; \frac{1}{3}\int\left[\frac{1}{x+1} + \frac{-x+2}{x^2-x+1}\right]dx$$

$$\frac{1}{3}\int\left[\frac{1}{x+1} - \frac{1}{2}\,\frac{2x-4}{(x^2-x+1)}\right]dx \;=\; \frac{1}{3}\int\left[\frac{1}{x+1} - \frac{1}{2}\,\frac{2x-1}{(x^2-x+1)} + \frac{3}{2}\,\frac{1}{(x^2-x+1)}\right]dx$$

$$= \tfrac{1}{3}I_1 - \tfrac{1}{6}I_2 + \tfrac{1}{2}I_1 .$$

$$I_1 = \ln(x+1) + C_1$$
$$I_2 = \ln(x^2 - x + 1) + C_2$$

$$I_3 = \int \frac{1}{x^2 - x + \frac{1}{4} + \frac{3}{4}}dx = \int \frac{1}{[(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2]}dx = \tfrac{2}{\sqrt{3}}\tan^{-1}\frac{2(x-1)}{\sqrt{3}} + C_3$$

Hence

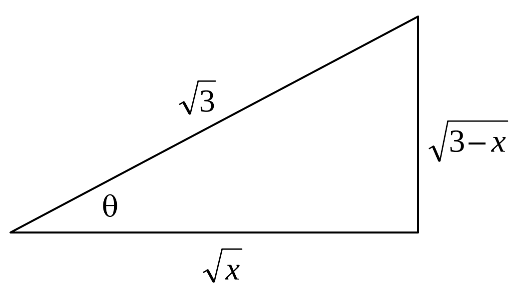
$$\int \frac{1}{x^3+1}dx \;=\; \underline{\underline{\tfrac{1}{3}\ln(x+1) - \tfrac{1}{6}\ln(x^2 - x + 1) + \tfrac{1}{\sqrt{3}}\tan^{-1}\frac{2(x-1)}{\sqrt{3}} + C}}$$

30.

$\int_0^1 \frac{x^3dx}{\sqrt{1-x}}$. With the substitution $x = \sin^2\theta$, this becomes

$$2\int_0^{\pi/2}\sin^7\theta d\theta = \underline{\underline{\frac{32}{35}}}$$

31.



$$x = 3\cos^2\theta \qquad \tan\theta = \sqrt{\frac{3-x}{x}} \qquad dx = -6\sin\theta\cos\theta d\theta$$

The integral becomes $-6\int \tan\theta\sin\theta\cos\theta d\theta = -6\int \sin^2\theta d\theta = 3\int(\cos 2\theta - 1)d\theta$

$= \frac{3}{2}\sin 2\theta - 3\theta + C = 3\sin\theta\cos\theta - 3\theta + C = 3.\sqrt{\frac{3-x}{3}}.\sqrt{\frac{x}{3}} - 3\tan^{-1}\sqrt{\frac{3-x}{x}} + C$

$$= \underline{\underline{\sqrt{x(3-x)} - 3\tan^{-1}\sqrt{\frac{3-x}{x}} + C}}$$

32. If you try the substitution $y = 1 + e^x$, you arrive at

$$\int \frac{dy}{(y-1)y} = \int \frac{dy}{(y-1)} - \int \frac{dy}{y} = \ln\frac{y-1}{y} = \ln\frac{e^x}{1+e^x} = \underline{\underline{-\ln(1+e^{-x}) + C}}$$

If you write the numerator as $1 + e^x - e^x$, you arrive at

$\int dx - \int \frac{e^x}{1+e^x}dx = x - \ln(1+e^x) + C$. And since $x = \ln e^x$, this simplifies to

$$\ln\frac{e^x}{1+e^x} = \underline{\underline{-\ln(1+e^{-x}) + C}}$$

33. The substitution $x = y + 1$ results in

$$\int \frac{y^3 + 3y^2 + 3y + 1}{y}dy = \frac{1}{3}y^3 + \frac{3}{2}y^2 + 3y + \ln y + C$$

Then $y = x - 1$ and a little algebra results in $\underline{\underline{\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln(x-1) + C}}$

Stuart McAlpine's solution: $x^3/(x-1) = x^2 + x + 1 + 1/(x-1)$, after which it is plain sailing.

34. I think the first thing I'd do would be to write $\sin^2 3x$ as $\frac{1}{2} - \frac{1}{2}\cos 6x$.

Then

$$\int e^{2x}\sin^2 3x dx = \frac{1}{2}\int e^{2x}dx - \frac{1}{2}\int e^{2x}\cos 6x dx = \frac{1}{4}e^{2x} - \frac{1}{2}I,$$

where $I = \int e^{2x}\cos 6x dx$

You could try integrating this by parts. I haven't tried it. Instead I tried

$$I = \text{Re} \int e^{(2+6i)x} dx = \text{Re} \frac{e^{(2+6i)x}}{2+6i} + C = \tfrac{1}{40} e^{2x} \text{Re}\,(2-6i)(\cos 6x + i\sin 6x) + C$$

$$I = \tfrac{1}{40} e^{2x}(2\cos 6x + 6\sin 6x) + C = \tfrac{1}{20} e^{2x}(\cos 6x + 3\sin 6x) + C$$

Hence

$$\int e^{2x}\sin^2 3x dx \left(= \tfrac{1}{4}e^{2x} - \tfrac{1}{2}I\right) = \underline{\underline{e^{2x}\left(\frac{1}{4} - \frac{\cos 6x + 3\sin 3x}{40}\right) + C}}$$

35.

$$I = \int_0^{\pi/4} \ln(1+\tan\theta) d\theta$$

Let $\theta = \frac{\pi}{4} - \phi$.. Then $d\theta = -d\phi$, and the limits become $\frac{\pi}{4}$ and 0.

Also, $1 + \tan\theta = 1 + \tan(\frac{\pi}{4} - \phi) = 1 + \dfrac{1-\tan\phi}{1+\tan\phi} = \dfrac{2}{1+\tan\phi}$

Thus $I = \int_0^{\pi/4} \ln\left(\dfrac{2}{1+\tan\phi}\right) d\phi = \int_0^{\pi/4} \ln 2 d\phi - \int_0^{\pi/4} \ln(1+\tan\phi) d\phi = \dfrac{\pi \ln 2}{4} - I$

Hence $\underline{\underline{I = \dfrac{\pi\ln 2}{8}}}$. Encouragingly, this is 0.272198261, which is what the numerical integration gave.

36. $\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

The substitution $x = \tan\theta$ gives, after a little algebra, $\int_0^{\pi/4} \ln(1+\tan\theta) d\theta$, which is the same as number **35**. Therefore the answer is $\underline{\underline{\dfrac{\pi \ln 2}{8}}}$.

Interestingly I tried both integrals (**35** and **36**) on Wolfram. Like me, it couldn't find a simple expression for the indefinite integral. For the definite integral, in both cases it gave both the numerical solution and the analytical one. Very impressive.

37.

We can deal with the case $\varepsilon = 1$ immediately, because it is the same as example 5, and there are several easy ways of dealing with it. We shall give that case no further thought.

If we make the substitution $t = \tan\frac{1}{2}\theta \quad \cos\theta = \dfrac{1-t^2}{1+t^2} \quad d\theta = \dfrac{2dt}{1+t^2}$ the integral becomes $2\displaystyle\int \frac{dt}{1+\varepsilon+(1-\varepsilon)t^2}$.

<u>If $\varepsilon < 1$</u>, we write this as

$$\frac{2}{1-\varepsilon}\int \frac{dt}{\frac{1+\varepsilon}{1-\varepsilon} + t^2}$$

so the answer is

$$\frac{2}{\sqrt{1-\varepsilon^2}}\tan^{-1}\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\,t\right) + C$$

where $t = \tan\frac{1}{2}\theta$.

<u>If $\varepsilon > 1$, we write it as</u>

$$\frac{2}{\varepsilon-1}\int \frac{dt}{\frac{\varepsilon+1}{\varepsilon-1} - t^2}$$

This can be written in terms of an inverse hyperbolic tangent if you are familiar with such things, but it is easier to split the integrand into partial fractions. Put

$$a = \sqrt{\frac{\varepsilon+1}{\varepsilon-1}}$$

to make this easier, if you like. The integral now becomes

$$\frac{2}{\varepsilon-1}\cdot\frac{1}{2a}\int\left[\frac{1}{a-t} + \frac{1}{a+t}\right]dt$$

$$= \underline{\underline{\frac{1}{\sqrt{\varepsilon^2 - 1}}\left[\ln\left(\frac{a+t}{a-t}\right)\right] + C}}$$, where $t = \tan\frac{1}{2}\theta$ and

$a \sqrt{\dfrac{\varepsilon + 1}{\varepsilon - 1}}$

38. $\int x\sqrt{1-x}\,dx$ Let $y = \sqrt{1-x}$, $x = 1 - y^2$, $dx = -2y\,dy$.
After that, it is straightforward, and you eventually arrive at the unlikely-looking

$$\underline{\underline{y = C - \tfrac{2}{15}(1-x)^{3/2}(2+3x).}}$$

39. $\int \dfrac{dx}{1 + a\cos x}$. For $a = 1$, see example **5.**

For $a = -1,\ 0,\ +1$ the answers are, respectively $\underline{\underline{-\cot\frac{1}{2}x + C}}$, $\underline{\underline{x + C}}$, $\underline{\underline{\tan\frac{1}{2}x + C}}$.

Otherwise, let $t = \tan\frac{1}{2}x$, and for a between -1 and $+1$ the integral becomes

$$2\int\frac{dt}{1+a + (1-a)t^2} = \frac{2}{1-a}\int\frac{dt}{\frac{1+a}{1-a} + t^2}.$$

Now let $t = \sqrt{\frac{1+a}{1-a}}\tan\psi$ and the answer will come out very quickly as

$$\underline{\underline{\frac{2}{\sqrt{1-a^2}}\tan^{-1}\left(\sqrt{\frac{1-a}{1+a}}\,t\right) + C}}, \text{ where } t = \tan^{-1}x.$$

This is good for either positive or negative a, as long as $|a| < 1$.

For $|a| > 1$, you'll need to write the integral as

$$2\int\frac{dt}{a + 1 - (a-1)t^2}.$$

You now have a choice, depending on which you dislike the least, hyperbolic functions or logarithms. If you don't mind hyperbolic functions, just proceed as we did in the case $|a| < 1$.

Thus $$2\int\frac{dt}{a+1-(a-1)t^2} = \frac{2}{a-1}\int\frac{dt}{\frac{a+1}{a-1}-t^2}$$

Now let $t = \sqrt{\frac{a+1}{a-1}}\tanh\psi$ and the answer will come out very quickly as

$$\underline{\underline{\frac{2}{\sqrt{a^2-1}}\tanh^{-1}\left(\sqrt{\frac{a-1}{a+1}}\,t\right) + C}}, \text{ where } t = \tan\tfrac{1}{2}x.$$

If you don't like $\tanh^{-1}$ (and most of us don't see things like that every day), try this:

Let $\frac{a+1}{a-1} = b^2$, so the integral becomes $\frac{2}{a-1}\int\frac{dt}{b^2-t^2}$.

Now $b^2 - t^2 = (b-t)(b+t)$, so split the integral into partial fractions so that it

becomes $$\frac{2}{b(a-1)}\int\left(\frac{1}{t+b} - \frac{1}{t-b}\right)dt = \underline{\underline{\frac{1}{\sqrt{a^2-1}}\ln\left(\frac{t+b}{t-b}\right) + C}}.$$

where $t = \tan\frac{1}{2}x, \quad b = \sqrt{\frac{a+1}{a-1}}$.

This looks like a nice, simple straightforward expression, and so it is. However, for $|a| > 1$, the expression $\frac{1}{1+a\cos x}$ is not a "well-behaved" function, in that it has some infinities. You will have to take care if you want the *definite* integral between two limits on either side of an infinity.

It might be noted in passing that $r = \frac{l}{1+\varepsilon\cos\theta}$ is the equation, in polar coordinates, of a conic section of eccentricity ε. It is unlikely in that context that one would want to calculate $\int r\,d\theta$, but you never know. In any case, if ε > 1, the conic section is a hyperbola - so you see why there are some infinities in this case.

40. $$\frac{1}{x^4+1} = \frac{1}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)} = \frac{-\frac{1}{4}\sqrt{2}x+\frac{1}{2}}{(x^2-\sqrt{2}x+1)} + \frac{\frac{1}{4}\sqrt{2}x+\frac{1}{2}}{(x^2+\sqrt{2}x+1)}$$

and so the integral is

$$-\tfrac{1}{4}\sqrt{2}\int\frac{xdx}{x^2-\sqrt{2}x+1}+\tfrac{1}{4}\sqrt{2}\int\frac{xdx}{x^2+\sqrt{2}x+1}+\tfrac{1}{2}\int\frac{dx}{x^2-\sqrt{2}x+1}+\tfrac{1}{2}\int\frac{dx}{x^2+\sqrt{2}x+1}$$

The first integral can be written

$$-\tfrac{1}{8}\sqrt{2}\int\frac{2x-\sqrt{2}+\sqrt{2}}{x^2-\sqrt{2}x+1}dx=-\tfrac{1}{8}\sqrt{2}\int\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx-\tfrac{1}{4}\int\frac{1}{x^2-\sqrt{2}x+1}dx$$

The second integral can be written

$$\tfrac{1}{8}\sqrt{2}\int\frac{2x+\sqrt{2}-\sqrt{2}}{x^2+\sqrt{2}x+1}dx=\tfrac{1}{8}\sqrt{2}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx-\tfrac{1}{4}\int\frac{1}{x^2-\sqrt{2}x+1}dx$$

So we now have $\displaystyle\int\frac{dx}{x^4+1}=$

$$-\tfrac{1}{8}\sqrt{2}\int\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx+\tfrac{1}{8}\sqrt{2}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx+\tfrac{1}{4}\int\frac{1}{x^2-\sqrt{2}x+1}dx+\tfrac{1}{4}\int\frac{1}{x^2+\sqrt{2}x+1}dx$$

Write the denominator of the third integral as $x^2-\sqrt{2}x+\tfrac{1}{2}+\tfrac{1}{2}=(x-\tfrac{1}{\sqrt{2}})^2+(\tfrac{1}{\sqrt{2}})^2$,

and similarly for the fourth integral, so we now have

$$-\tfrac{1}{8}\sqrt{2}\int\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}dx+\tfrac{1}{8}\sqrt{2}\int\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}dx$$
$$+\tfrac{1}{4}\int\frac{1}{(x-\tfrac{1}{\sqrt{2}})^2+(\tfrac{1}{\sqrt{2}})^2}dx+\tfrac{1}{4}\int\frac{1}{(x+\tfrac{1}{\sqrt{2}})^2+(\tfrac{1}{\sqrt{2}})^2}dx$$

$$=-\tfrac{1}{8}\sqrt{2}\ln(x^2-\sqrt{2}x+1)+\tfrac{1}{8}\sqrt{2}\ln(x^2+\sqrt{2}x+1)-\tfrac{1}{4}\sqrt{2}\tan^{-1}(1-\sqrt{2}x)+\tfrac{1}{4}\sqrt{2}\tan^{-1}(1+\sqrt{2}x)$$
$$+C$$

$$=\underline{\underline{\tfrac{1}{4}\sqrt{2}\left[\tan^{-1}(1+\sqrt{2}x)-\tan^{-1}(1-\sqrt{2}x)+\ln\sqrt{\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}}\right]+C}}$$

Difficult Integrals

$$\int x^2 e^{-2x^2} dx \quad \int \cos x^2 dx \quad \int \frac{dx}{\sqrt{1-4\sin^2\theta}} \quad \int \frac{dx}{\sqrt{3+2\cos\theta}} \quad \int x^{-2}e^{-3x}dx \quad \int \frac{x^3}{e^x-1}dx$$

All of these integrals look at first glance as though they are about the same order of difficulty as the others we have met earlier in this Chapter, and you could probably do them in half-an-hour or so. Go ahead and try! If you give up, read on.

These - and many others that we may come across - cannot be expressed in terms of simple familiar functions. If they are to be integrated as definite integrals between definite lower and upper integrals, they will usually have to be integrated numerically, although it is possible that, in some cases, if they are to be integrated, say from 0 to 1, or 0 to ∞, or 0 to $\pi/2$, they may have exact values. I do not deal with numerical integration here; I merely draw attention to some integrals that cannot be expressed analytically. For methods of numerical integration, see Chapter 1, Section 2, Celestial Mechanics orca.phys.uvic.ca/-tatum/celmechs.html

We dealt with $\int x^n e^{-ax^2} dx$ earlier in this chapter, so we don't repeat it here, other than to give the following brief table:

$$\int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty xe^{-ax^2} dx = \frac{1}{2a} \qquad \int_0^\infty x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$

$$\int_0^\infty x^3 e^{-ax^2} dx = \frac{1}{2a^2} \qquad \int_0^\infty x^4 e^{-ax^2} dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$$

$$\int_0^\infty x^5 e^{-ax^2} dx = \frac{1}{a^3}. \qquad \int_0^\infty x^6 e^{-ax^2} dx = \frac{15\sqrt{\pi}}{16a^{7/2}}$$

It is related to the *error function* erf(x). $\int \cos(-x^2)dx$ and $\int \sin(-x^2)dx$ can be dealt with by writing $\cos(-x^2)$ as $\mathrm{Re}(e^{-x^2})$ and $\sin(-x^2)$ as $\mathrm{Im}(e^{-x^2})$. Things like $\int x^n e^{+ax^2} dx$, $\int \cos(+x^2)dx$ and $\int \sin(+x^2)dx$ are not closely related to erf(x), but can be integrated numerically between limits. $\int \cos(+x^2)dx$ and $\int \sin(+x^2)dx$ are *Fresnel integrals*. In physics they turn up in the theory of the Cornu spiral.

Integrals involving $\sqrt{1-k^2\sin^2\theta}$, or expressions that can be written like this, are called *elliptic integrals*. In particular

$$\int \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\ , \quad \int \sqrt{1-k^2\sin^2\theta}\, d\theta, \quad \int \frac{d\theta}{\left(1-k^2\sin^2\theta\right)^{3/2}}$$

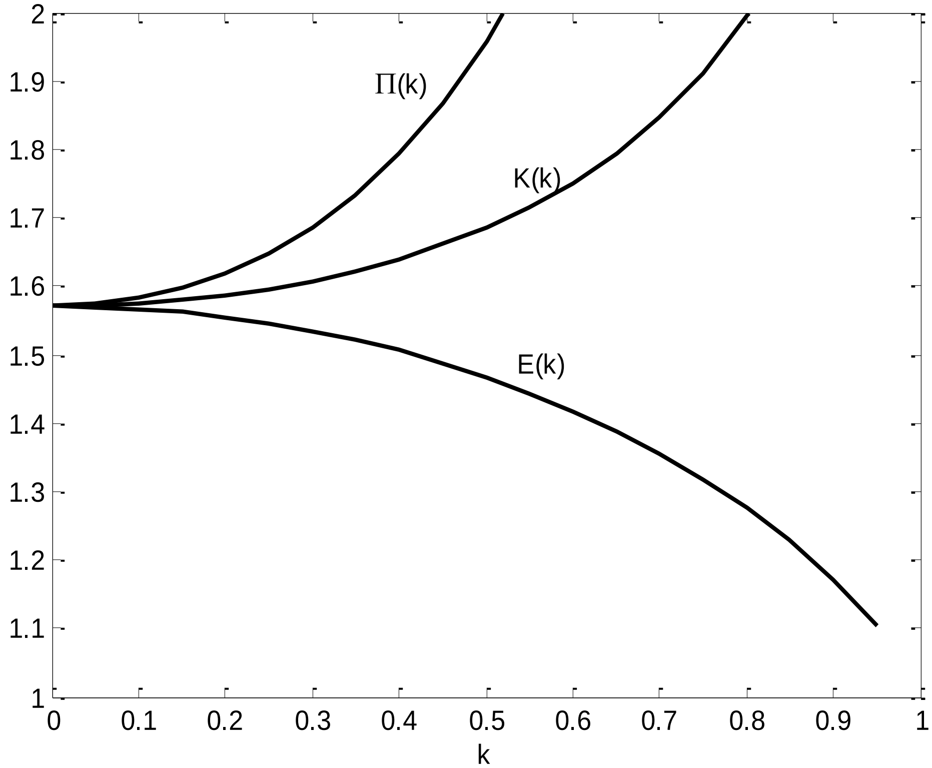
are *elliptic integrals of the first, second and third kinds* respectively. The definite integrals

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\ , \quad E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta, \quad \Pi(k) = \int_0^{\pi/2} \frac{d\theta}{\left(1-k^2\sin^2\theta\right)^{3/2}}$$

are *complete elliptic integrals of the first, second and third kinds* respectively. Here are

brief tables and graphs of them:

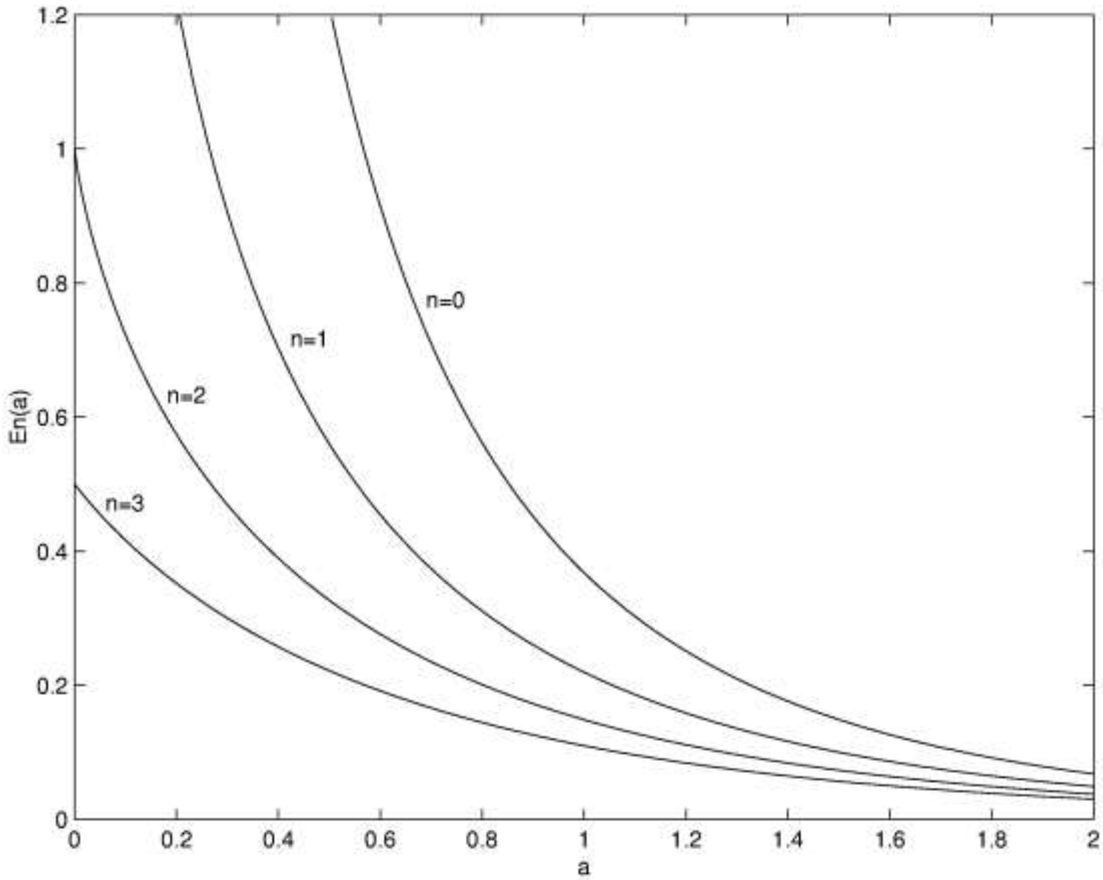
k	$K(k)$	$E(k)$	$\Pi(k)$
0.00	1.5708	1.5708	1.5708
0.05	1.5718	1.5698	1.5737
0.10	1.5747	1.5669	1.5827
0.15	1.5797	1.5619	1.5979
0.20	1.5869	1.5550	1.6198
0.25	1.5962	1.5460	1.6490
0.30	1.6080	1.5348	1.6866
0.35	1.6225	1.5215	1.7339
0.40	1.6400	1.5059	1.7928
0.45	1.6609	1.4880	1.8658
0.50	1.6858	1.4675	1.9566
0.55	1.7154	1.4442	2.0706
0.60	1.7508	1.4181	2.2158
0.65	1.7935	1.3887	2.4047
0.70	1.8457	1.3887	2.6582
0.75	1.9110	1.3185	3.0137
0.80	1.9953	1.2763	3.5454
0.85	2.1099	1.2281	4.4256
0.90	2.2805	1.1717	6.1668
0.95	2.5900	1.1027	11.3100



Integrals involving expressions such as $\sqrt{a + b\cos\theta}$, $\sqrt{\sin^2\alpha - \sin^2\theta}$, $\sqrt{\sec\theta}$ can be written in terms of elliptic integrals. Expressions such as these occur in physics in calculating the gravitational potential and field of a massive ring, or the oscillations of a pendulum through a large angle. For $\sqrt{a + b\cos\theta}$, write $\theta = 2\phi$ and $\cos 2\phi = 1 - 2\sin^2\phi$.

$\underline{\underline{\int x^{-2}e^{-3x}dx}}$ You'd think you could just integrate this by parts - but each time you do, it gets worse and worse instead of better and better. Between definite limits, it can be integrated numerically. If the limits are 1 and ∞, it is an example of a *exponential integral function*:

$$E_n(a) = \int_1^{\infty} x^{-n} e^{-ax} dx.$$



For further details on this function, see http://orca.phys.uvic.ca/~tatum/stellatm/atm3.pdf

Other integrals. Some people collect stamps. Others collect difficult integrals. Here are a few from my collection. I may add more later if I think of it.

$$\int_0^{\infty} x^2 \ln(1 - e^{-x})dx = -\frac{\pi^4}{45}$$

$$\int_0^{\infty} \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

$$\int_0^{\infty} \frac{x^4 e^x}{(e^x - 1)^2} dx = \frac{4\pi^4}{15}$$

$$\int_0^{\infty} \frac{dx}{x^5 (e^{1/x} - 1)} = \frac{\pi^4}{15}$$