CHAPTER 4
RIGID BODY ROTATION

4.1 *Introduction*

No real solid body is perfectly rigid. A rotating nonrigid body will be distorted by centrifugal force* or by interactions with other bodies. Nevertheless most people will allow that in practice some solids are fairly rigid, are rotating at only a modest speed, and any distortion is small compared with the overall size of the body. No excuses, therefore, are needed or offered for analysing to begin with the rotation of a rigid body.

*I do not in this chapter delve deeply into whether there really is "such thing" as "centrifugal force". Some would try to persuade us that there is no such thing. But is there "such thing" as a "gravitational force"? And is one any more or less "real" than the other? These are deep questions best left to the philosophers. In physics we use the concept of "force" – or indeed any other concept – according as to whether it enables us to supply a description of how physical bodies behave. Many of us would, I think, be challenged if we were faced with an examination question: "Explain, without using the term *centrifugal force*, why Earth bulges at its equator."

We have already discussed some aspects of solid body rotation in Chapter 2 on Moment of Inertia, and indeed the present Chapter 4 should not be plunged into without a good understanding of what is meant by "moment of inertia". One of the things that we found was that, while the comfortable relation $L = I\omega$ which we are familiar with from elementary physics is adequate for problems in two dimensions, in three dimensions the relation becomes $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, where $\mathbf{I}$ is the *inertia tensor*, whose properties were discussed at some length in Chapter 2. We also learned in Chapter 2 about the concepts of *principal moments of inertia*, and we introduced the notion that, unless a body is rotating about one of its principal axes, the equation $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ implies that the angular momentum and angular velocity vectors *are not in the same direction.* We shall discuss this in more detail in this chapter.

A full treatment of the rotation of an *asymmetric top* (whose three principal moments of inertia are unequal and which has as its momental ellipsoid a triaxial ellipsoid) is very lengthy, since there are so many cases to consider. I shall restrict consideration of the motion of an asymmetric top to a qualitative argument that shows that rotation about the principal axis of greatest moment of inertia or about the axis of least moment of inertia is stable, whereas rotation about the intermediate axis is unstable.

I shall treat in more detail the free rotation of a *symmetric top* (which has two equal principal moments of inertia) and we shall see how it is that the angular velocity vector *precesses* while the angular momentum vector (in the absence of external torques) remains fixed in magnitude and direction.

I shall also discuss the situation in which a symmetric top is subjected to an external torque (in which case $\mathbf{L}$ is certainly not fixed), such as the motion of a top. A similar situation, in which Earth is subject to external torques from the Sun and Moon, causes Earth's axis to precess with a period of 26,000 years, and this will be dealt with in a chapter of the notes on Celestial Mechanics.

Before discussing these particular problems, there are a few preparatory topics, namely, angular velocity and Eulerian angles, kinetic energy, Lagrange's equations of motion, and Euler's equations of motion.

4.2 *Angular Velocity and Eulerian Angles*

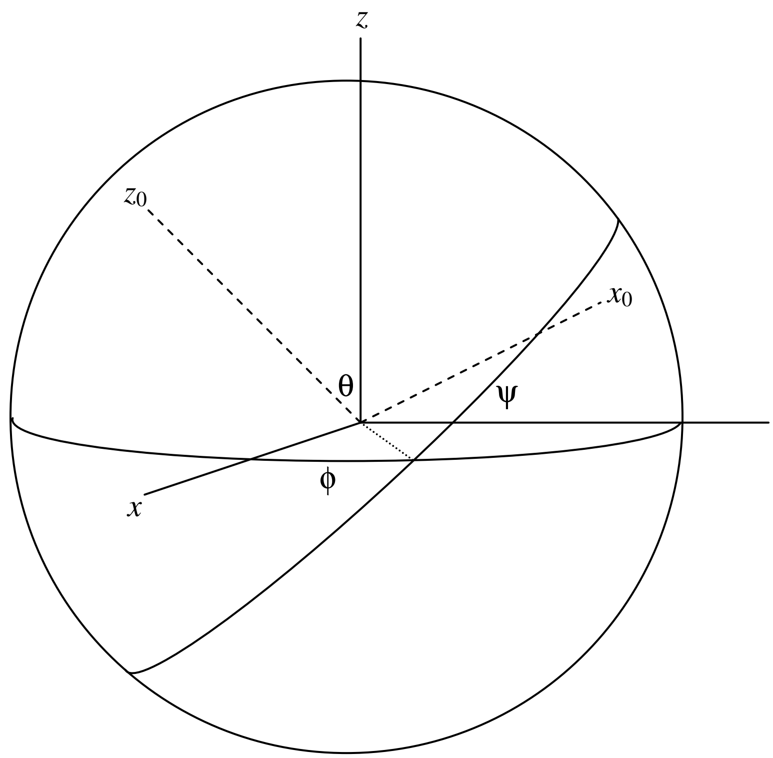


FIGURE IV.1a

Let Oxyz be a set of space-fixed axis, and let O$x_0y_0z_0$ be the body-fixed principal axes of a rigid body. The orientation of the body-fixed principal axes Ox_0 y_0 $z_0$with respect to the space-fixed axes Oxyz can be described by the three Euler angles θ, ϕ, ψ.. These are illustrated in Figure IV.1a. Those who are not familiar with Euler angles or who would like a reminder can refer to their detailed description in Chapter 3 of my notes on Celestial Mechanics.

We are going to examine the motion of a body that is rotating about a nonprincipal axis. If the body is freely rotating in space with no external torques acting upon it, its angular momentum $\mathbf{L}$ will be constant in magnitude and direction. The angular velocity vector $\boldsymbol{\omega}$, however, will not be constant, but will wander with respect to both the space-fixed and body-fixed axes, and we shall be examining this motion. I am going to call the instantaneous components of $\boldsymbol{\omega}$ relative to the body-fixed axes $\omega_1, \omega_2, \omega_3$, and its magnitude ω. As the body tumbles over and over, its Euler angles will be changing continuously. We are going to establish a geometrical relation between the instantaneous rates of change of the Euler angles and the instantaneous components of $\boldsymbol{\omega}$. That is, we are going to find how ω_1, ω_2 and ω_3 are related to $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$.

I have indicated, in figure VI.2a, the angular velocities $\dot{\boldsymbol{\theta}}$, $\dot{\boldsymbol{\phi}}$ and $\dot{\boldsymbol{\psi}}$ as vectors in what I hope will be agreed are the appropriate directions.

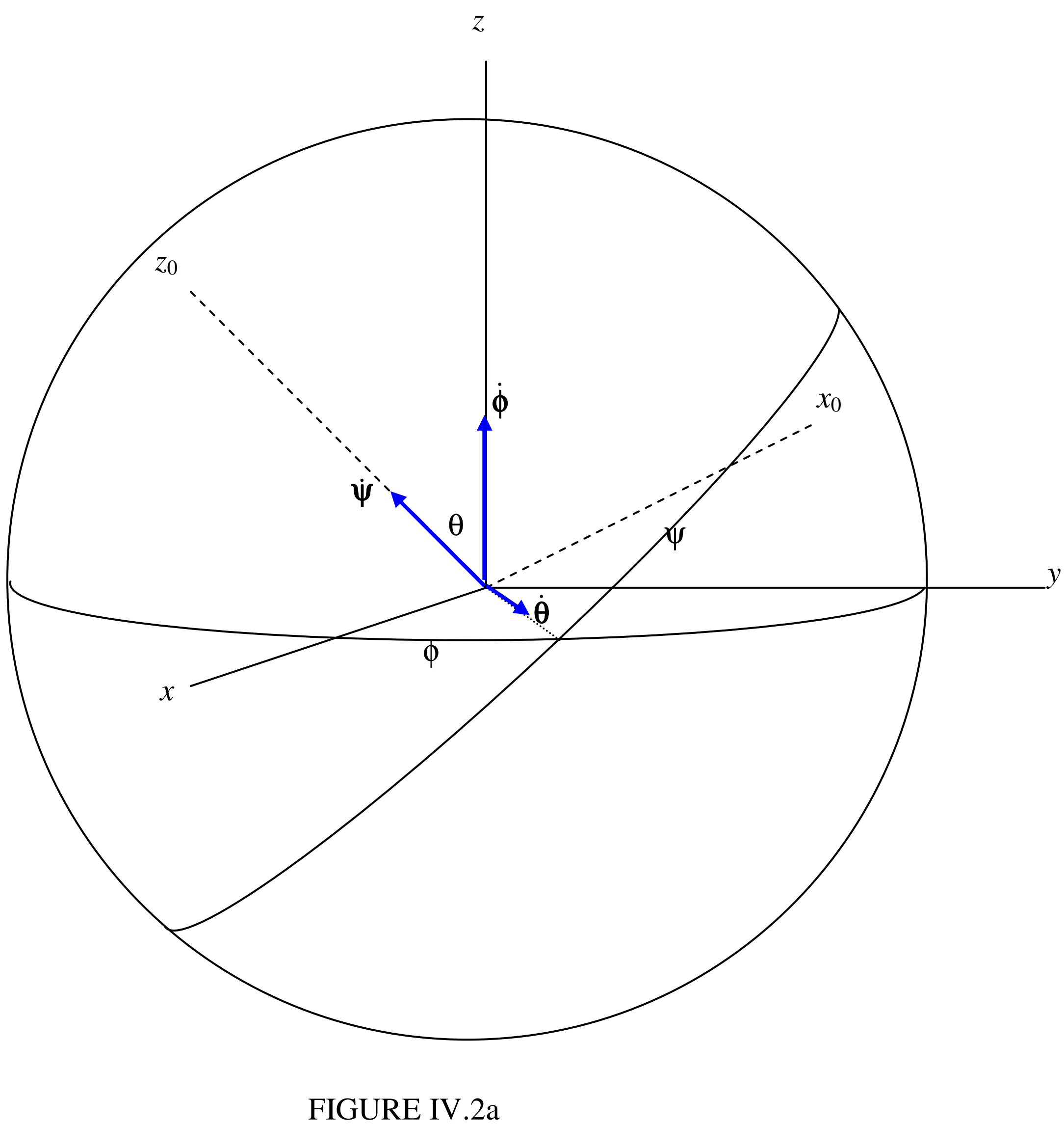


FIGURE IV.2a

It should be clear that ω_1 is equal to the x_0-component of $\dot{\boldsymbol{\phi}}$ plus the x_0-component of $\dot{\boldsymbol{\theta}}$ and that ω_2 is equal to the y_0-component of $\dot{\boldsymbol{\phi}}$ plus the y_0-component of $\dot{\boldsymbol{\theta}}$ and that ω_3 is equal to the z_0-component of $\dot{\boldsymbol{\phi}}$ plus $\dot{\boldsymbol{\psi}}$.

Let us look at Figure IV.2b

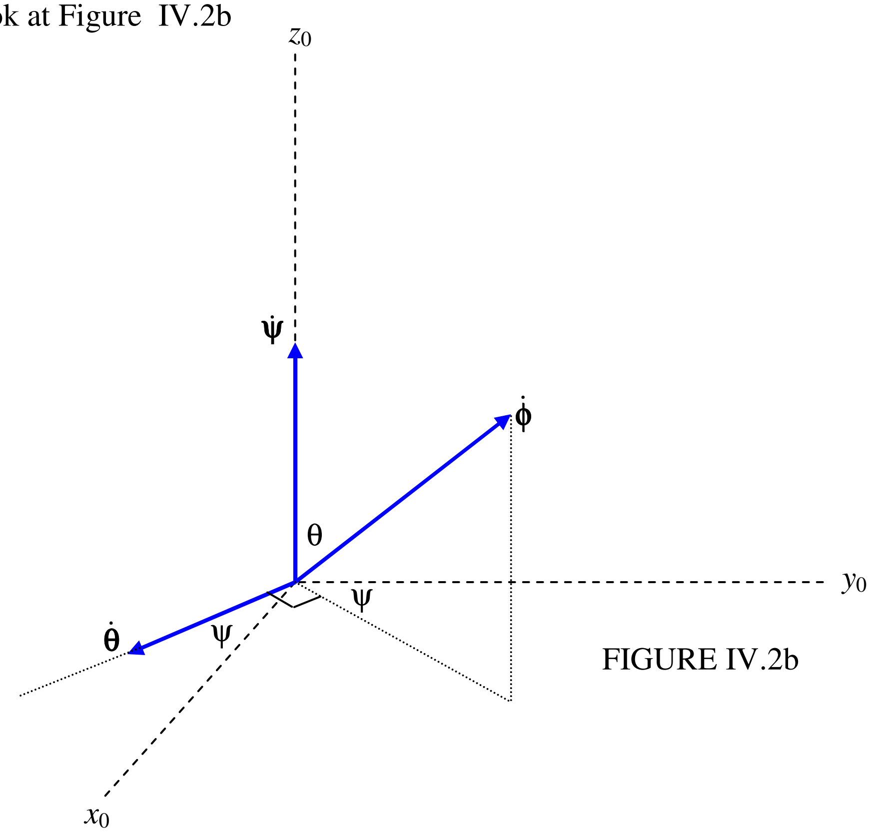


FIGURE IV.2b

We see that the x_0 and y_0 components of $\dot{\boldsymbol{\theta}}$ are $\dot{\theta}\cos\psi$ and $-\dot{\theta}\sin\psi$ respectively. The x_0, y_0 and z_0 components of $\dot{\boldsymbol{\phi}}$ are, respectively, $\dot{\phi}\sin\theta\sin\psi$, $\dot{\phi}\sin\theta\cos\psi$ and $\dot{\phi}\cos\theta$.

Hence we arrive at

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi. \qquad 4.2.1$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \qquad 4.2.2$$

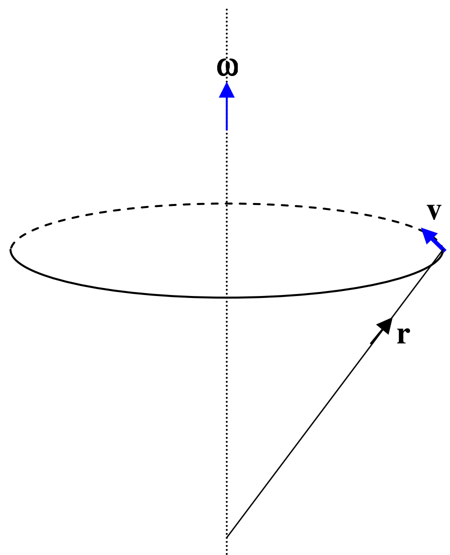
$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}. \qquad 4.2.3$$

4.3 *Kinetic Energy*

Most of us are familiar with the formula $\frac{1}{2}I\omega^2$ for the rotational kinetic energy of a rotating solid body. This formula is adequate for simple situations in which a body is rotating about a principal axis, but is not adequate for a body rotating about a nonprincipal axis.

I am going to think of a rotating solid body as a collection of point masses, fixed relative to each other, but all revolving with the same angular velocity about a common axis – and those who believe in atoms assure me that this is indeed the case. (If you believe that a solid is a continuum, you can still divide it in your imagination into lots of small mass elements.)

FIGURE IV.3



In figure IV.3, I show just one particle of the rotating body. The position vector of the particle is **r**. The body is rotating at angular velocity $\boldsymbol{\omega}$. I hope you'll agree that the linear velocity **v** of the particle is (now think about this carefully) $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$..

The rotational kinetic energy of the solid body is

$$\mathbf{T}_{\mathrm{rot}} = \tfrac{1}{2}\sum \mathbf{m}\upsilon^2 = \tfrac{1}{2}\sum \mathbf{v}\bullet\mathbf{m}\mathbf{v} = \tfrac{1}{2}\sum(\boldsymbol{\omega}\times\mathbf{r})\bullet\mathbf{p}$$

The triple scalar product is the volume of a parallelepiped, which justifies the next step:

$$= \tfrac{1}{2}\sum \boldsymbol{\omega}\bullet(\mathbf{r}\times\mathbf{p}).$$

All particles have the same angular velocity, so:

$$= \tfrac{1}{2}\boldsymbol{\omega}\bullet\sum(\mathbf{r}\times\mathbf{p}) = \tfrac{1}{2}\boldsymbol{\omega}\bullet\mathbf{L} = \tfrac{1}{2}\boldsymbol{\omega}\bullet\mathbf{I}\boldsymbol{\omega}.$$

Thus we arrive at the following expressions for the rotational kinetic energy:

$$T_{\text{rot}} = \tfrac{1}{2}\boldsymbol{\omega}\bullet\mathbf{L} = \tfrac{1}{2}\boldsymbol{\omega}\bullet\mathbf{I}\boldsymbol{\omega}. \qquad 4.3.1$$

If the body is rotating about a nonprincipal axis, the vectors $\boldsymbol{\omega}$ and **L** are not parallel (we shall discuss this in more detail in later sections). If it is rotating about a principal axis, they *are* parallel, and the expression reduces to the familiar $\frac{1}{2}I\omega^2$.

In matrix notation, this can be written

$$\mathbf{T}_{\text{rot}} = \tfrac{1}{2}\tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega}. \qquad 4.3.2$$

Here **I** is the inertia tensor, $\boldsymbol{\omega}$ is a column vector containing the rectangular components of the angular velocity and $\tilde{\boldsymbol{\omega}}$ is its transpose, namely a row vector.

That is:

$$T_{\text{rot}} = \begin{pmatrix}\omega_x & \omega_y & \omega_z\end{pmatrix}\begin{pmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C\end{pmatrix}\begin{pmatrix}\omega_x \\ \omega_y \\ \omega_z\end{pmatrix}, \qquad 4.3.3$$

or $$T_{\text{rot}} = \tfrac{1}{2}\left(A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - 2F\omega_y\omega_z - 2G\omega_z\omega_x - 2H\omega_x\omega_y\right). \qquad 4.3.4$$

This expression gives the rotational kinetic energy when the components of the inertia tensor and the angular velocity vector are referred to an arbitrary set of axes. If we refer them to the *principal* axes, the off-diagonal elements are zero. I am going to call the principal moments of inertia I_1, I_2 and I_3. (I could call them *A*, *B* and *C*, but I shall often use the convention that $A < B < C$, and I don't want to specify at the present which of the three moments is the greatest and which is the greatest, so I'll call them I_1, I_2 and I_3, with $I_1 = \sum m(y^2 + z^2)$, etc.). I'll also call the angular velocity components referred to the principal axes ω_1, ω_2, ω_3. Referred, then, to the principal axes, the rotational kinetic energy is

$$T = \tfrac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right). \qquad 4.3.5$$

I have now dropped the subscript "rot", because in this chapter I am dealing entirely with rotational motion, and so *T* can safely be understood to mean rotational kinetic energy.

We can also now write the kinetic energy in terms of the rates of change of the Eulerian angles, and the expression we obtain will be useful later when we derive Euler's equations of motion:

$$T = \tfrac{1}{2} I_1 \left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + \tfrac{1}{2} I_2 \left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2 + \tfrac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2 .$$

4.3.6

You will probably want a concrete example in order to understand this properly, so let us imagine that we have a concrete brick of dimensions 10 cm × 15 cm × 20 cm, and of density 4 g cm^{-3} , and that it is rotating about a body diameter (the ends of which are fixed) at an angular speed of 6 rad s^{-1} .

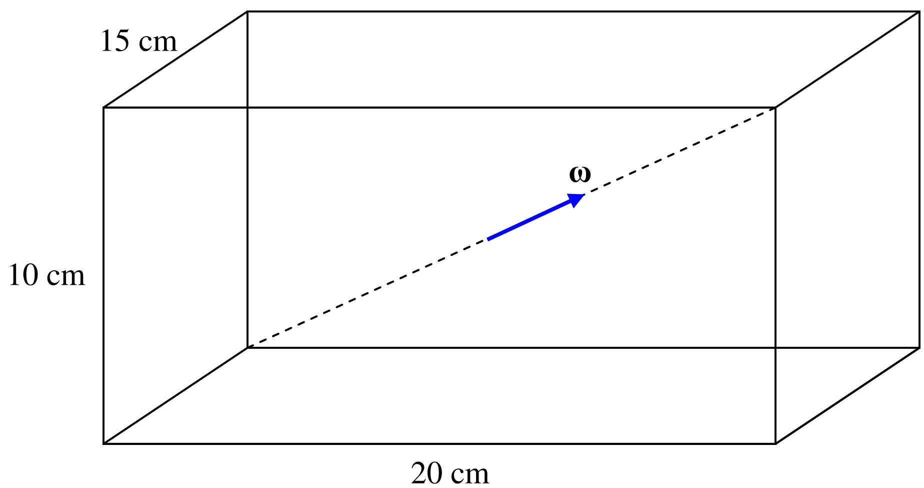


FIGURE IV.4

I hope you'll agree that the mass is 12000 g = 12 kg.

The principal moment of inertia about the vertical axis is

$$I_3 = \tfrac{1}{3} \times 12000 \times (10^2 + 7.5^2) = 625{,}000 \text{ g cm}^2 = 0.0625 \text{ kg m}^2 .$$

Similarly the other principal moments are

$$I_1 = 0.0500 \text{ kg m}^2 \quad \text{and} \quad I_2 = 0.0325 \text{ kg m}^2 .$$

The direction cosines of the vector $\boldsymbol{\omega}$ are

$$\frac{15}{\sqrt{15^2 + 20^2 + 10^2}} , \quad \frac{20}{\sqrt{15^2 + 20^2 + 10^2}} , \quad \frac{10}{\sqrt{15^2 + 20^2 + 10^2}} .$$

Therefore $\omega_1 = 3.34252$ rad s^{-1}, $\omega_2 = 4.45669$ rad s^{-1}, $\omega_3 = 2.22834$ rad s^{-1} .

Hence $\underline{T\ =\ 0.02103\ \text{J}}$

4.4 *Lagrange's Equations of Motion*

In Section 4.5 I want to derive Euler's equations of motion, which describe how the angular velocity components of a body change when a torque acts upon it. In deriving Euler's equations, I find it convenient to make use of Lagrange's equations of motion. This will cause no difficulty to anyone who is already familiar with Lagrangian mechanics. Those who are not familiar with Lagrangian mechanics may wish just to understand what it is that Euler's equations are dealing with and may wish to skip over their derivation at this stage. Later in this series, I hope to add a longer chapter on Lagrangian mechanics, when all will be made clear (maybe). In the meantime, for those who are not content just to accept Euler's equations but must also understand their derivation, this section gives a five-minute course in Lagrangian mechanics.

To begin with, I have to introduce the idea of *generalized coordinates* and *generalized forces.*

The geometrical description of a mechanical system at some instant of time can be given by specifying a number of *coordinates.* For example, if the system consists of just a single particle, you could specify its rectangular coordinates xyz, or its cylindrical coordinates $\rho\phi z$, or its spherical coordinates $r\theta\phi$. Certain theorems to be developed will be equally applicable to any of these, so we can think of *generalized coordinates* $q_1q_2q_3$, which could mean any one of the rectangular, cylindrical of spherical set.

In a more complicated system, for example a polyatomic molecule, you might describe the geometry of the molecule at some instant by a set of interatomic distances plus a set of angles between bonds. A fairly large number of distances and angles may be necessary. These distances and angles can be called the *generalized coordinates.* Notice that generalized coordinates need not always be of dimension L. Some generalized coordinates, for example, may have the dimensions of angle.

[See Appendix of this Chapter for a brief discussion as to whether angle is a dimensioned or a dimensionless quantity.]

While the generalized coordinates at an instant of time describe the geometry of a system at an instant of time, they alone do not predict the future behaviour of the system.

I now introduce the idea of *generalized forces*. With each of the generalized coordinates there is associated a *generalized force.* With the generalized coordinate q_i there is associated a corresponding generalized force P_i. It is defined as follows. If, when the generalized coordinate q_i increases by δq_i, the work done <u>on</u> the system is $P_i\delta q_i$, then P_i is the generalized force associated with the generalized coordinate q_i. For example, in our simple example of a single particle, if one of the generalized coordinates is merely the x-

coordinate, the generalized force associated with x is the x-component of the force acting on the particle.

Note, however, that often one of the generalized coordinates might be an *angle*. In that case the generalized force associated with it is a *torque* rather than a force. In other words, a generalized force need not necessarily have the dimensions MLT^{-2}.

Before going on to describe Lagrange's equations of motion, let us remind ourselves how we solve problems in mechanics using Newton's law of motion. We may have a ladder leaning against a smooth wall and smooth floor, or a cylinder rolling down a wedge, the hypotenuse of which is rough (so that the cylinder does not slip) and the smooth base of which is free to obey Newton's third law of motion on a smooth horizontal table, or any of a number of similar problems in mechanics that are visited upon us by our teachers. The way we solve these problems is as follows. We draw a large diagram using a pencil , ruler and compass. Then we mark in red all the *forces*, and we mark in green all the *accelerations*. If the problem is a two-dimensional problem, we write $F = ma$ in any two directions; if it is a three-dimensional problem, we write $F = ma$ in any three directions. Usually this is easy and straightforward. Sometimes it doesn't seem to be as easy as it sounds, and we may prefer to solve the problem by Lagrangian methods.

To do this, as before, we draw a large diagram using a pencil , ruler and compass. But this time we mark in blue all the *velocities* (including angular velocities).

Lagrange, in the Introduction to his book *La méchanique analytique* (modern French spelling omits the *h*) pointed out that there were no diagrams at all in his book, since all of mechanics could be done analytically – hence the title of the book. Not all of us, however, are as mathematically gifted as Lagrange, and we cannot bypass the step of drawing a large, neat and clear diagram.

Having drawn in the velocities (including angular velocities), we now calculate the *kinetic energy*, which in advanced texts is often given the symbol T, presumably because potential energy is traditionally written U or V. There would be no harm done if you prefer to write E_k , E_p and E for kinetic, potential and total energy. I shall stick to T , U or V, and E.

Now, instead of writing $F = ma$, we write, for each generalized coordinate, the Lagrangian equation (whose proof awaits a later chapter):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = P_i . \qquad 4.4.1$$

The only further intellectual effort on our part is to determine what is the generalized force associated with that coordinate. Apart from that, the procedure goes quite automatically. We shall use it in use in the next section.

That ends our five-minute course on Lagrangian mechanics.

4.5 *Euler's Equations of Motion*

In our first introduction to classical mechanics, we learn that when an external torque acts on a body its angular momentum changes (and if no external torques act on a body its angular momentum does not change.) We learn that the rate of change of angular momentum is equal to the applied torque. In the first simple examples that we typically meet, a symmetrical body is rotating about an axis of symmetry, and the torque is also applied about this same axis. The angular momentum is just $I\omega$, and so the statement that torque equals rate of change of angular momentum is merely $\tau = I\dot{\omega}$, and that's all there is to it.

Later, we learn that $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, where $\mathbf{I}$ is a tensor, and $\mathbf{L}$ and $\boldsymbol{\omega}$ are not parallel. There are three principal moments of inertia, and $\mathbf{L}$, $\boldsymbol{\omega}$ and the applied torque $\boldsymbol{\tau}$ each have three components, and the statement "torque equals rate of change of angular momentum" somehow becomes much less easy.

Euler's equations sort this out, and give us a relation between the components of the $\boldsymbol{\tau}$, $\mathbf{I}$ and $\boldsymbol{\omega}$.

For figure IV.5, I have just reproduced, with some small modifications, figure III.19 from my notes on this Web site on Celestial Mechanics, where I defined *Eulerian angles*. Again it is suggested that those who are unfamiliar with Eulerian angles consult Chapter III of Celestial Mechanics.

In figure IV.5, Oxyz are space-fixed axes, and O$x_0y_0z_0$ are the *body-fixed principal axes.* The axis Oy_0 is behind the plane of your screen; you will have to look inside your monitor to find it.

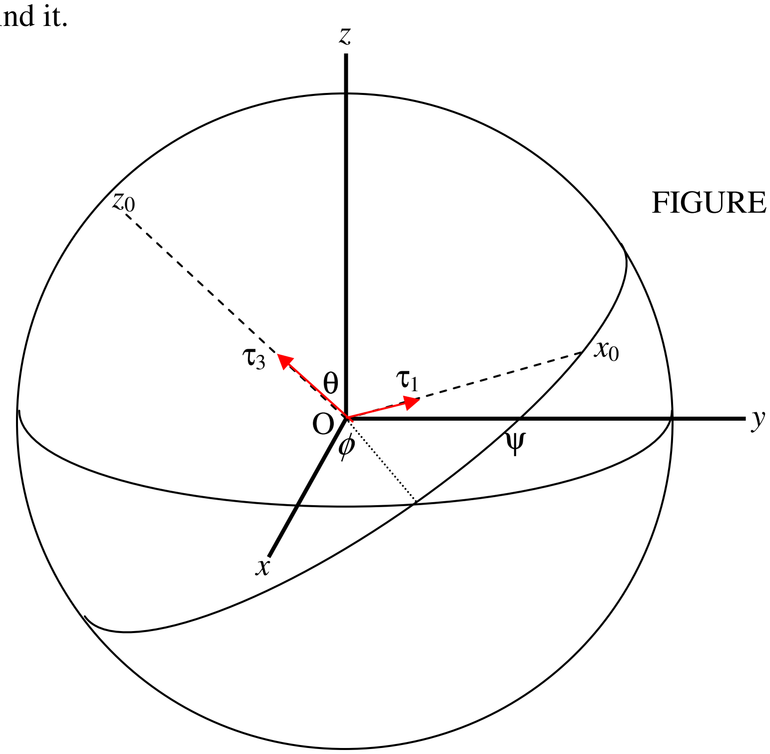


FIGURE IV.5

I suppose an external torque $\boldsymbol{\tau}$ acts on the body, and I have drawn the components τ_1 and τ_3. Now let's suppose that the body rotates in such a manner that the Eulerian angle ψ were to increase by $\delta\psi$. I think it will be readily agreed that the work done on the body is $\tau_3\delta\psi$. This means, following our definition of generalized force in section 4.4, that τ_3 *is the generalized force associated with the generalized coordinate* ψ. Having established that, we can now apply the Lagrangian equation 4.4.1:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = \tau_3 . \qquad 4.5.1$$

Here the kinetic energy is the expression that we have already established in equation 4.3.6. In spite of the somewhat fearsome aspect of equation 4.3.6, it is quite easy to apply equation 4.5.1 to it. Thus

$$\frac{\partial T}{\partial \dot{\psi}} = I_3(\dot{\phi}\cos\theta + \dot{\psi}) = I_3\omega_3 , \qquad 4.5.2$$

where I have made use of equation 4.2.3.

Therefore $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = I_3\dot{\omega}_3 . \qquad 4.5.3$$

And, if we make use of equations 4.2.1,2,3, it is easy to obtain

$$\frac{\partial T}{\partial \psi} = I_1\omega_1\omega_2 - I_2\omega_2\omega_1 = \omega_1\omega_2(I_1 - I_2). \qquad 4.5.4$$

Thus equation 4.5.1 becomes:

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = \tau_3 . \qquad 4.5.5$$

This is one of the *Eulerian equations of motion*.

Now, although we saw that τ_3 is the generalized force associated with the coordinate ψ, it will we equally clear that τ_1 is *not* the generalized force associated with θ, nor is τ_2 the generalized force associated with ϕ. However, we do not have to think about what the generalized forces associated with these two coordinates are; it is much easier than that. To obtain the remaining two Eulerian equations, all that is necessary is to carry out a cyclic permutation of the subscripts in equation 4.5.5. Thus the three Eulerian equation are:

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = \tau_1 , \qquad 4.5.6$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = \tau_2 \ , \qquad 4.5.7$$

$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = \tau_3 \ . \qquad 4.5.8$$

These take the place of $\tau = I\dot{\omega}$, which we are more familiar with in elementary problems in which a body is rotating about a principal axis and a torque is applied around that principal axis.

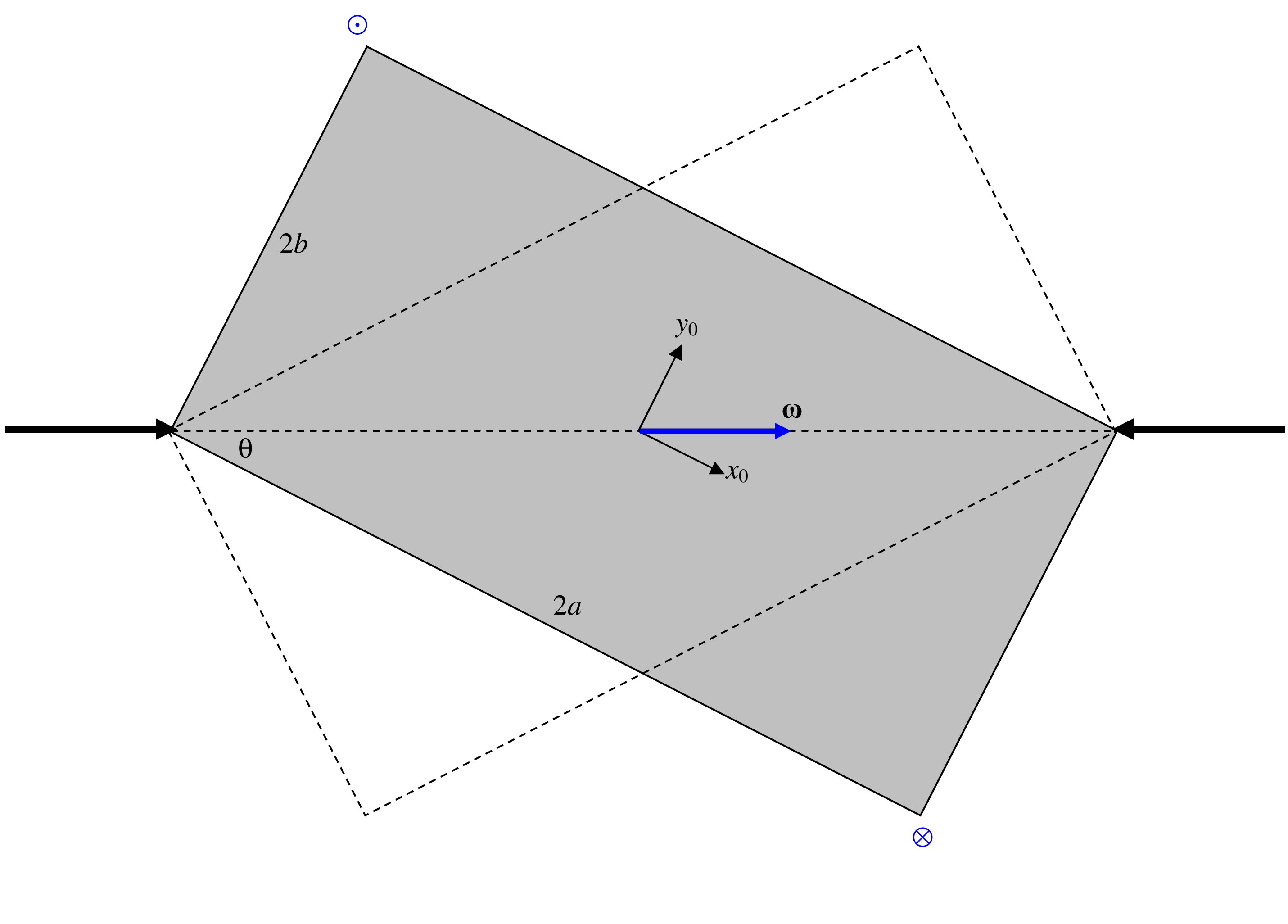
If there are no external torques acting on the body, then we have Euler's equations of free rotation of a rigid body:

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 \ , \qquad 4.5.9$$

$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 \ , \qquad 4.5.10$$

$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 \ . \qquad 4.5.11$$

Example.



In the above drawing, a rectangular lamina is spinning with constant angular velocity $\boldsymbol{\omega}$ between two frictionless bearings. We are going to apply Euler's equations of motion to it. We shall find that the bearings are exerting a torque on the rectangle, and the rectangle is exerting a torque on the bearings. The angular momentum of the rectangle is not constant – at least it is not constant in *direction*. We shall calculate the torque (its magnitude and its direction) and see what is happening to the angular momentum.

We note that the principal (second) moments of inertia are

$$I_1 = \tfrac{1}{3}mb^2 \qquad I_2 = \tfrac{1}{3}ma^2 \qquad I_3 = \tfrac{1}{3}m(a^2 + b^2)$$

and that the components of angular velocity are

$$\omega_1 = \omega\cos\theta \qquad \omega_2 = \omega\sin\theta \qquad \omega_3 = 0.$$

Also, $\dot{\boldsymbol{\omega}}$ and all of its components are zero. We immediately obtain, from Euler's equations, that τ_1 and τ_2 are zero, and that the torque exerted **on** the rectangle **by** the bearings is

$$\tau_3 = (I_2 - I_1)\omega_1\omega_2 = \tfrac{1}{3}m(a^2 - b^2)\omega^2\sin\theta\cos\theta.$$

And since $\sin\theta = \dfrac{b}{\sqrt{a^2 + b^2}}$ and $\cos\theta = \dfrac{a}{\sqrt{a^2 + b^2}}$,

we obtain $\tau_3 = \dfrac{m(a^2 - b^2)ab}{3(a^2 + b^2)}\omega^2.$

Thus $\boldsymbol{\tau}$, the torque exerted **on** the rectangle **by** the bearings is directed normal to the plane of the rectangle (out of the plane of the paper in the instantaneous snapshot above).

The angular momentum is given by $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$. That is to say:

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \tfrac{1}{3}m \begin{pmatrix} b^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \begin{pmatrix} \omega\cos\theta \\ \omega\sin\theta \\ 0 \end{pmatrix}$$

$$L_1 = \tfrac{1}{3}mb^2\omega\cos\theta = \tfrac{1}{3}m\frac{ab^2}{\sqrt{a^2+b^2}}\omega$$

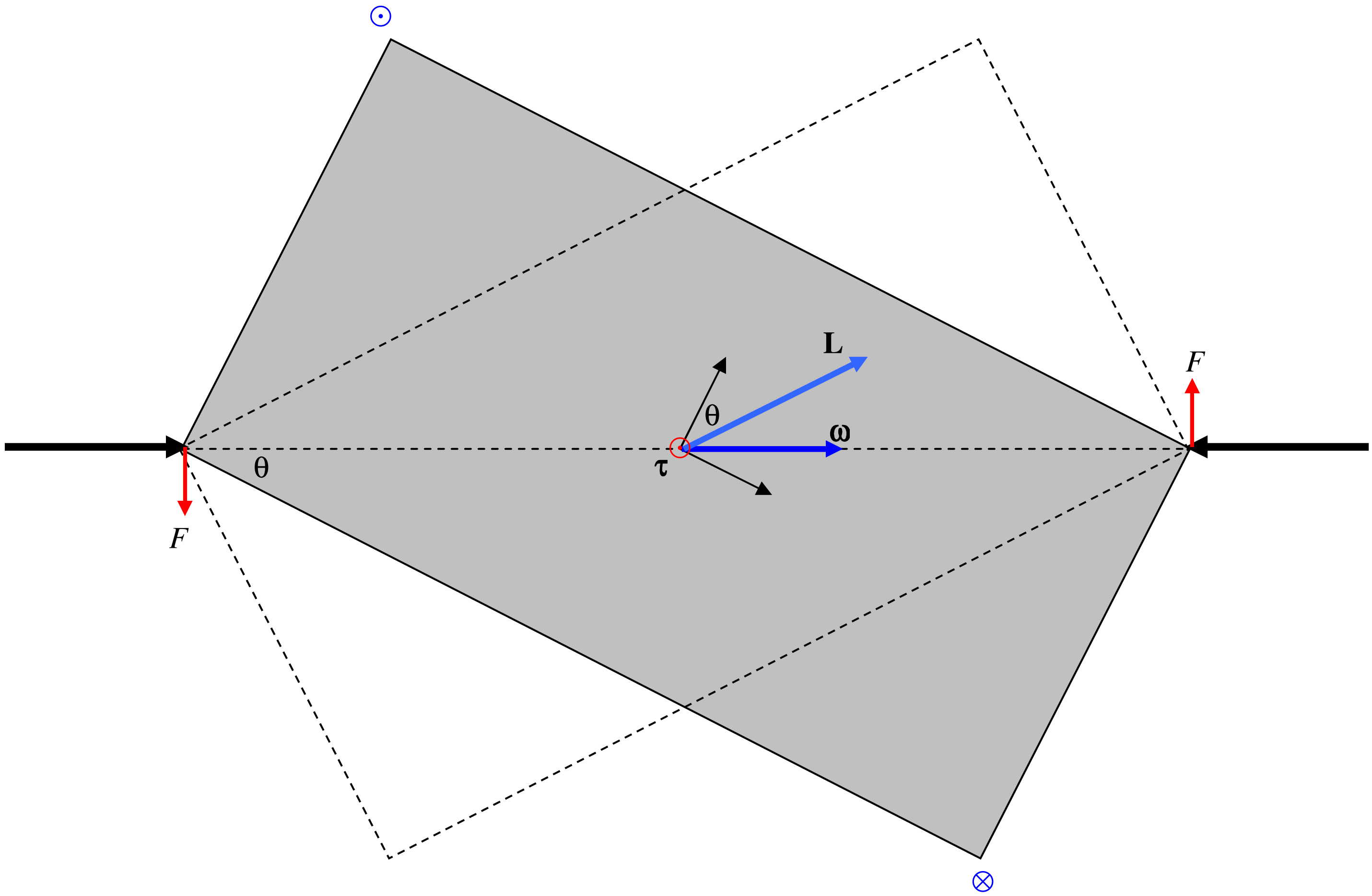
$$L_2 = \tfrac{1}{3}ma^2\omega\sin\theta = \tfrac{1}{3}m\frac{a^2b}{\sqrt{a^2+b^2}}\omega$$

$$L_3 = 0$$

$$L = \tfrac{1}{3}mab\omega$$

$$L_2 / L_1 = \frac{a^2\sin\theta}{b^2\cos\theta} = \cot\theta = \tan(90^\circ - \theta).$$

This tells us that **L** is in the plane of the rectangle, and makes an angle $90^\circ - \theta$ with the *x*-axis, or θ with the *y*-axis, and it rotates around the vector $\boldsymbol{\tau}$. $\boldsymbol{\tau}$ is perpendicular to the plane of the rectangle, and of course the change in **L** takes place in that direction. The torque does no work, and $\boldsymbol{\omega}$ and *T* are constant. The reader might find an analogy in the situation of a planet in orbit around the Sun in a cicular orbit.. The planet experiences a force that is always perpendicular to its velocity. The force does no work, and the speed and kinetic energy remain constant.



The torque on the plate can be represented as a *couple* of forces exerted by the bearings on the plate, each of magnitude $\dfrac{\tau_3}{2\sqrt{a^2+b^2}}$, or $\dfrac{m(a^2-b^2)ab}{6(a^2+b^2)^{3/2}}\omega^2$. Forces exerted by the plate on the bearings are, of course, in the opposite direction.

Example.

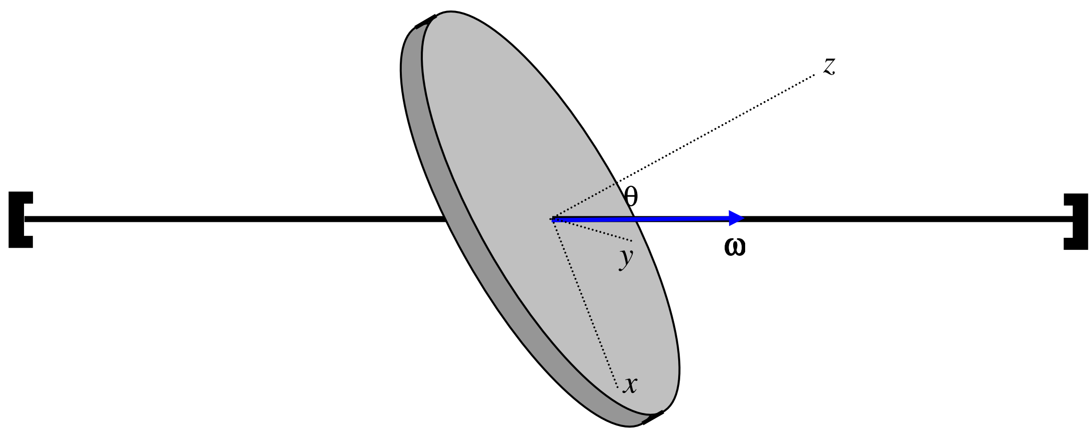


FIGURE IV.6

Figure IV.6 shows a disc of mass *m*, radius *a*, spinning at a constant angular speed ω about at axle that is inclined at an angle θ to the normal to the disc. I have drawn three body-fixed principal axes. The *x*- and *y*- axes are in the plane of the disc; the direction of the *x*-axis is chosen so that the axle (and hence the vector $\boldsymbol{\omega}$) is in the *zx*-plane. The disc is evidently unbalanced and there must be a torque on it to maintain the motion.

Since $\boldsymbol{\omega}$ is constant, all components of $\dot{\boldsymbol{\omega}}$ are zero, so that Euler's equations are

$$\tau_1 = (I_3 - I_2)\omega_3\omega_2\,,$$

$$\tau_2 = (I_1 - I_3)\omega_1\omega_3\,,$$

$$\tau_3 = (I_2 - I_1)\omega_2\omega_1\,.$$

Now $\omega_1 = \omega\sin\theta$, $\omega_2 = 0$, $\omega_3 = \omega\cos\theta$, $I_1 = \frac{1}{4}ma^2$, $I_2 = \frac{1}{4}ma^2$, $I_3 = \frac{1}{2}ma^2$.
Therefore $\tau_1 = \tau_3 = 0$, *and* $\tau_2 = -\frac{1}{4}ma^2\omega^2\sin\theta\cos\theta = -\frac{1}{8}ma^2\omega^2\sin 2\theta$.

(Check, as always, that this expression is dimensionally correct.) Thus the torque acting on the disc is in the negative *y*-direction.

Can you reconcile the fact that there is a torque acting on the disc with the fact that is it moving with constant angular velocity? Yes, most decidedly! What is *not* constant is the *angular momentum* **L**, which is moving around the axle in a cone such that $\dot{\mathbf{L}} = -\tau_2\mathbf{j}$, where **j** is the unit vector along the *y*-axis.

4.6 *Force-Free Rotation of a Rigid Asymmetric Top*

By "asymmetric top" I mean a body whose three principal moments of inertia are unequal. While we often think of a "top" as a symmetric body spinning on a table, in this section the "top" will not necessarily be symmetric, and it will not be in contact with any table, nor indeed subjected to any external forces or torques.

A complete description of the motion of an asymmetric top is quite complicated, and therefore all that we shall attempt in this chapter is a qualitative description of certain aspects of the motion. That our description is going to be "qualitative" does not by any means imply that this section is not going to be replete with equations or that we can give our poor brains a rest.

The first point that we can make is that, *provided that no external torques act on the body*, its angular momentum **L** is constant in magnitude and direction. A second point is that, *provided the body is rigid and has no internal degrees of freedom*, the rotational kinetic energy T is constant. I deal briefly with nonrigid bodies in section 4.7. Although the angular velocity vector $\boldsymbol{\omega}$ is by no means fixed in either magnitude and direction, and the body can tumble over and over, these two conditions impose some constraints of the magnitude and direction of $\boldsymbol{\omega}$.

We are going to examine these two conditions to see what constraints are imposed on $\boldsymbol{\omega}$. One of the things we shall find is that rotation of a body about a principal axis of greatest or of least moment of inertia is stable against small displacements, whereas rotation about the principal axis of intermediate moment of inertia is unstable.

Absence of an external torque means that the angular momentum is constant:

$$L^2 = L_1^2 + L_2^2 + L_3^2 = \text{constant}, \qquad 4.6.1$$

so that, at all times,

$$I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = L^2 . \qquad 4.6.2$$

Thus, for a given L. the angular velocity components always satisfy

$$\frac{\omega_1^2}{(L/I_1)^2} + \frac{\omega_2^2}{(L/I_2)^2} + \frac{\omega_3^2}{(L/I_3)^2} = 1. \qquad 4.6.3$$

That is to say, the angular velocity vector is constrained such that the tip of the vector $\boldsymbol{\omega}$ is always on the surface of an ellipsoid of semi axes L/I_1 , L/I_2 , L/I_3 .

In addition to the constancy of angular momentum, the kinetic energy is also constant:

$$\tfrac{1}{2}I_1\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2 + \tfrac{1}{2}I_3\omega_3^2 = T. \qquad 4.6.4$$

Thus the tip of the angular velocity vector must also be on the surface of the ellipsoid

$$\frac{\omega_1^2}{\left(\sqrt{2T/I_1}\right)^2} + \frac{\omega_2^2}{\left(\sqrt{2T/I_2}\right)^2} + \frac{\omega_3^2}{\left(\sqrt{2T/I_3}\right)^2} = 1. \qquad 4.6.5$$

This ellipsoid (which is similar in shape to the momental ellipsoid) has semi axes $\sqrt{2T/I_1}$, $\sqrt{2T/I_2}$, $\sqrt{2T/I_3}$.

Thus, how ever the body tumbles over and over, $\boldsymbol{\omega}$ is constrained in magnitude and direction so that its tip is on the curve where these two ellipses intersect.

Suppose, for example, that we have a rigid body with

$$I_1 = 0.2 \text{ kg m}^2 , \quad I_2 = 0.3 \text{ kg m}^2 , \quad I_3 = 0.5 \text{ kg m}^2 ,$$

and that we set it in motion such that the angular momentum and kinetic energy are

$$L = 4 \text{ J s} , \quad T = 20 \text{ J} .$$

(The angular momentum and kinetic energy will be determined by the magnitude and direction of the initial velocity vector by which it is set in motion.)

The tip of $\boldsymbol{\omega}$ is constrained to be on the curve of intersection of the two ellipsoids

$$\frac{\omega_1^2}{20^2} + \frac{\omega_2^2}{13.\dot{3}^2} + \frac{\omega_3^2}{8^2} = 1 \qquad 4.6.6$$

and

$$\frac{\omega_1^2}{14.14^2} + \frac{\omega_2^2}{11.55^2} + \frac{\omega_3^2}{8.94^2} = 1. \qquad 4.6.7$$

It is not easy (or I don't find it so) to imagine what this curve of intersection looks like in three-dimensional space, but one of my students, Leif Petersen, prepared the drawing below, and I am grateful to him for permission to reproduce it here. You can see that the curve of intersection is not a plane curve.

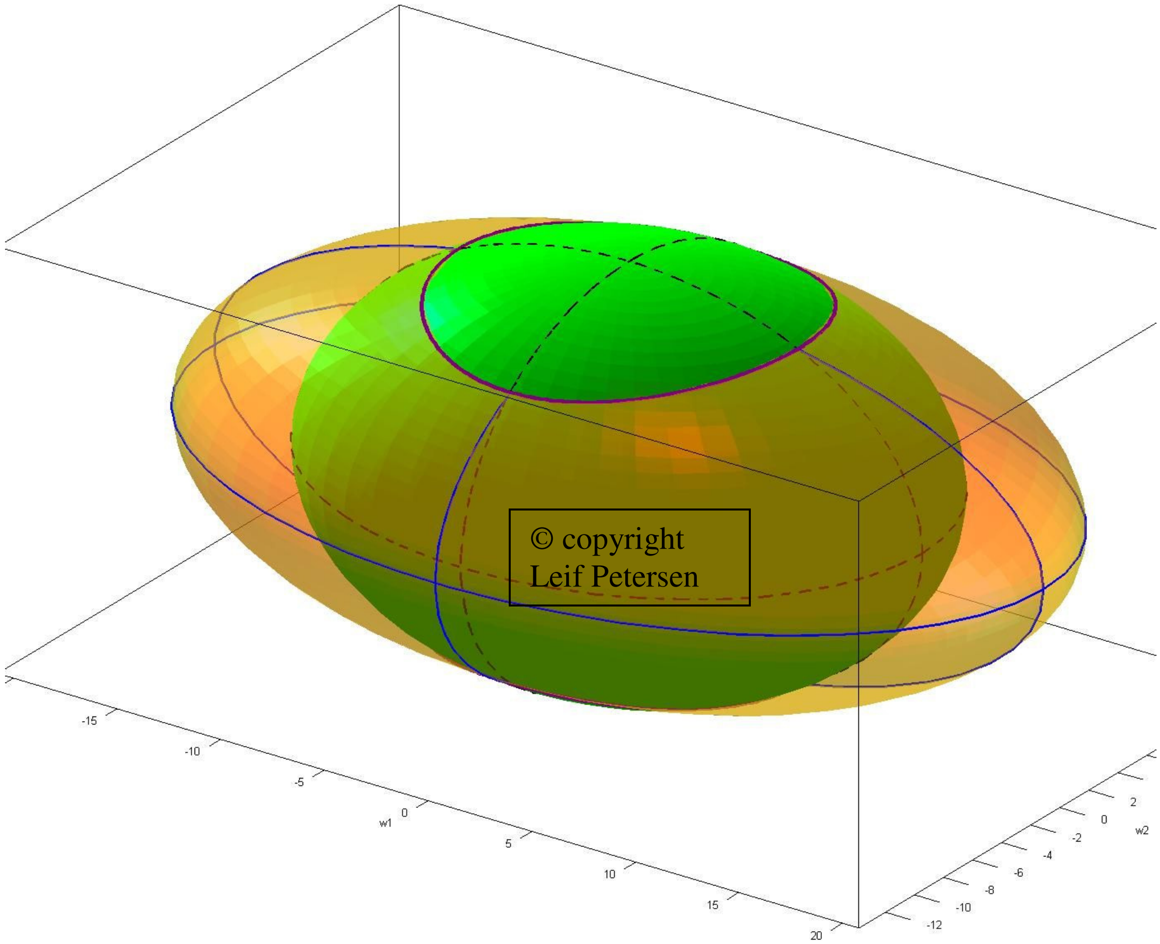
In case it's of any help, you might want to note that equations 4.6.6 and 4.6.7 can be written

$$4\omega_1^2 + 9\omega_2^2 + 25\omega_3^2 = 1600 \qquad 4.6.8$$

and

$$2\omega_1^2 + 3\omega_2^2 + 5\omega_3^3 = 400, \qquad 4.6.9$$

but I'm going to leave the equations in the form 4.6.6 and 4.6.7, and in figure IV.7, I'll sketch one octant of the two ellipsoidal surfaces.



L - Blue, T-Dashed Black, Orange mesh - L ellipsoid, Blue mesh - T ellipsoid

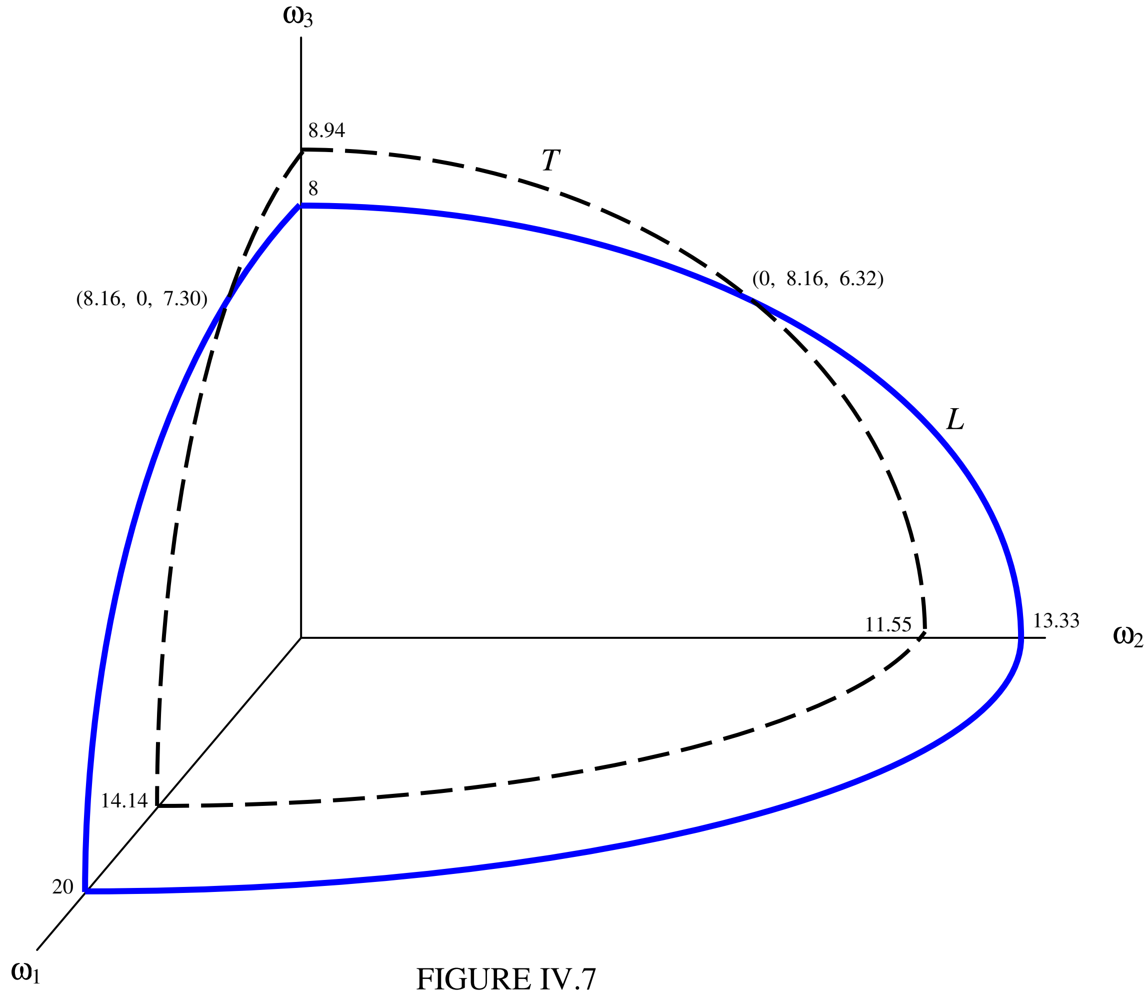


FIGURE IV.7

The continuous blue curve shows an octant of the ellipsoid L = constant, and the dashed black curve shows an octant of the ellipsoid T = constant. The angular momentum vector can end only on the curve (not drawn) where the two ellipsoids intersect. Two points on the curve are indicated in Figure IV.7. If, for example, $\boldsymbol{\omega}$ is oriented so that $\omega_1 = 0$, the other two components must be $\omega_2 = 8.16$ and $\omega_3 = 6.32$. If it is oriented so that $\omega_2 = 0$, the other two components must be $\omega_3 = 7.30$ and $\omega_1 = 8.16$. If $\omega_3 = 0$, there are no real solutions for ω_1 and ω_2. This means that, for the given values of L and T, ω_3 cannot be zero.

Now I'm going to address myself to the stability of rotation when a symmetric top is initially set to spin about one of its principal axes, which I'll take to be the z-axis. We'll suppose that initially $\omega_1 = \omega_2 = 0$, and $\omega_3 = \Omega$. In that case the angular momentum and the kinetic energy are $L = I_3\Omega$ and $T = \frac{1}{2}I_3\Omega^2$. In any subsequent motion, the tip of $\boldsymbol{\omega}$ is restricted to move along the curve of intersection of the ellipsoids given by equations 4.6.3 and 4.6.5. That is to say, along the curve of intersection of the ellipsoids

$$\frac{\omega_1^2}{\left(\frac{I_3}{I_1}\Omega\right)^2} + \frac{\omega_2^2}{\left(\frac{I_3}{I_2}\Omega\right)^2} + \frac{\omega_3^2}{\Omega^2} = 1 \qquad 4.6.10$$

and

$$\frac{\omega_1^2}{\left(\sqrt{\frac{I_3}{I_1}}\Omega\right)^2} + \frac{\omega_2^2}{\left(\sqrt{\frac{I_3}{I_2}}\Omega\right)^2} + \frac{\omega_3^2}{\Omega^2} = 1. \qquad 4.6.11$$

For a specific example, I'll suppose that the moments of inertia are in the ratio 2 : 3 : 5, and we'll consider three cases in turn.

Case I. Rotation about the axis of *least* moment of inertia. That is, we'll take $I_3 = 2$, $I_1 = 3$, $I_2 = 5$. Since I_3 is the smallest moment of inertia, each of the ratios I_3/I_1 and I_3/I_2 are less than 1, and $\sqrt{\frac{I_3}{I_1}} > \frac{I_3}{I_1}$ and $\sqrt{\frac{I_3}{I_2}} > \frac{I_3}{I_2}$. The two ellipsoids are

$$\frac{\omega_1^2}{(0.667\Omega)^2} + \frac{\omega_2^2}{(0.400\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1 \qquad 4.6.12$$

and

$$\frac{\omega_1^2}{(0.816\Omega)^2} + \frac{\omega_2^2}{(0.632\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1. \qquad 4.6.13$$

I'll try and sketch these:

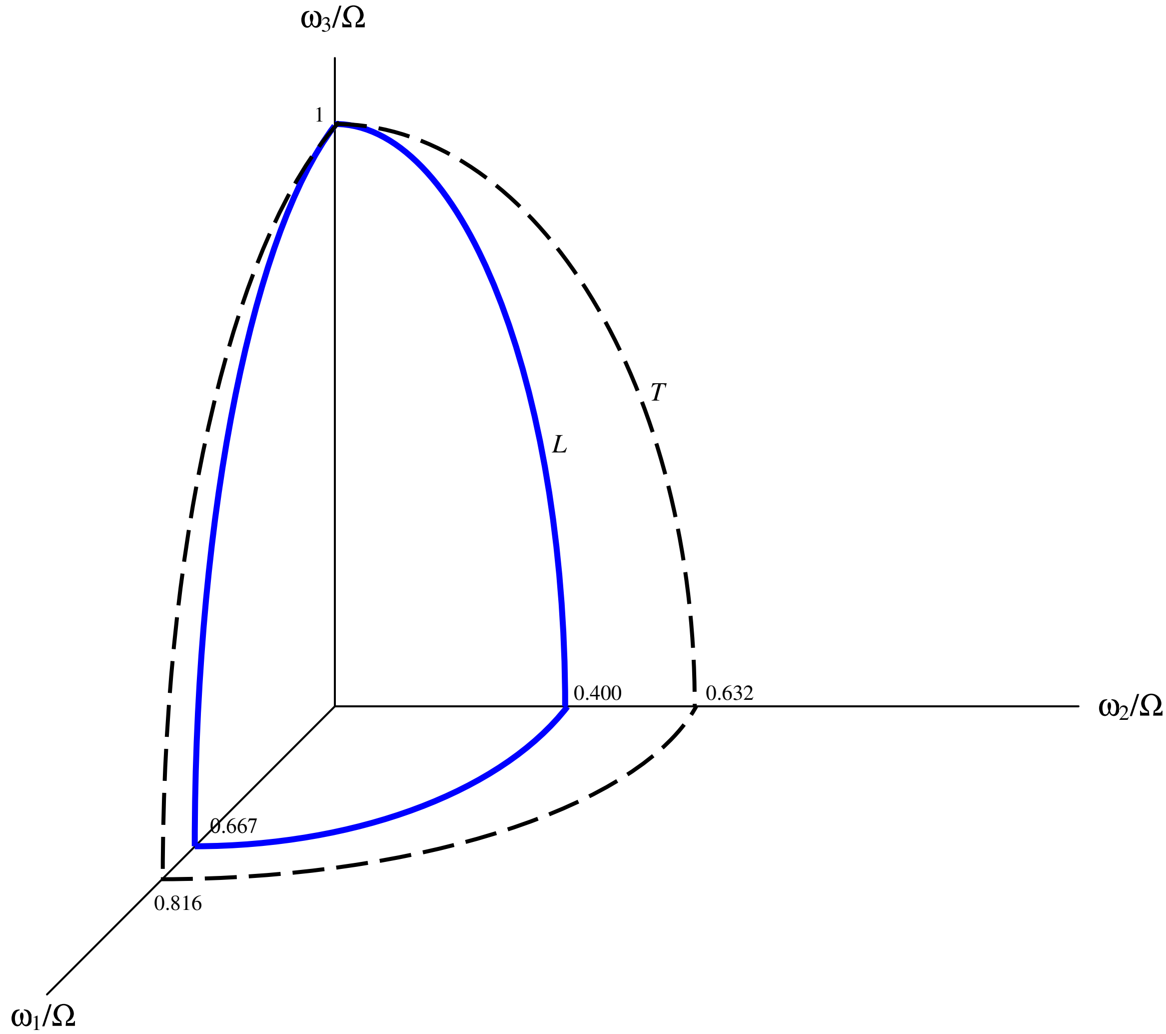


FIGURE IV.8

Initially, we suppose, the body was set in motion rotating about the *z*-axis with angular speed Ω, which determines the values of L and T, which will remain constant. The tip of the vector $\boldsymbol{\omega}$ is constrained to remain on the surface of the ellipsoid $L = 0$ and on the ellipsoid $T = 0$, and hence on the intersection of these two surfaces. But these two surface touch only at one point, namely $(\omega_1, \omega_2, \omega_3) = (0, 0, \Omega)$. Thus there the vector $\boldsymbol{\omega}$ remains, and the rotation is stable.

Case II. Rotation about the axis of *greatest* moment of inertia. That is, we'll take $I_3 = 5$, $I_1 = 2$, $I_2 = 3$. Since I_3 is the greatest moment of inertia, each of the ratios I_3/I_1 and I_3/I_2 are greater than 1, and $\sqrt{\frac{I_3}{I_1}} < \frac{I_3}{I_1}$ and $\sqrt{\frac{I_3}{I_2}} < \frac{I_3}{I_2}$. The two ellipsoids are

$$\frac{\omega_1^2}{(2.50\Omega)^2} + \frac{\omega_2^2}{(1.67\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1 \qquad 4.6.14$$

and $$\frac{\omega_1^2}{(1.58\Omega)^2} + \frac{\omega_2^2}{(1.29\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1. \qquad 4.6.15$$

I'll try and sketch these:

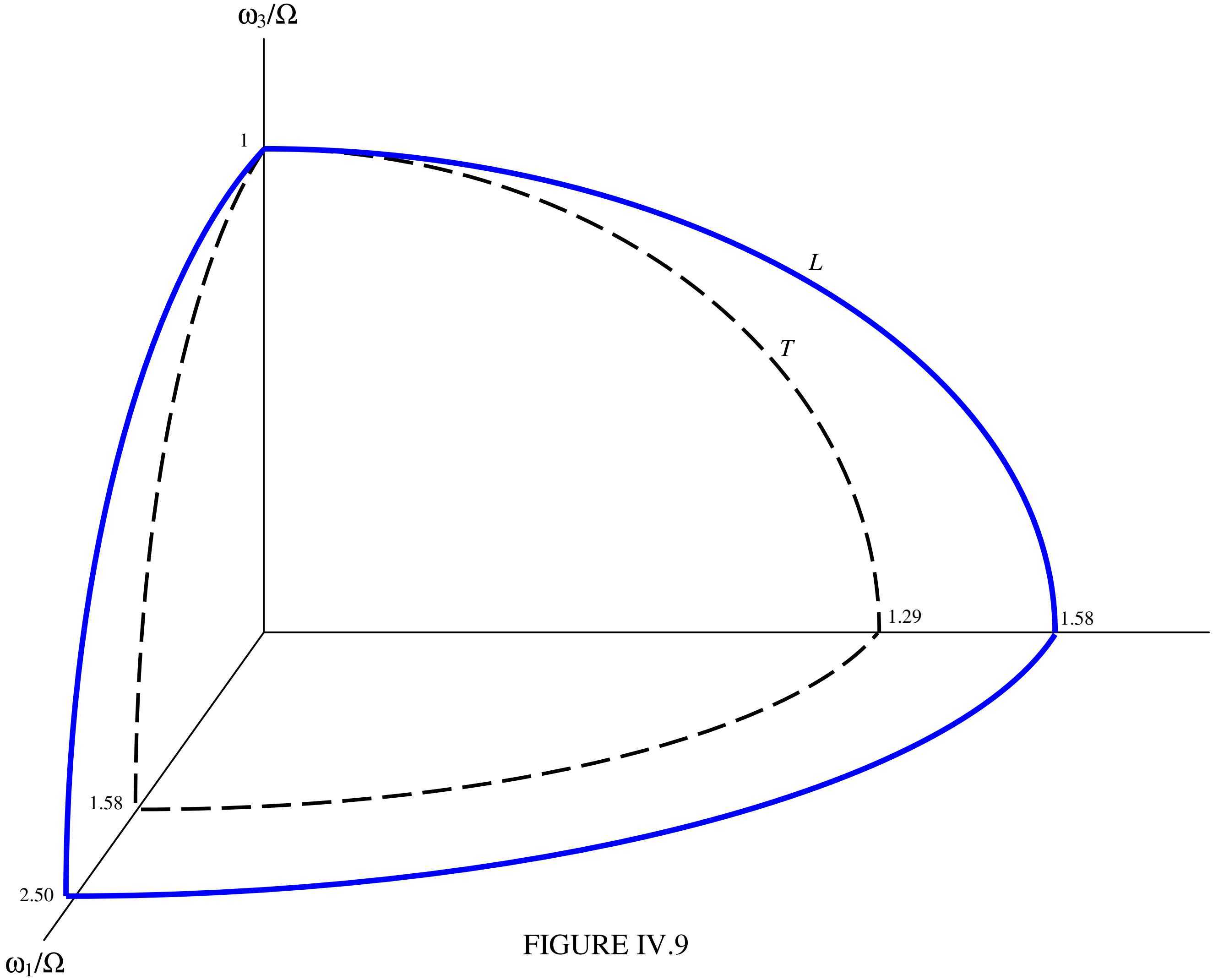


FIGURE IV.9

Again, and for the same reason as for Case I, we see that this motion is stable.

Case III. Rotation about the *intermediate* axis. That is, we'll take $I_3 = 3$, $I_1 = 5$, $I_2 = 2$. This time I_3/I_1 is less than 1 and I_3/I_2 is less than 1, and $\sqrt{\frac{I_3}{I_1}} > \frac{I_3}{I_1}$ and $\sqrt{\frac{I_3}{I_2}} < \frac{I_3}{I_2}$. The two ellipsoids are

$$\frac{\omega_1^2}{(0.60\Omega)^2} + \frac{\omega_2^2}{(1.50\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1 \qquad 4.6.16$$

and

$$\frac{\omega_1^2}{(0.77\Omega)^2} + \frac{\omega_2^2}{(1.22\Omega)^2} + \frac{\omega_3^2}{\Omega^2} = 1. \qquad 4.6.17$$

I'll try and sketch these:

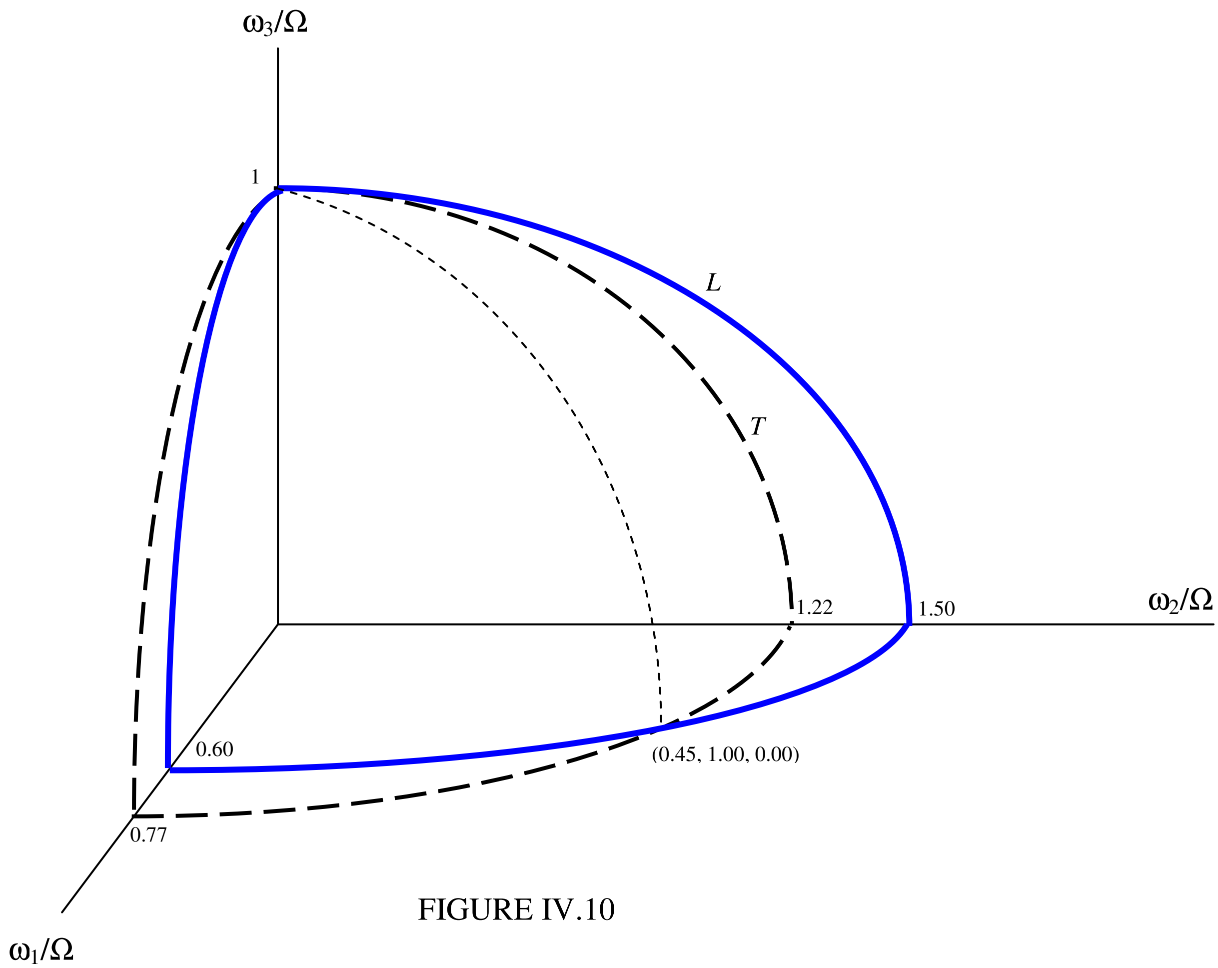


FIGURE IV.10

Unlike the situation for Cases I and II, in which the two ellipsoids touch at only a single point, the two ellipses for Case III intersect in the curve shown as a dotted line in figure IV.10. Thus $\boldsymbol{\omega}$ is not restricted to lying along the z-axis, but it can move anywhere along the dotted line. The motion, therefore, is not stable.

You should experiment by throwing a body in the air in such a manner as to let it spin around one of its principal axes. A rectangular block will do, though the effect is particularly noticeable with something like a table-tennis bat or a tennis racket.

Here is another approach to reach the same result. We imagine an asymmetric top spinning about one of its principal axes with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$. It is then given a small perturbation, so that its angular velocity is now

$$\boldsymbol{\omega} = \varepsilon\hat{\mathbf{x}} + \eta\hat{\mathbf{y}} + \omega_z\hat{\mathbf{z}}. \qquad 4.6.18$$

Here the "hatted" quantities are the unit orthogonal vectors; ε and η are supposed small compared with ω_z. Euler's equations are :

$$I_1\dot{\varepsilon} = \eta\omega_z(I_2 - I_3), \qquad 4.6.19$$

$$I_2\dot{\eta} = \omega_z\varepsilon(I_3 - I_1), \qquad 4.6.20$$

$$I_3\dot{\omega}_z = \varepsilon\eta(I_1 - I_2). \qquad 4.6.21$$

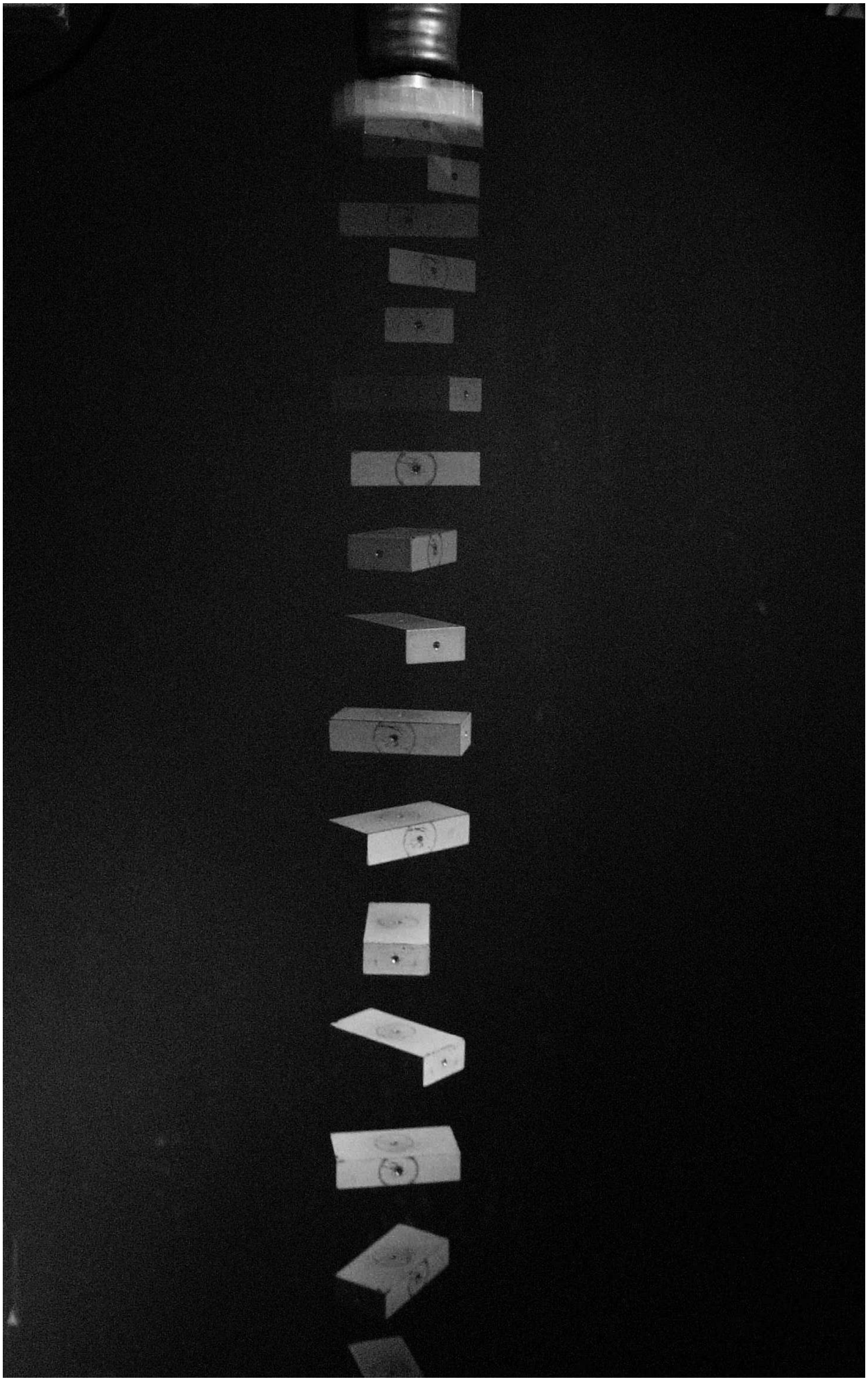
If $\varepsilon\eta << \dot{\omega}_z$, then ω_z is approximately constant. Elimination of η from the first two equations yields

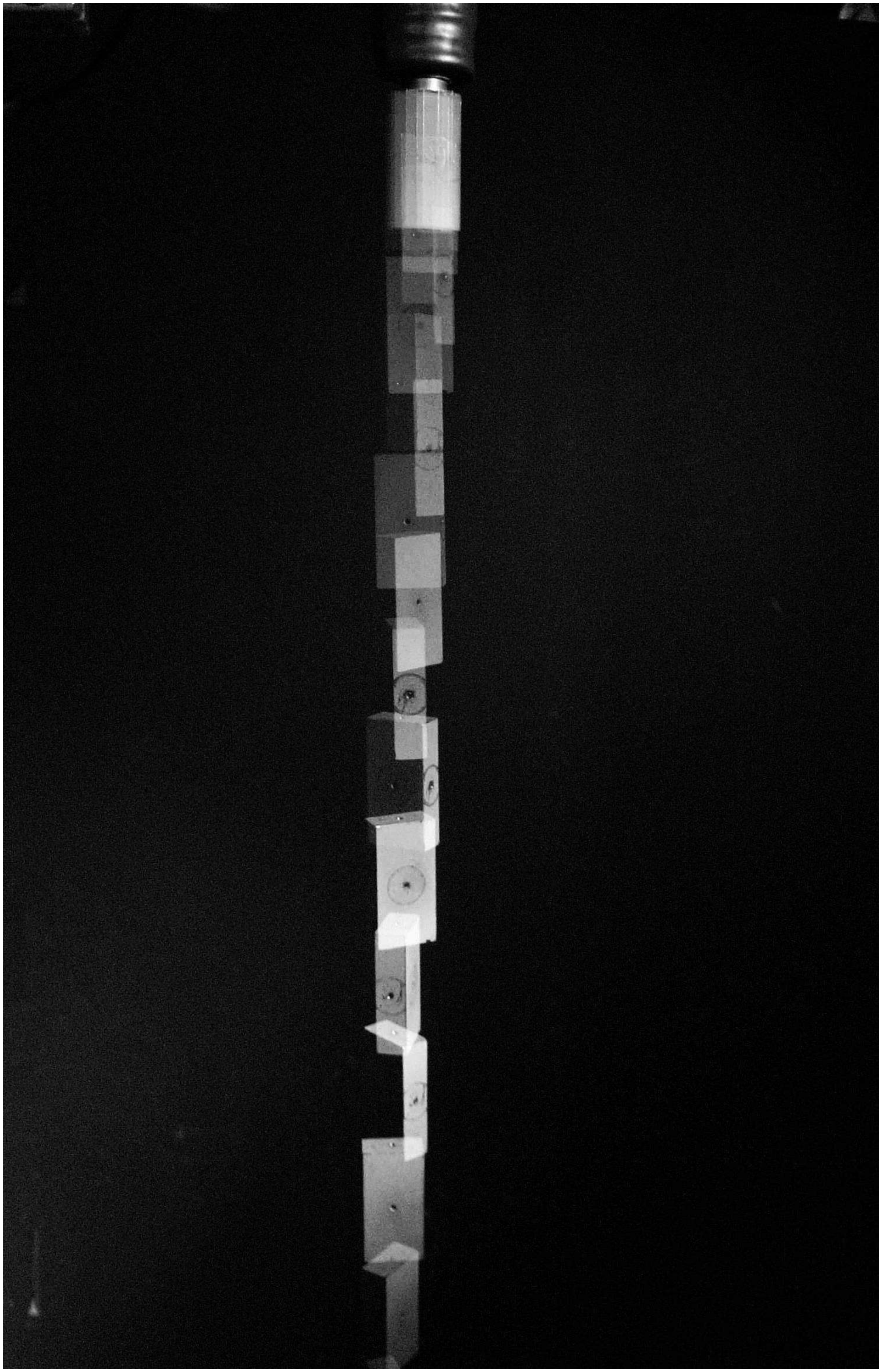
$$\ddot{\varepsilon} = -\left[\frac{(I_2 - I_3)(I_1 - I_3)\omega_z^2}{I_1 I_2}\right]\varepsilon. \qquad 4.6.22$$

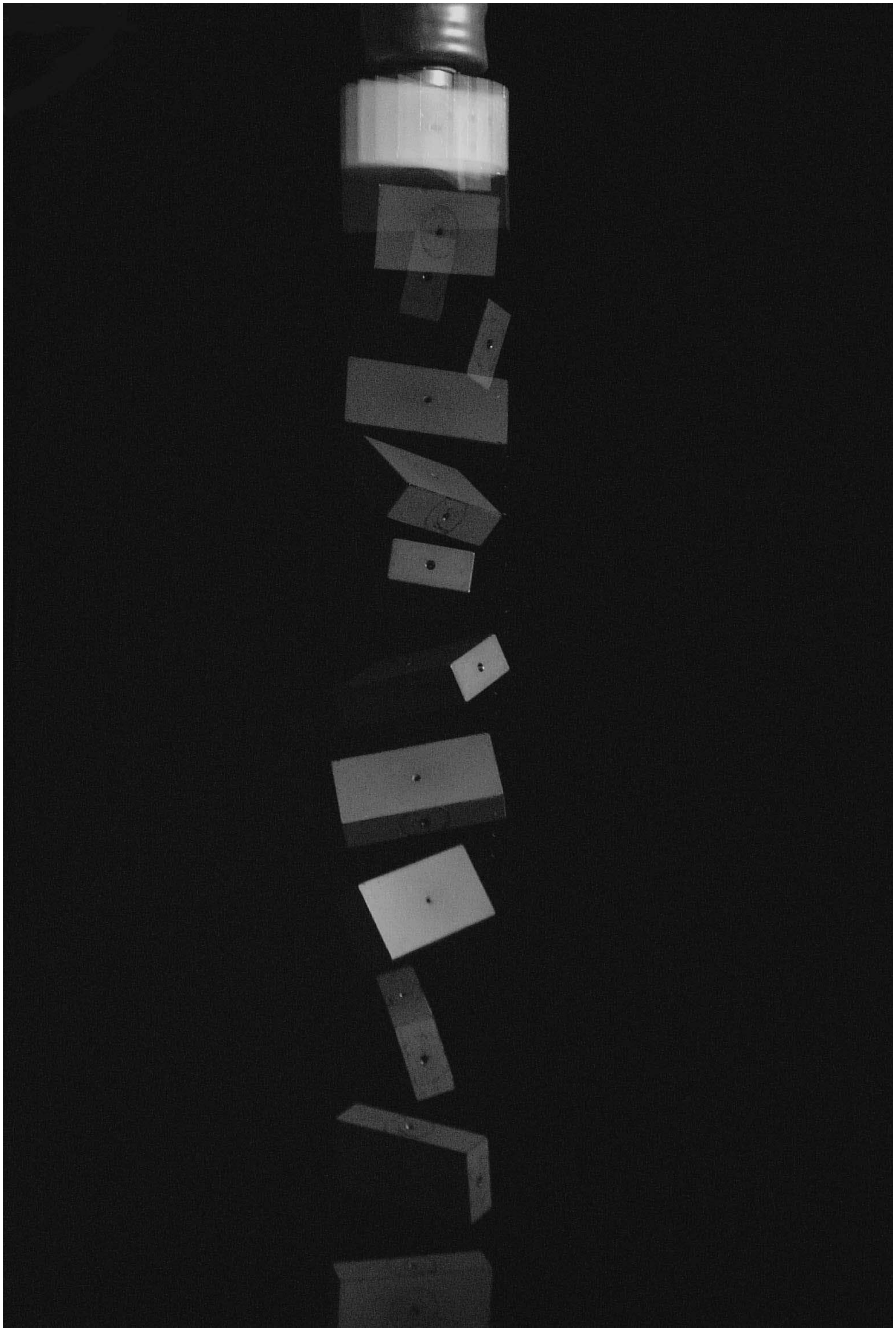
Elimination of ε instead results in a similar equation in η.

If I_3 is either the largest or the smallest of the three moments of inertia, the two parentheses in the denominator have the same sign, so the expression in the brackets is positive. Equation 4.6.22 is then the equation for simple harmonic motion, and the motion is stable. If, however, I_3 is intermediate between the other two, the two parentheses have opposite sign, and the expression in brackets in negative. In that case ε and η increase exponentially, and the motion is unstable.

Mr Neil Honkanen of the University of Victoria conducted an experiment to illustrate the stability of rotation about the three principal axes. The body in question was a small "brick" of mild steel (density 7.83 g/cm^3) of dimensions $\frac{3}{8}$ inch × $\frac{3}{4}$ inch × $1\frac{1}{2}$ inch, mass 54.1 g. In round figures, this corresponds to principal moments of inertia $A_0 = 2 \times 10^{-6}$ kg m^2, $B_0 = 7 \times 10^{-6}$ kg m^2, $C_0 = 8 \times 10^{-6}$ kg m^2. He suspended it from an electromagnet, which he set in rotation at about 25 revolutions per second, and then let it fall, while photographing it stroboscopically. He did three experiments rotation respectively about the three principle axes. You can see from the photographs below that the rotation is stable when the rotation is about the axes of greatest or least moment of inertia, but is unstable when the rotation is about the axis of intermediate moment of inertia.







4.7 *Nonrigid rotator*

The rotational kinetic energy of a body rotating about a principal axis is $\frac{1}{2}I\omega^2$, where I is the moment of inertia about that principal axis, and the angular momentum is $L = I\omega$. (For rotation about a nonprincipal axis, see section 4.3.) Thus the rotational kinetic energy can be written as $L^2/(2I)$.

When an asymmetric top is rotating about a nonprincipal axes, the body experiences internal stresses, which, if the body is nonrigid, result in periodic strains which periodically distort the shape of the body. As a result of this, rotational kinetic energy becomes degraded into heat; the rotational kinetic energy of the body gradually decreases. In the absence of external torques, however, the angular momentum is constant. The expression $L^2/(2I)$ for the kinetic energy shows that the kinetic energy is least for a given angular momentum when the moment of inertia is greatest. Thus eventually the body rotates about its principal axis of greatest moment of inertia. After that, it no longer loses kinetic energy to heat, because, when the body is rotating about a principal axis, it is no longer subject to internal stresses.

The time taken (the "relaxation time") for a body to reach its final state of rotation about its principal axis of greatest moment of inertia depends, among other things, on how fast the body is rotating. A fast rotator will reach its final state relatively soon, whereas it takes a long time for a slow rotator to reach its final state. Thus it is not surprising to find that, among the asteroids, most of the fast rotators are principal axis rotators, whereas many slow rotators are also nonprincipal axis rotators. There are, however, a few fast rotators that are still rotating about a nonprincipal axis. It is assumed that such asteroids may have suffered a collision in the recent past.

4.8 *Force-free Motion of a Symmetric Top*

Notation: I_1, I_1, I_3 are the principal moments of inertia. I_3 is the unique moment. If it is the largest of the three, the body is an oblate symmetric top; if it is the smallest, it is a prolate spherical top.

Ox_0, Oy_0, Oz_0 are the corresponding body-fixed principal axes.

ω_1, ω_2, ω_3 are the components of the angular velocity vector $\boldsymbol{\omega}$ with respect to the principal axes.

In the analysis that follows, we are going to have to think about three vectors. There will be the angular momentum vector **L**, which, in the absence of external torques, is fixed in magnitude and in direction in laboratory space. There will be the direction of the axis of symmetry, the Oz_0 axis, which is fixed in the body, but not necessarily in space, unless the body happens to be rotating about its axis of symmetry; we'll denote a unit vector in this direction by $\hat{\mathbf{z}}_0$. And there will be the instantaneous angular velocity vector $\boldsymbol{\omega}$ which is neither space- nor body-fixed.

What we are going to find is the following. We shall find that $\boldsymbol{\omega}$ precesses in the body about the body-fixed symmetry axis in a cone called the *body cone.* The angle between $\boldsymbol{\omega}$ and $\hat{\mathbf{z}}_0$ is constant (we'll be calling this angle α), and the magnitude ω of $\boldsymbol{\omega}$ is constant. We shall find that the sense of the precession is the same as the sense of the spin if the body is oblate, but opposite if it is prolate. The direction of the symmetry axis, however, is not fixed in space, but it precesses about the space-fixed angular momentum vector **L** in another cone. This cone is narrower than the body cone if the body is oblate, but broader than the body cone if the body is prolate. The net result of these two precessional motions is that $\boldsymbol{\omega}$ precesses in space about the space-fixed angular momentum vector in a cone called the *space cone*. For a prolate top, the semi vertical angle of the space cone can be anything from 0° to 90°; for an oblate top, however, the semi vertical angle of the space cone cannot exceed $19^\circ\ 28'$. That's quite a lot to take in in one breath!

We can start with Euler's equations of motion for force-free rotation of a symmetric top:

$$I_1\dot{\omega}_1 = -\omega_2\omega_3(I_3 - I_1), \qquad 4.8.1$$

$$I_1\dot{\omega}_2 = \quad \omega_1\omega_3(I_3 - I_1), \qquad 4.8.2$$

$$I_3\dot{\omega}_3 = 0. \qquad 4.8.3$$

From the first of these we obtain the result

$$\omega_3 = \text{constant}. \qquad 4.8.4$$

For brevity, I am going to let

$$\frac{(I_3 - I_1)\omega_3}{I_1} = \Omega, \qquad 4.8.5$$

although in a moment Ω will have a physical meaning.

Equations 4.8.1 and 4.8.2 become:

$$\dot{\omega}_1 = -\Omega\omega_2 \qquad 4.8.6$$

and

$$\dot{\omega}_2 = +\Omega\omega_1. \qquad 4.8.7$$

Eliminate ω_2 from these to obtain

$$\ddot{\omega}_1 = -\Omega^2 \omega_1 . \tag{4.8.8}$$

This is the equation for simple harmonic motion and its solution is

$$\omega_1 = \omega_0 \cos(\Omega t + \varepsilon), \tag{4.8.9}$$

in which ω_0 and ε, the two constants of integration, whose values depend on the initial conditions in the usual fashion, are the amplitude and initial phase angle. On combining this with equation 4.8.6, we obtain

$$\omega_2 = \omega_0 \sin(\Omega t + \varepsilon). \tag{4.8.10}$$

From these we see that $(\omega_1^2 + \omega_2^2)^{1/2}$, which is the magnitude of the component of $\boldsymbol{\omega}$ in the x_0y_0-plane, is constant, equal to ω_0; and since ω_3 is also constant, it follows that $(\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2}$, which is the magnitude of $\boldsymbol{\omega}$, is also constant. The cosine of the angle α between $\hat{\mathbf{z}}_0$ and $\boldsymbol{\omega}$ is $\omega_3/(\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2}$, and its sine is $\omega_0/(\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2}$, so that α is constant. Equations 4.8.9 and 4.8.10 tell us, then, that the vector $\boldsymbol{\omega}$ is precessing around the symmetry axis at an angular speed Ω. Making use of equation 4.8.5, we find that

$$\cos\alpha = \frac{\omega_3}{\omega} = \frac{I_1\Omega}{(I_3 - I_1)\omega} . \tag{4.8.11}$$

If we take the direction of the z_0 axis to be the direction of the component of $\boldsymbol{\omega}$ along the symmetry axis, then $\boldsymbol{\Omega}$ is in the same direction as $\mathbf{z_0}$ if $I_3 > I_1$ (that is, if the top is oblate) and it is in the opposite direction if the top is prolate. The situation for oblate and prolate tops is shown in figure IV.11.

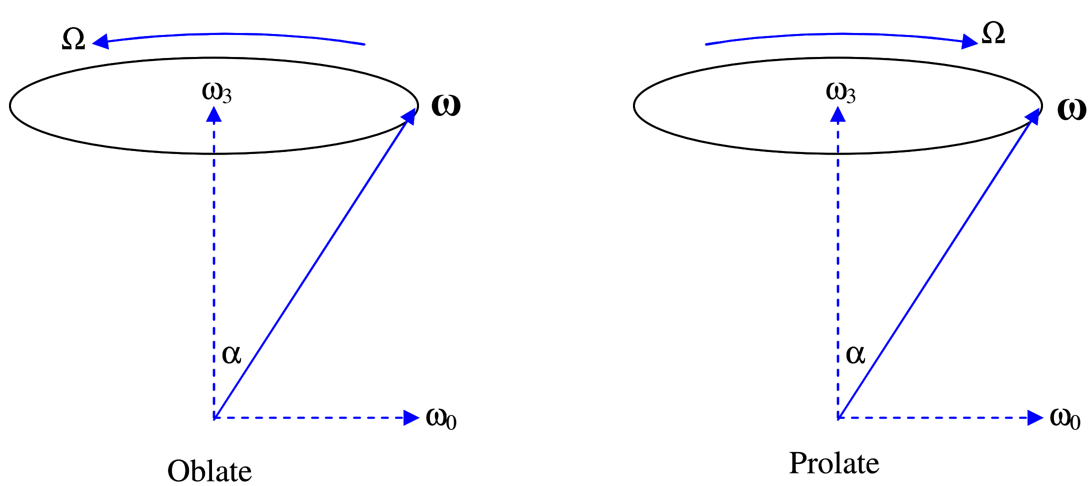


FIGURE IV.11

We have just dealt with how the instantaneous axis of rotation precesses about the body-fixed symmetry axis, describing the *body cone* of semi vertical angle α.

Now we are going to consider the precession of the body-fixed symmetry axis about the space-fixed angular momentum vector **L**. I am going to make use of the idea of Eulerian angles for expressing the orientation of one three-dimensional set of axes with respect to another. If you are not already familiar with Eulerian angles or would like a refresher, you can go to

http://astrowww.phys.uvic.ca/~tatum/celmechs/celm3.pdf

especially section 3.7, page 29.

Recall that we are using O$x_0\ y_0\ z_0$ for *body-fixed* coordinates, referred to the principal axes. I shall use Oxyz for *space-fixed* coordinates, and there is no loss of generality if I choose the Oz axis to coincide with the angular momentum vector **L .** Let me try to draw the situation in figure IV.12a. The axes Oxyz are the *space-fixed axes*. The axes O$x_0y_0z_0$ are the *body-fixed principal axes.* The angular momentum vector **L** is directed along the axis Oz. The symmetry axis of the body is directed along the axis Oz_0. The Eulerian angles of the body-fixed axes relative to the space fixed axes are (ϕ, θ, ψ).

Recall, with the aid of figure IV.12b, how these Euler angles are formed:

First, a rotation by ϕ about Oz. Second, a rotation by θ about the dashed line Ox' to form an intermediate set of axes O$x'y'z'$. Third, a rotation by ψ about Oz' to form the body-fixed principal axes O$x_0y_0z_0$.

Spend a little time trying to visualize these three sets of axes. Please also convince yourself, from the way the Euler angles were formed through three rotations, that the vector **L** is in the $y'z'$ plane and has no x' component. It is also in the y_0z_0 plane and has no x_0 component.

You will then agree that

$$L_{x'} = 0, \qquad L_{y'} = L\sin\theta, \qquad L_{z'} = L\cos\theta \ . \qquad 4.8.12$$

Now if $L_{x'} = 0$, then $\omega_{x'}$ is also zero, which means that $\boldsymbol{\omega}$, like **L** , is in the $y'z'$ plane. We have seen that $\boldsymbol{\omega}$ makes an angle α with the symmetry axis Oz_0, where α is given by equation 4.8.11.

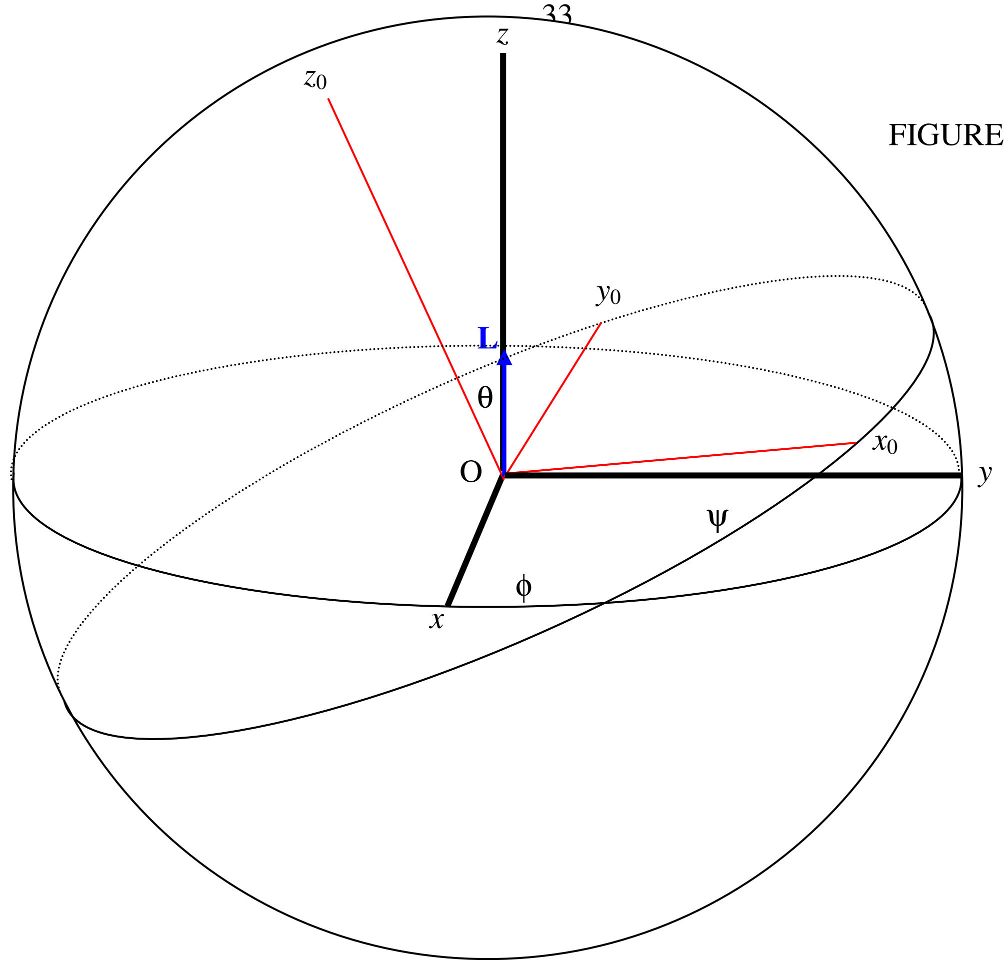


FIGURE IV.12a

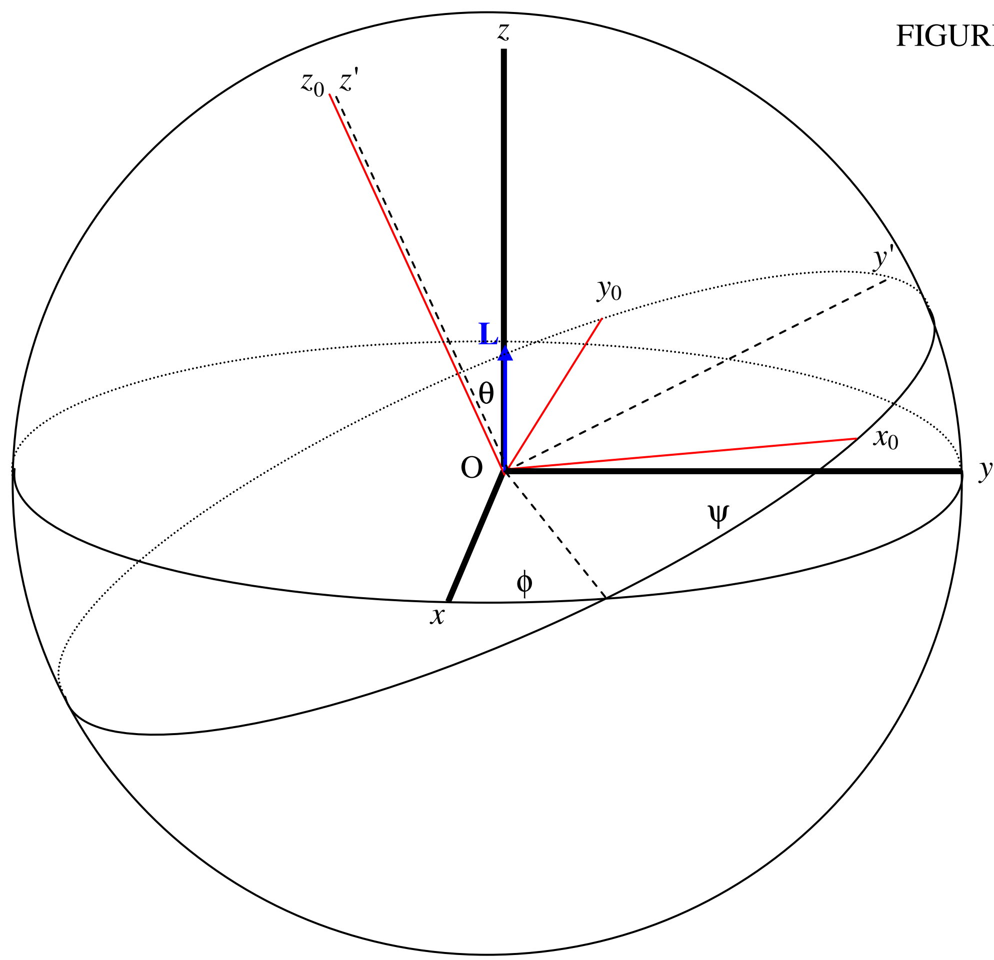


FIGURE IV.12b

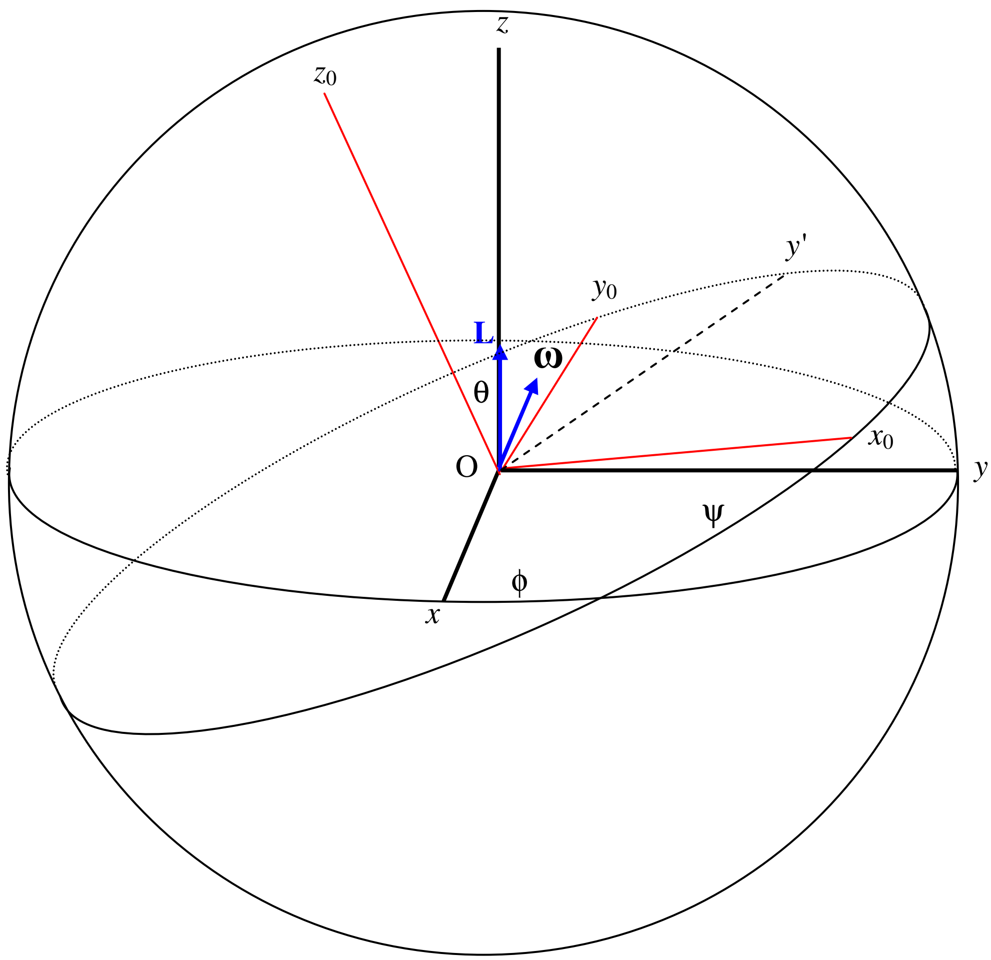
I'll now add $\boldsymbol{\omega}$ to the drawing to make figure IV.13. Like **L**, it is in the $y'z'$ plane and has no x' component. I haven't marked in the angle α. I leave it to your imagination. It is the angle between $\boldsymbol{\omega}$ and z_0. You should easily agree that

$$\omega_{x'} = 0, \qquad \omega_{y'} = \omega \sin\alpha, \qquad \omega_{z'} = \omega\cos\alpha. \qquad 4.8.13$$

From these, together with $L_{y'} = I_1\,\omega_{y'}$ and $L_{z'} = I_3\,\omega_{z'}$ we obtain

$$I_1 \tan\alpha \;=\; I_3 \tan\theta. \qquad 4.8.14$$

For an oblate symmetric top, $I_3 > I_1$, $\alpha > \theta$.
For a prolate symmetric top, $I_3 < I_1$, $\alpha < \theta$.



Now $\boldsymbol{\omega}$ can be written as the vector sum of the rates of change of the three Euler angles:

$$\boldsymbol{\omega} \;=\; \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\phi}} + \dot{\boldsymbol{\psi}}\,. \qquad 4.8.15$$

The components of $\dot{\boldsymbol{\theta}}$ and $\dot{\boldsymbol{\psi}}$ along Oy' are each zero, and therefore the component of $\boldsymbol{\omega}$ along Oy' is equal to the component of $\dot{\boldsymbol{\psi}}$ along Oy' .

$$\therefore \qquad \omega \sin\alpha = \dot{\phi} \sin\theta. \qquad 4.8.16$$

In summary, then:

1. The instantaneous axis of rotation, which makes an angle α with the symmetry axis, precesses around it at angular speed

$$\Omega = \frac{I_3 - I_1}{I_1} \omega \cos\alpha, \qquad 4.8.17$$

which is in the same sense as $\boldsymbol{\omega}$ if the top is oblate and opposite if it is prolate.

2. The symmetry axis makes an angle θ with the space-fixed angular momentum vector **L**, where

$$\tan\theta = \frac{I_1}{I_3} \tan\alpha. \qquad 4.8.18$$

For an oblate top, $\theta < \alpha$. For a prolate top, $\theta > \alpha$.

3. The speed of precession of the symmetry axis about **L** is

$$\dot{\phi} = \frac{\sin\alpha}{\sin\theta} \omega, \qquad 4.8.19$$

or, by elimination of θ between 4.8.18 and 4.8.19,

$$\dot{\phi} = \left[1 + \frac{I_3^2 - I_1^2}{I_1^2} \cos^2\alpha\right]^{1/2} \omega. \qquad 4.8.20$$

The net result of this is that $\boldsymbol{\omega}$ precesses about **L** at a rate $\dot{\phi}$ in the *space cone*, which has a semi-vertical angle $\alpha - \theta$ for an oblate rotator, and $\theta - \alpha$ for a prolate rotator. The space cone is fixed in space, while the body cone rolls around it, always in contact, $\boldsymbol{\omega}$ being a mutual generator of both cones. If the rotator is oblate, the space cone is smaller than the body cone and is inside it. If the rotator is prolate, the body cone is outside the space cone and can be larger or smaller than it.

Write $$c = I_3/I_1 \qquad 4.8.21$$

for the ratio of the principal moments of inertia. Note that for a pencil, $c = 0$; for a sphere, $c = 1$; for a plane disc or any regular plane lamina, $c = 2$. (The last of these follows from the perpendicular axes theorem – see Chapter 2.) The range of c, then, is from 0 to 2, 0 to 1 being prolate, 1 to 2 being oblate.

Equations 4.8.17 and 4.8.20 can be written

$$\frac{\Omega}{\omega} = (c-1)\cos\alpha \qquad 4.8.22$$

and

$$\frac{\dot{\phi}}{\omega} = \left[1 + (c^2 - 1)\cos^2\alpha\right]^{1/2}. \qquad 4.8.23$$

Figures IV.15 and IV.16 show, for an oblate and a prolate rotator respectively, the instantaneous rotation vector $\boldsymbol{\omega}$ precessing around the body-fixed symmetry axis at a rate Ω in the *body cone* of semi vertical angle α; the symmetry axis precessing about the space-fixed angular momentum vector **L** at a rate $\dot{\phi}$ in a cone of semi vertical angle θ (which is less than α for an oblate rotator, and greater than α for a prolate rotator); and consequently the instantaneous rotation vector $\boldsymbol{\omega}$ precessing around the space-fixed angular momentum vector **L** at a rate $\dot{\phi}$ in the *space cone* of semi vertical angle $\alpha - \theta$ (oblate rotator) or $\theta - \alpha$ (prolate rotator).

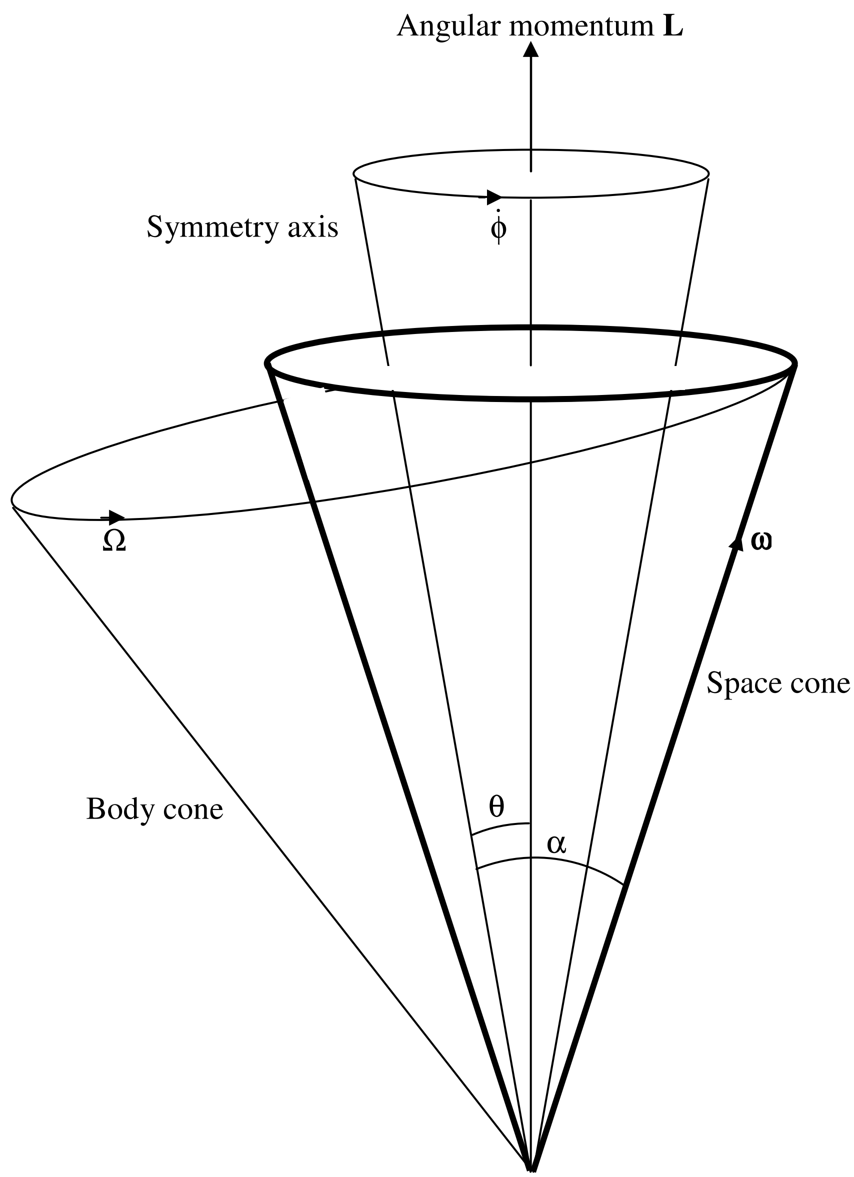

Angular momentum L
Symmetry axis
ϕ̇
Ω
ω
Space cone
Body cone
θ
α

FIGURE IV.15

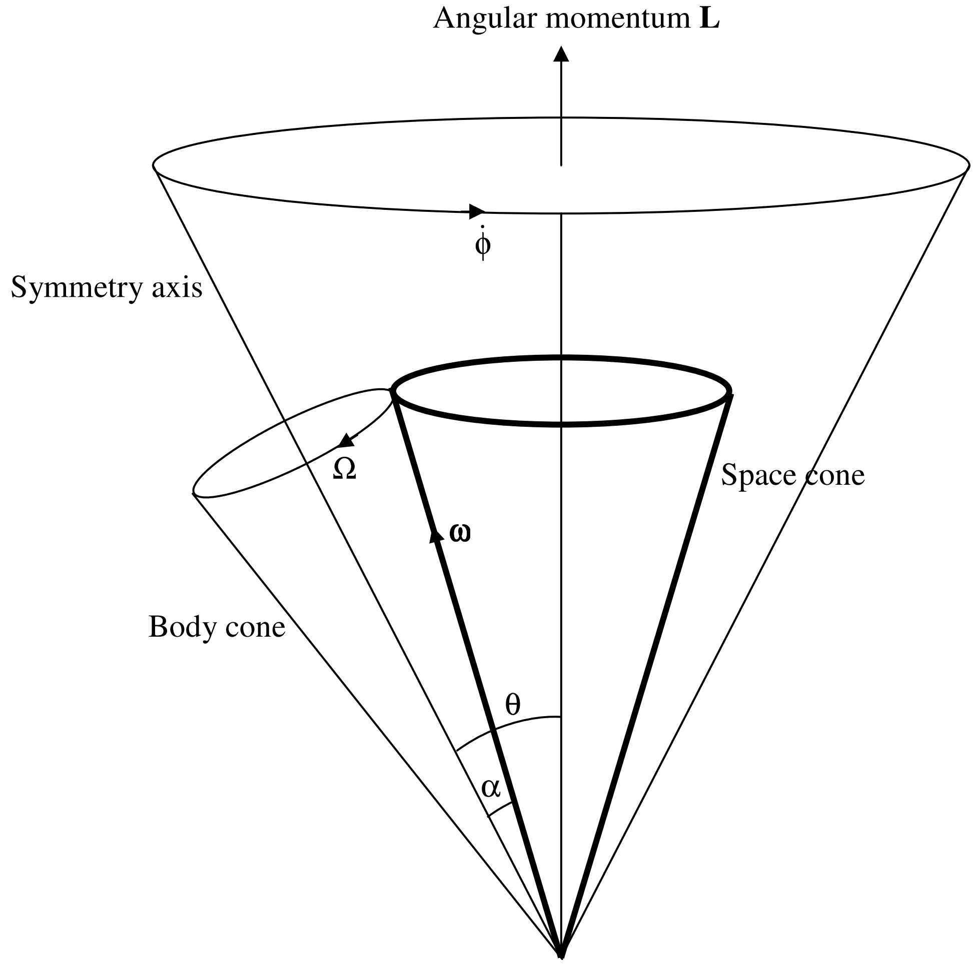


FIGURE IV.16

One can see from figures IV.15 and 16 that the angle between **L** and **ω** is limited for an oblate rotator, but it can be as large as 90° for a prolate rotator. The angle between **L** and **ω** is $\theta - \alpha$ (which is negative for an oblate rotator). We have

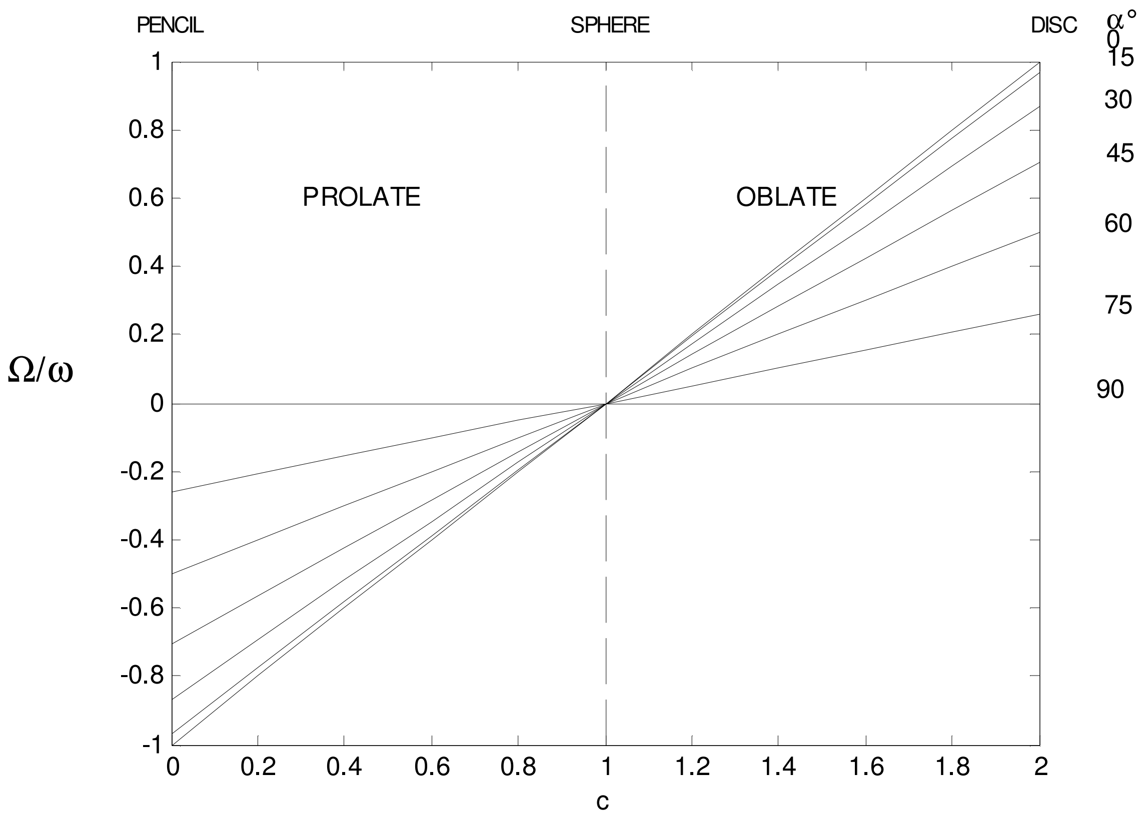
$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = \frac{(1-c)\tan\alpha}{c + \tan^2\alpha}. \qquad 4.8.24$$

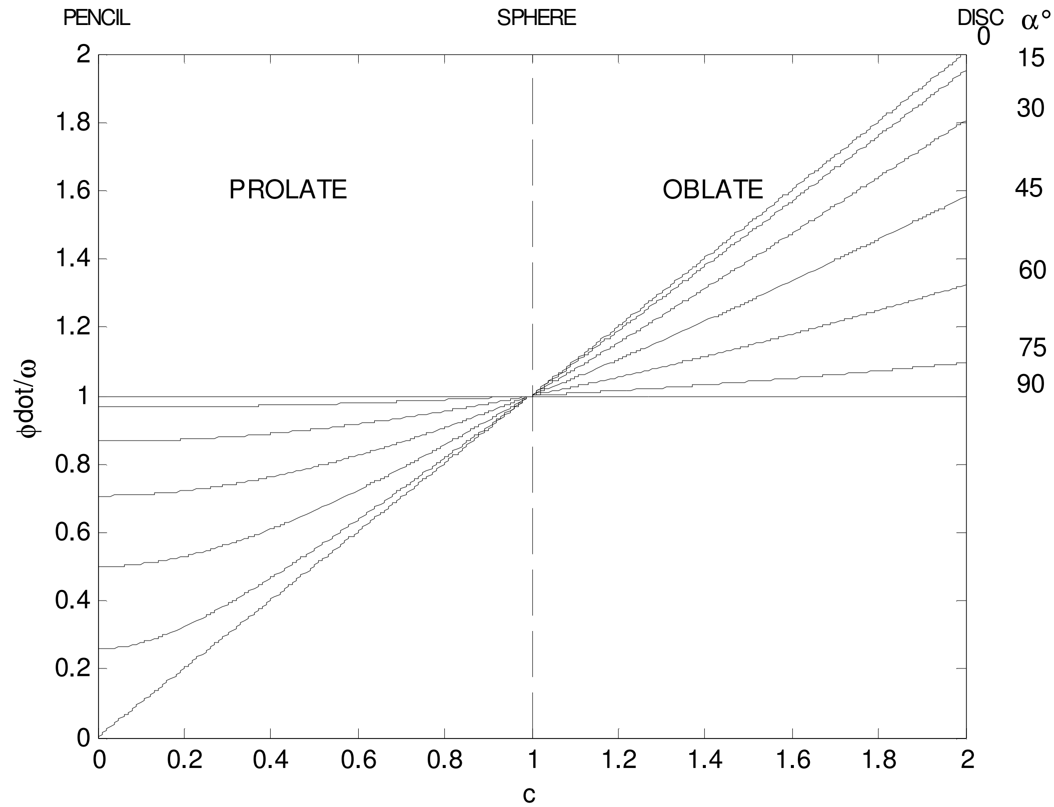
By calculus this reaches a maximum value of $\dfrac{1-c}{2\sqrt{c}}$ for $\tan\alpha = \sqrt{c}$.

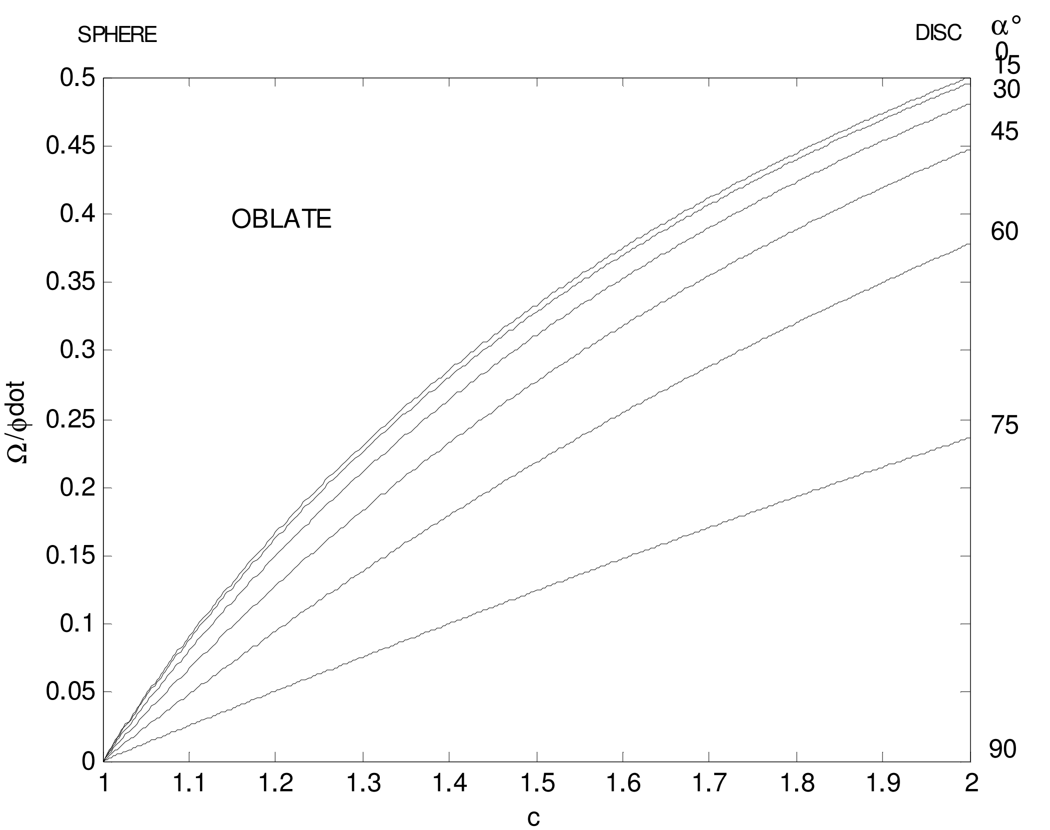
For a rod or pencil (prolate), in which $c = 0$, the angle between **L** and **ω** can be as large as 90°. Recalling exactly what are meant by the vectors **L** and **ω**, the reader should try now and imagine in his or her mind's eye a pencil rotating so that **L** and **ω** are at right angles. The spin vector **ω** is along the length of the pencil and the angular momentum vector **L** is at right angles to the length of the pencil.

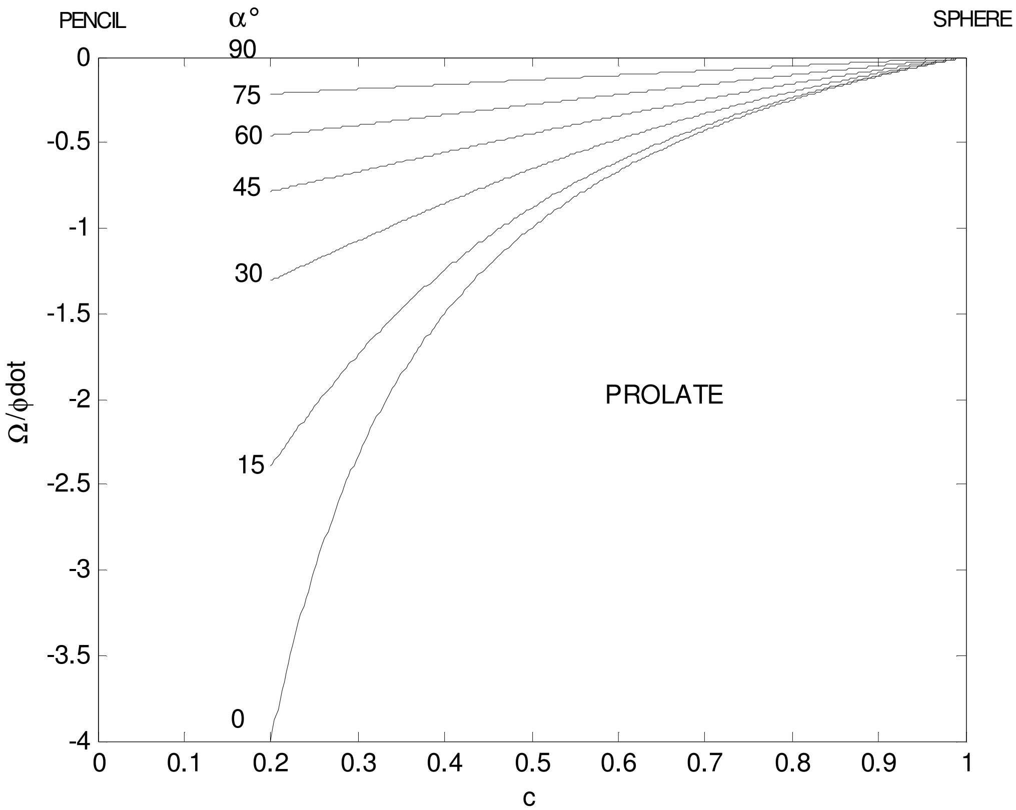
For an oblate rotator, the angle between **L** and **ω** is limited. The most oblate rotator is a flat disc or any regular flat lamina. The parallel axis theorem shows that for such a body, $c = 2$. The greatest angle between **L** and **ω** for a disc occurs when $\tan \alpha = \sqrt{2}$ ($\alpha = 54^\circ$ 44'), and then $\tan(\alpha - \theta) = 1/\sqrt{8}$, $\alpha - \theta = 19^\circ$ 28'.

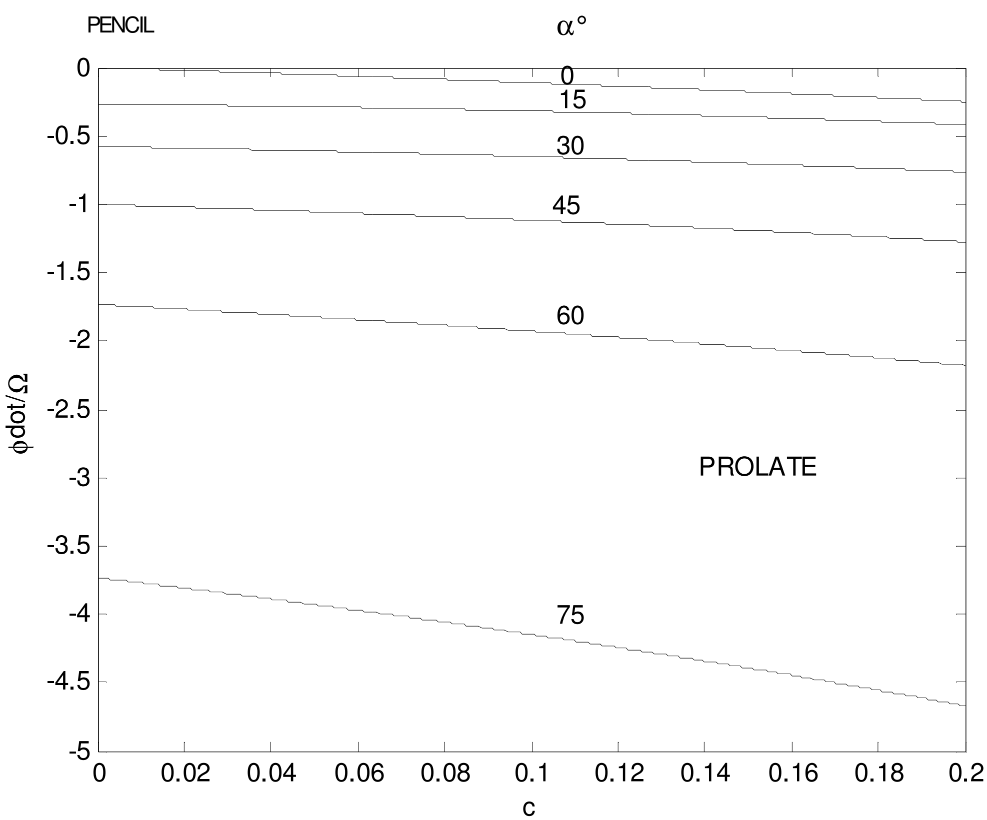
In the following figures I illustrate some of these results graphically. The ratio I_3/I_1 goes from 0 for a pencil through 1 for a sphere to 2 for a disc.

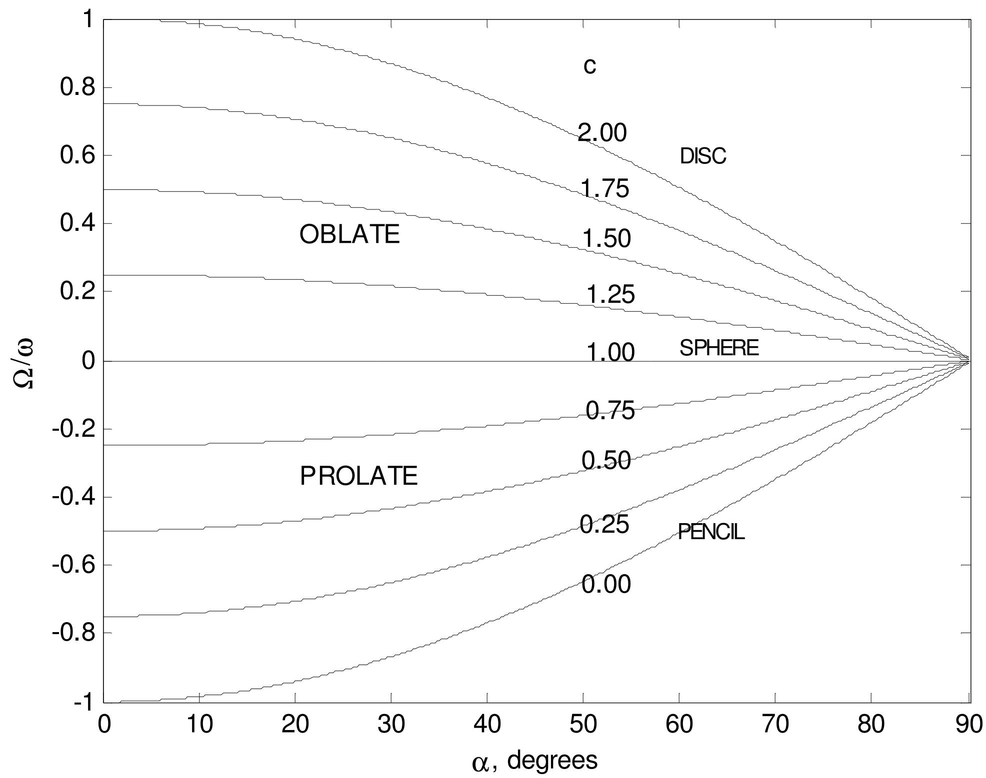


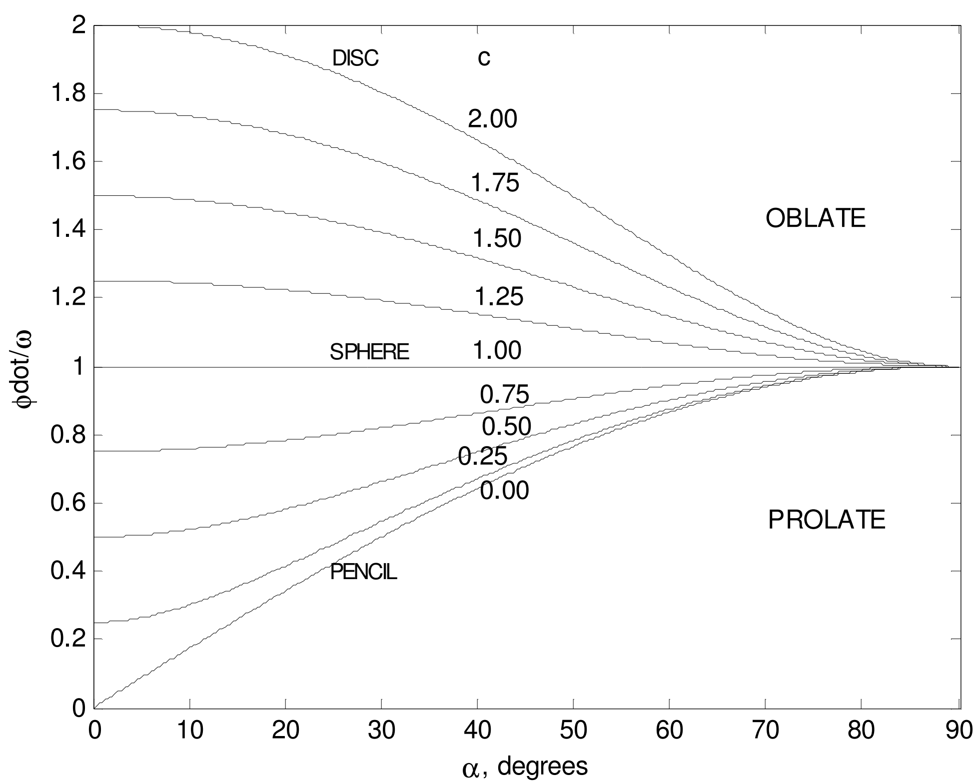

PENCIL
SPHERE
DISC
α°
0
15
30
45
60
75
90
PROLATE
OBLATE
φdot/ω
c
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

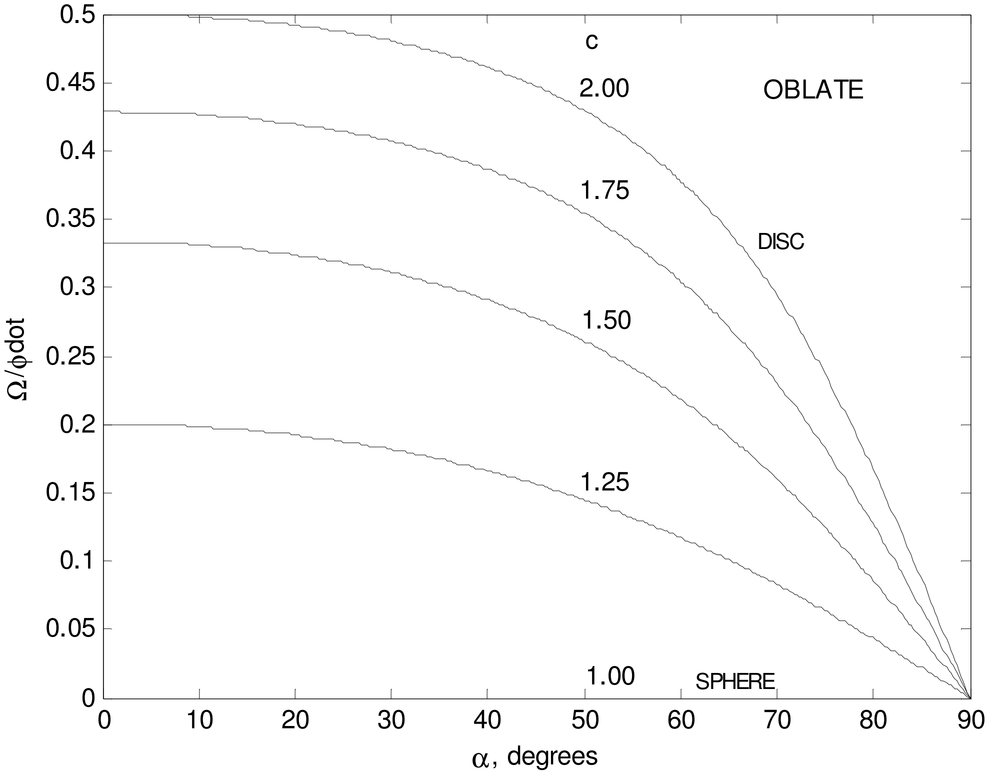

SPHERE
DISC
α°
0
15
30
45
60
75
90
OBLATE
Ω/φdot
c
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2


PENCIL
α°
SPHERE
90
75
60
45
30
15
0
PROLATE
Ω/φdot
c
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

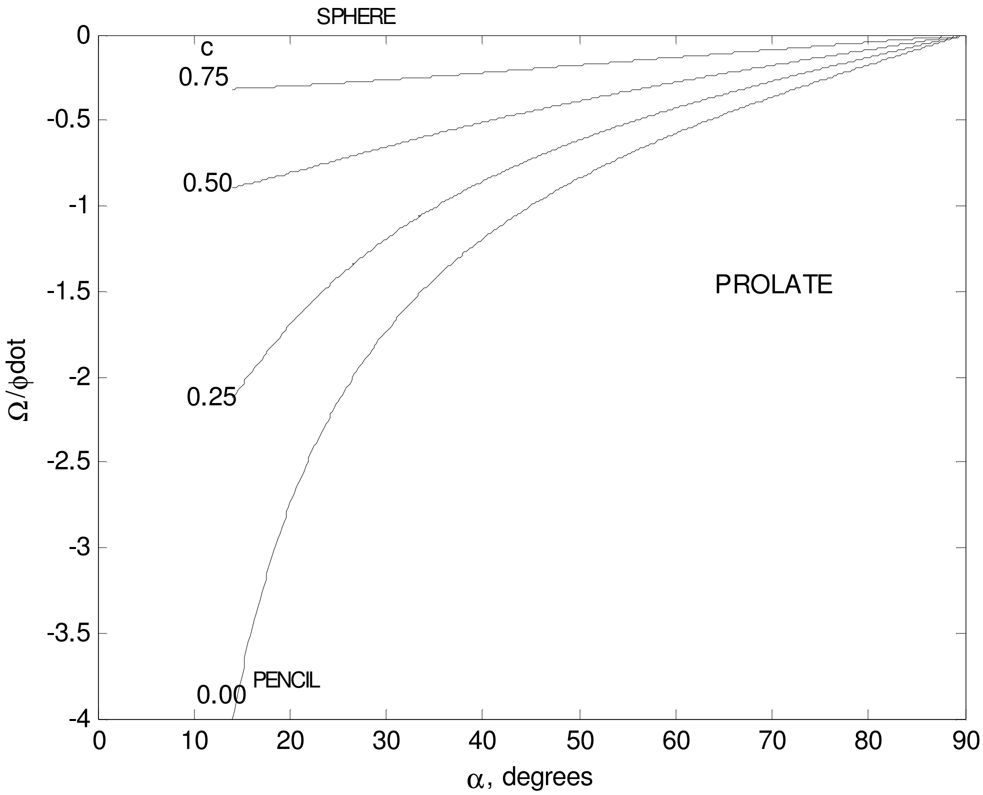

PENCIL
α°
0
15
30
45
60
75
PROLATE
φdot/Ω
c
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2

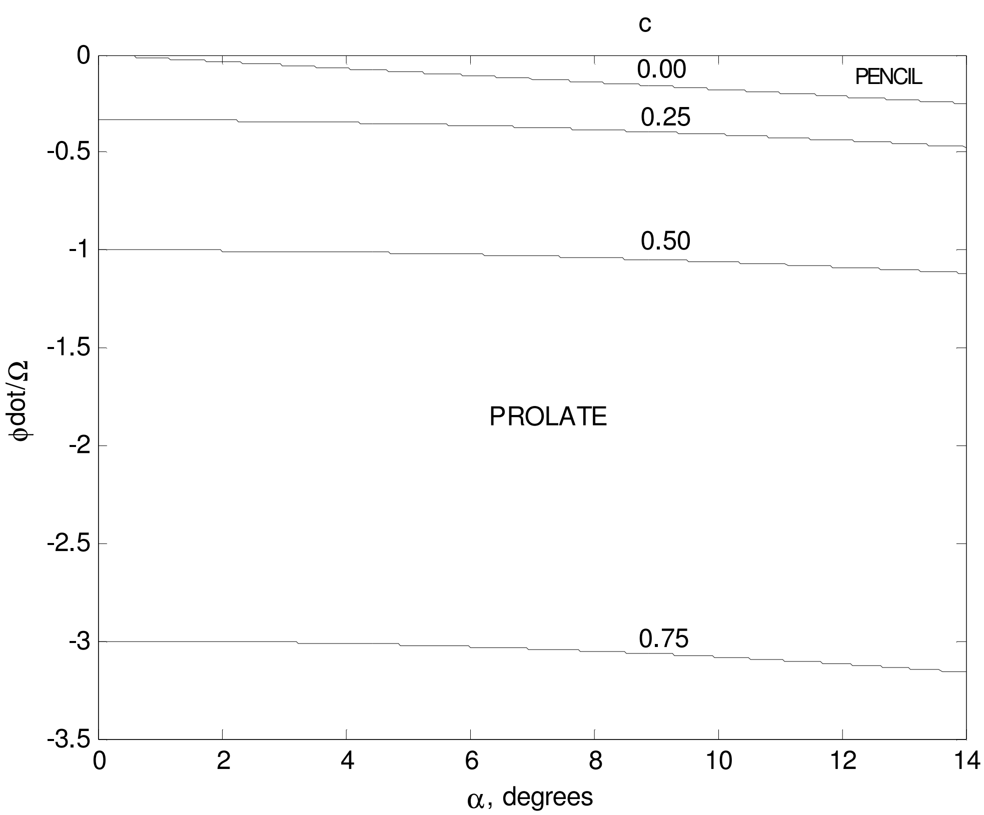

c
2.00
DISC
1.75
OBLATE
1.50
1.25
1.00
SPHERE
0.75
PROLATE
0.50
0.25
PENCIL
0.00
Ω/ω
α, degrees

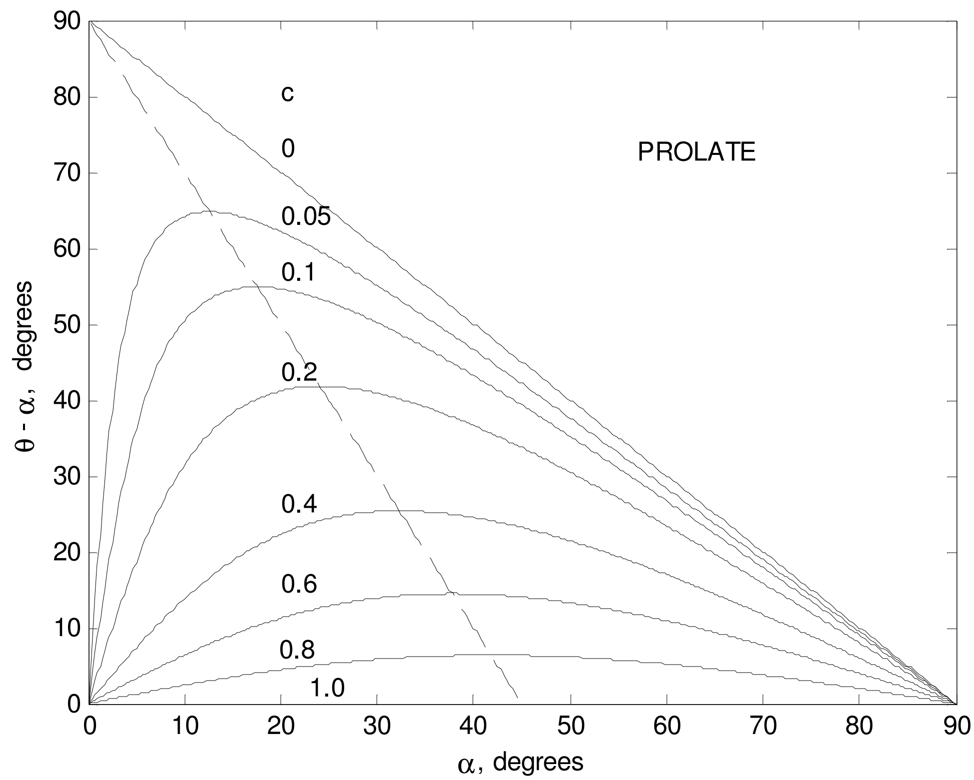

DISC
c
2.00
1.75
OBLATE
1.50
1.25
SPHERE
1.00
0.75
0.50
0.25
0.00
PROLATE
PENCIL
φdot/ω
α, degrees

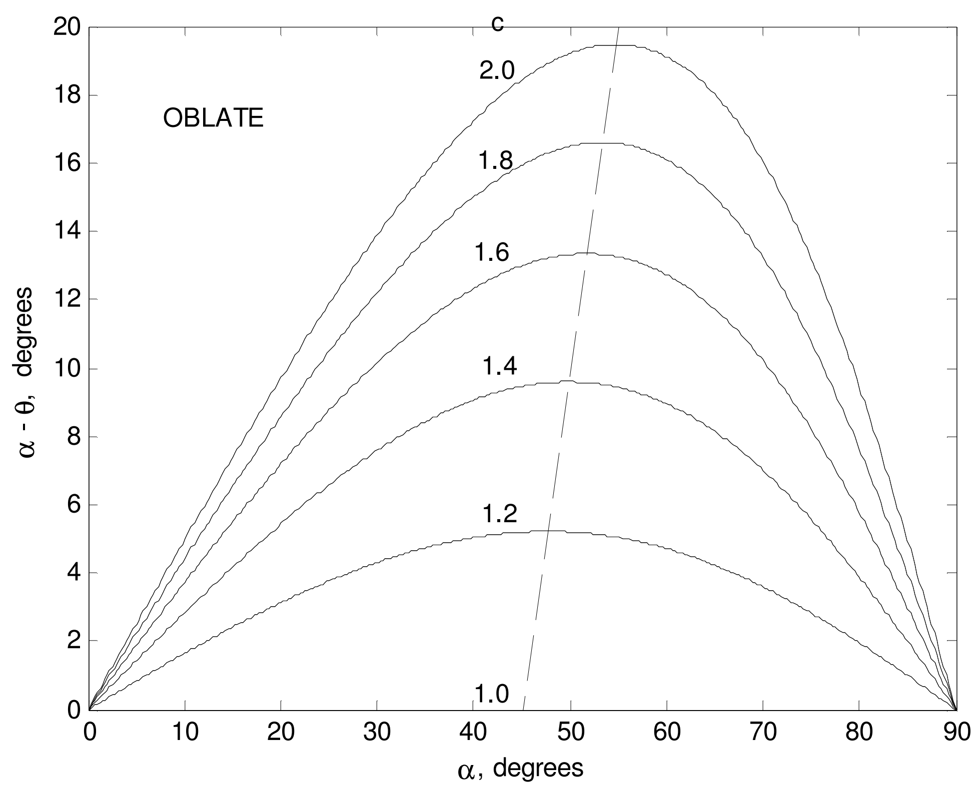


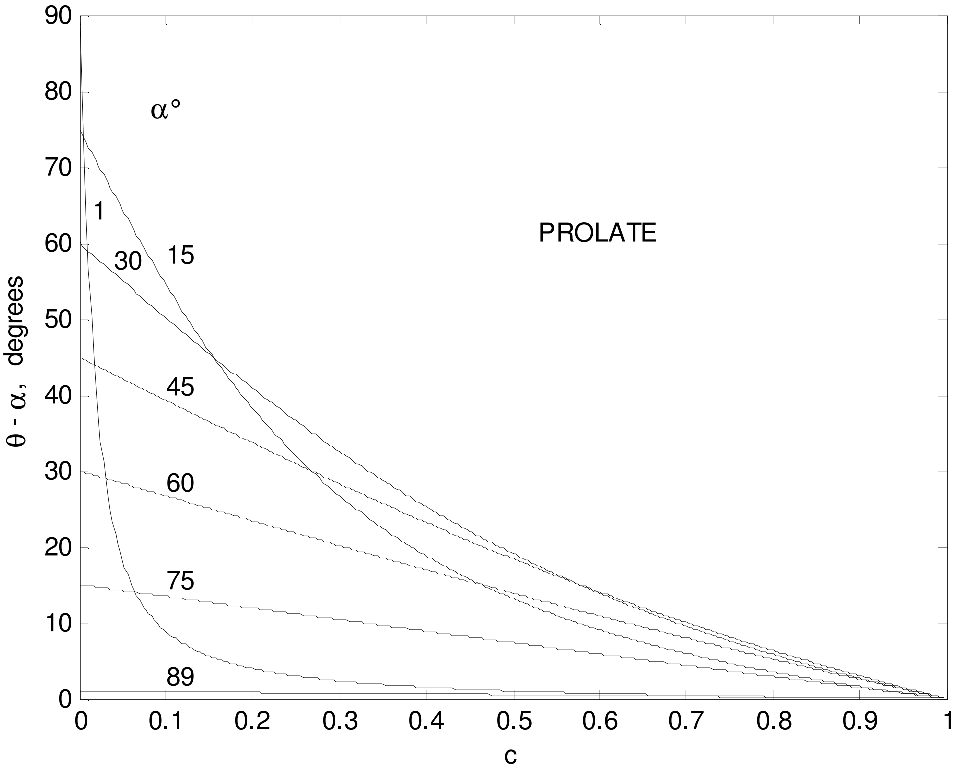
c
2.00
OBLATE
1.75
DISC
1.50
1.25
1.00
SPHERE
Ω/φdot
α, degrees
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
10
20
30
40
50
60
70
80
90


SPHERE
c
0.75
0.50
0.25
0.00
PENCIL
PROLATE
Ω/φdot
α, degrees
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
0
10
20
30
40
50
60
70
80
90

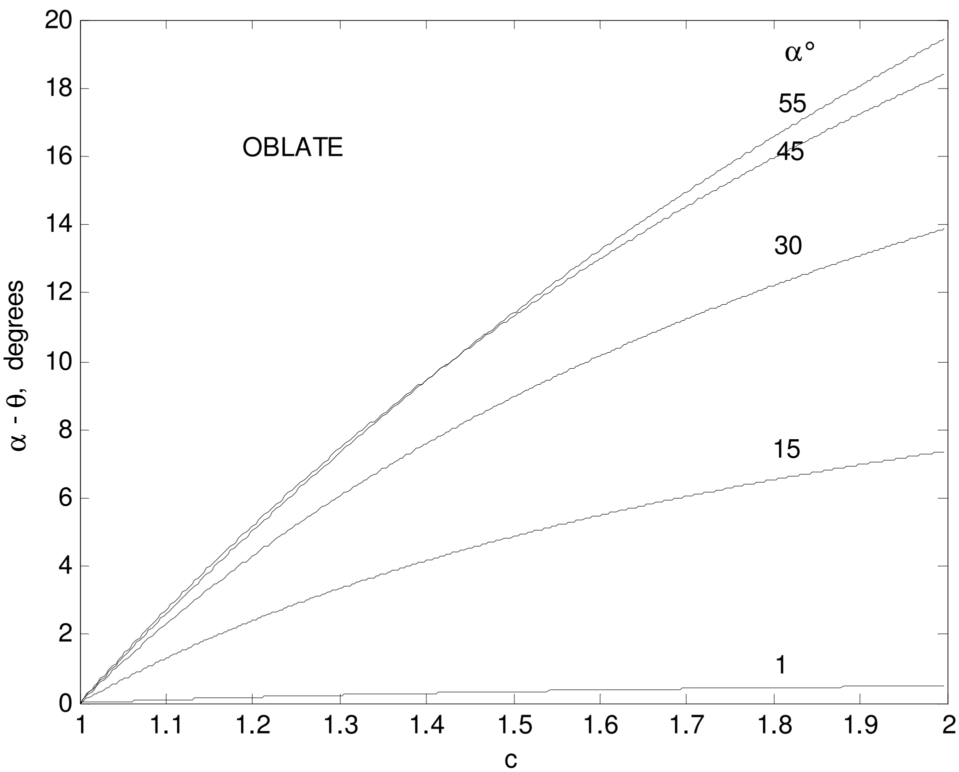

c
0.00
PENCIL
0.25
0.50
PROLATE
0.75
φdot/Ω
α, degrees
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
0
2
4
6
8
10
12
14


c
0
0.05
0.1
0.2
0.4
0.6
0.8
1.0
PROLATE
θ - α, degrees
α, degrees

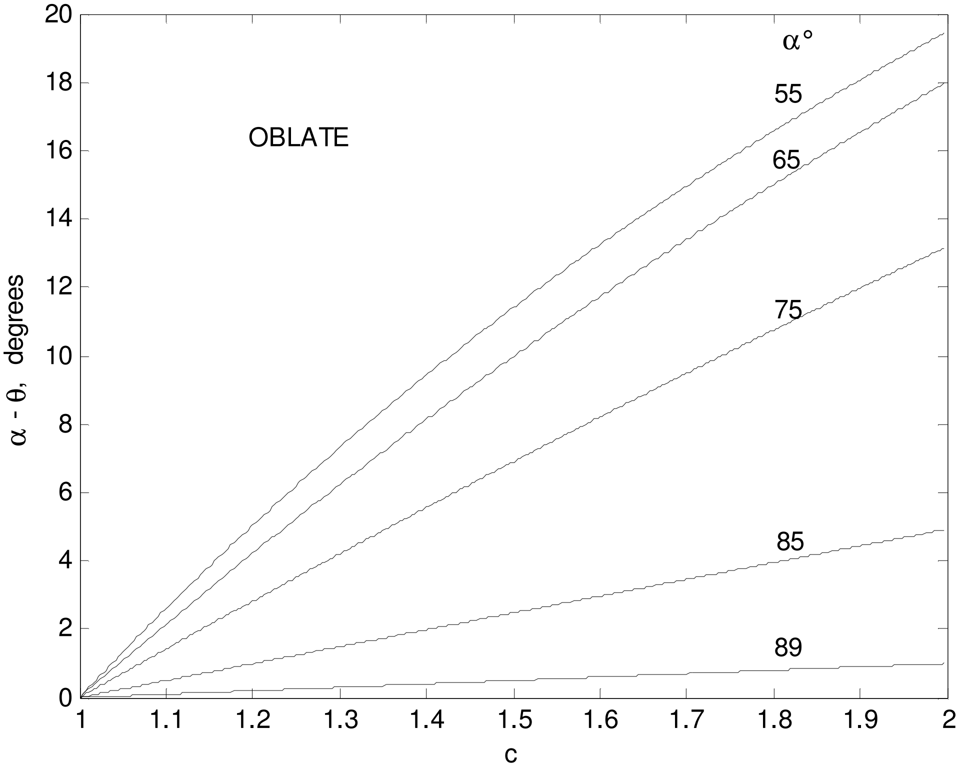

c
2.0
1.8
1.6
1.4
1.2
1.0
OBLATE
α - θ, degrees
α, degrees

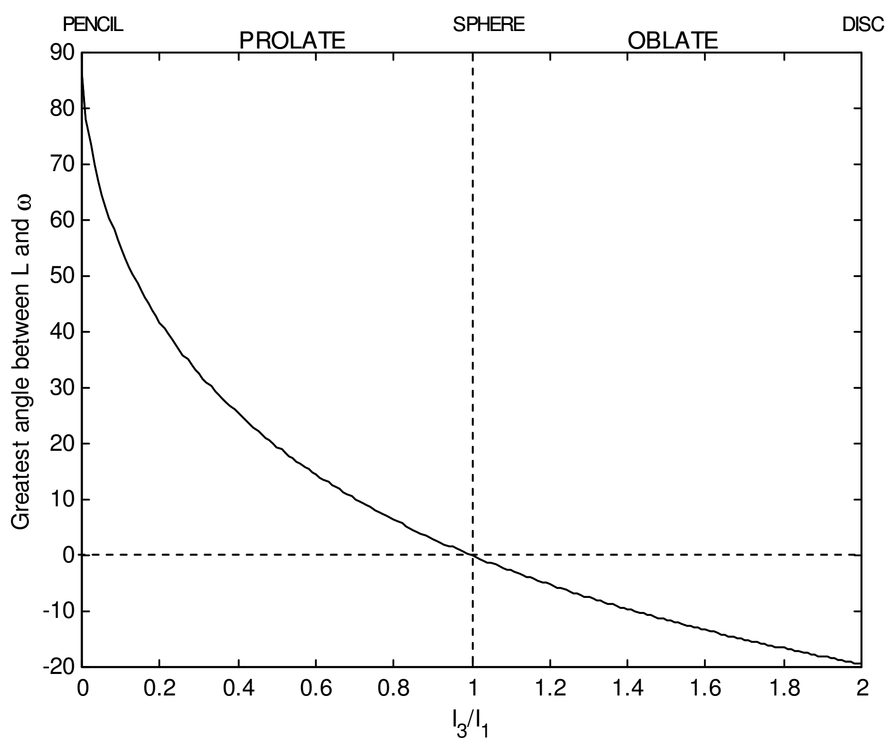


90
80
70
60
50
40
30
20
10
0
θ - α, degrees
α°
1
30
15
45
60
75
89
PROLATE
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
c



20
18
16
14
12
10
8
6
4
2
0
α - θ, degrees
α°
55
45
30
15
1
OBLATE
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2
c


α°
55
65
75
85
89
OBLATE
α - θ, degrees
c
20
18
16
14
12
10
8
6
4
2
0
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2


PENCIL
PROLATE
SPHERE
OBLATE
DISC
Greatest angle between L and ω
I_3/I_1
90
80
70
60
50
40
30
20
10
0
-10
-20
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Our planet Earth is approximately an oblate spheroid, its *dynamical ellipticity* $(I_3 - I_1)/I_1$ being about 3.285×10^{-3}. It is not rotating exactly abut its symmetry axis; the angle α between $\boldsymbol{\omega}$ and the symmetry axis being about one fifth of an arcsecond, which is about six metres on the surface. The rotation period is one sidereal day (which is a few minutes shorter than 24 solar hours.) Equation 4.8.17 tells us that the spin axis precesses about the symmetry axis in a period of about 304 days, all within the area of a tennis court. The actual motion is a little more complicated than this. The period is closer to 432 days because of the nonrigidity of Earth, and superimposed on this is an annual component caused by the annual movement of air masses. This precessional motion of a symmetric body spinning freely about an axis inclined to the symmetry axis gives rise to *variations of latitude* of amplitude about a fifth of an arcsecond. It is not to be confused with the 26,000 year period of the *precession of the equinoxes*, which is caused by *external torques* from the Moon and the Sun.

4.9 *Centrifugal and Coriolis Forces*

We are usually told in elementary books that there is "no such thing" as *centrifugal force*. When a satellite orbits around Earth, it is not held in equilibrium between two equal and opposite forces, namely gravity acting towards Earth and centrifugal force acting outwards. In reality, we are told, the satellite is accelerating (the *centripetal acceleration*); there is only one force, namely the gravitational force, which is equal to the mass times the centripetal acceleration.

Yet when we drive round a corner too fast and we feel ourselves flung away from the centre of curvature of our path, the "centrifugal force" certainly feels real enough, and indeed we can solve problems referred to rotating coordinate systems as if there "really" were such a thing as "centrifugal force".

Let's look at an even simpler example, not even involving rotation. A car is accelerating at a rate *a* towards the right. See figure IV.20 – but forgive my limited artistic abilities. Drawing a motor car is somewhat beyond my skills.

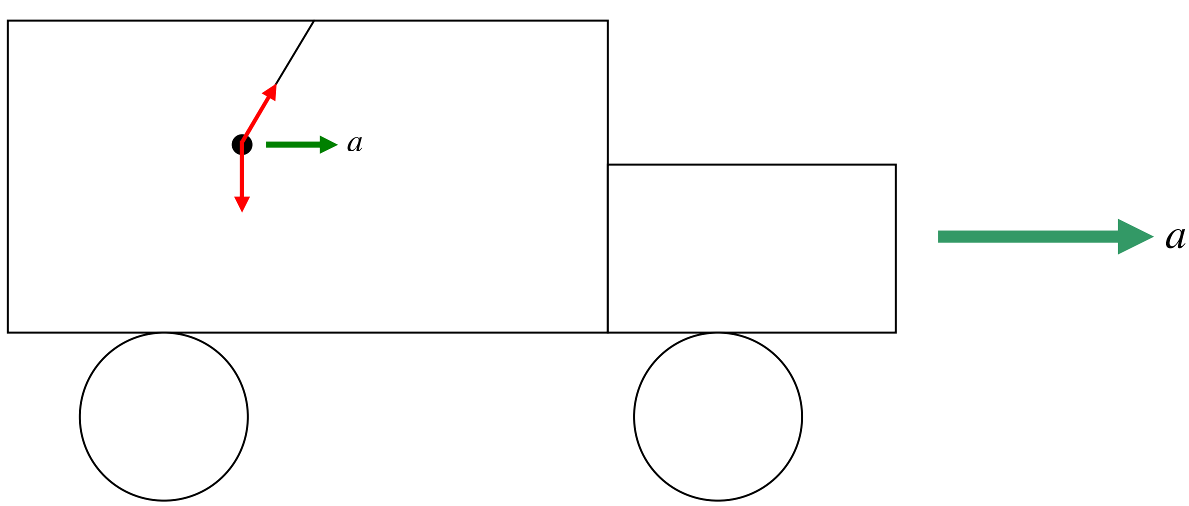


FIGURE IV.20

There is a plumb-bob hanging from the roof of the car, but, because of the acceleration of the car, it is not hanging vertically. Some would say that there are but two forces on the plumb-bob – its weight and the tension in the string – and, as a result of these, the bob is accelerating towards the right. By application of $F = ma$ it is easily possible to find the tension in the string and the angle that the string makes with the vertical.

The passenger in the car, however, sees things rather differently (figure IV.21.)

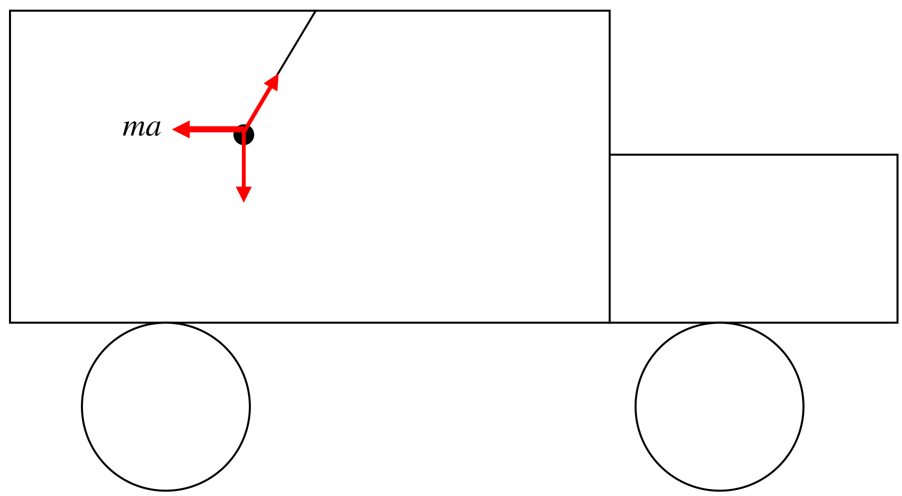


FIGURE IV.21

To the passenger in the car, nothing is accelerating. The plumb-bob is merely in static equilibrium under the action of three forces, one of which is a force *ma* towards the left. To the question "But what is the agent that is causing this so-called force?" I counter with the question "What is the agent that is causing the downward force that you attribute to some mysterious 'gravity' "?

It seems that, when referred to the reference frame of figure IV.20, there are only two forces, but when referred to the accelerating reference frame of figure IV.21, the system can be described perfectly well by postulating the existence of a force *ma* pulling towards the left. This is in fact a principle in classical mechanics, known as d'Alembert's principle, whereby, if one refers the description of a system to an accelerating reference frame, one can replace an acceleration with a force in the opposite direction. It results in a perfectly valid description of the behavior of a system, and will accurately predict how the system will behave. So who's to say which forces are "real" and which are "fictitious", and which reference frame is better than another?

The situation is similar with respect to centrifugal force. If you consider a satellite in orbit around Earth, some would say that there is only one force acting on the satellite, namely the gravitational force towards Earth. The satellite, being in a circular orbit, is accelerating towards the centre of the circle, and all is as expected - $F = ma$. The acceleration is the *centripetal acceleration* (*peto* – I desire). An astronaut on board the satellite may have a different point of view. He is at a constant distance from Earth, not accelerating; he is in static equilibrium, and he feels no net force on him whatever – he feels weightless. If he has been taught that Earth exerts a gravitational force, then this must be balanced by a force away from Earth. This force, which becomes apparent when referred to a corotating reference frame, is the *centrifugal force* (*fugo* – I flee, like a *fugitive*). It would need a good lawyer to argue that the invisible gravitational force towards Earth is a real force, while the equally invisible force acting away from Earth is imaginary. In truth, all forces are "imaginary" – in that they are only devices or concepts used in physics to describe and predict the behaviour of matter.

I mentioned earlier a possible awful examination question: Explain why Earth bulges at the equator, without using the term "centrifugal force". Just thank yourself lucky if you are not asked such a question! People who have tried to answer it have come up with some interesting ideas. I have heard (I don't know whether it is true) that someone once offered a prize of $1000 to anyone who could prove that Earth is rotating, and that the prize has never been claimed! Some have tried to imagine how you would determine whether Earth is rotating if it were the only body in the universe. There would be no external reference points against which one could measure the orientation of Earth. It has been concluded (by some) that even to think of Earth rotating in the absence of any external reference points is meaningless, so that one certainly could not determine how fast, or even whether and about what axis, Earth was rotating. Since "rotation" would then be meaningless, there would be no centrifugal force, Earth would not bulge, nor would the Foucault pendulum rotate, nor would naval shells deviate from their paths, nor would cyclones and anticyclones exist in the atmosphere.. Centrifugal force comes into existence only when there is an external universe. It is the external universe, then, revolving around the stationary Earth, that *causes* centrifugal force and all the other effects that we have mentioned. These are deep waters indeed, and I do not pursue this aspect further here. We shall merely take the pragmatic view that problems in mechanics can often be solved by referring motions to a corotating reference frame, and that the behaviour of mechanical systems can successfully and accurately be described and predicted by postulating the "existence" of "inertial" forces such as centrifugal and Coriolis forces, which make themselves apparent only when referred to a rotating frame. Thus, rather than involving ourselves in difficult questions about whether such forces are real, we shall take things easy with just a few simple equations.

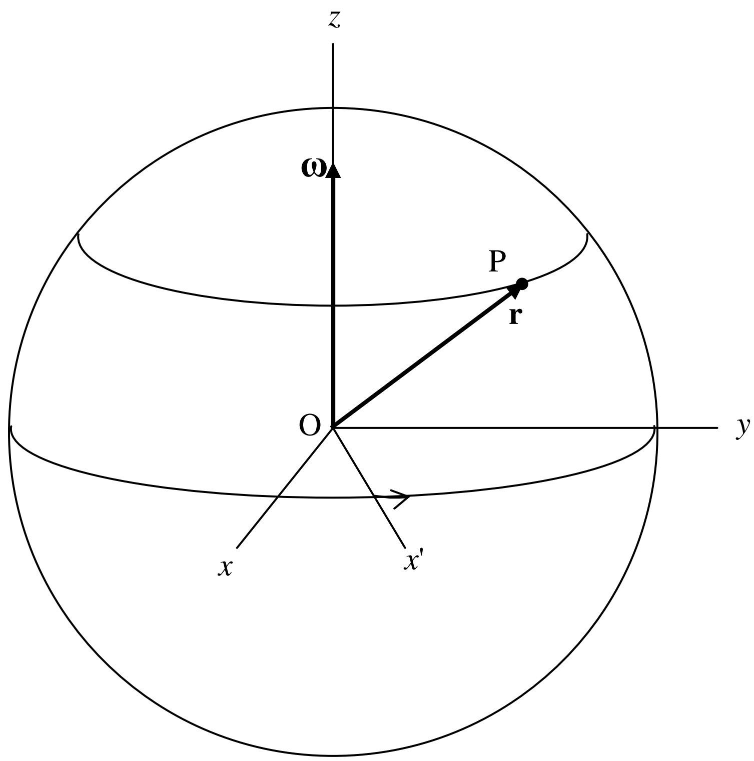


FIGURE IV.22

Σ = Oxyz is an inertial reference frame (i.e. not accelerating or rotating).

Σ' = O$x'y'z'$ is a frame that is rotating about the z-axis with angular velocity $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$.

Three questions:

1. If P is a point that is fixed with respect to Σ', what is its velocity **v** with respect to Σ?

2. If P is a point that is moving with velocity **v'** with respect to Σ', what is its velocity **v** with respect to Σ?

3. If P is a point that has an acceleration **a'** with respect to Σ', what is its acceleration **a** with respect to Σ?

1. The answer to the first question is, I think, fairly easy. Just by inspection of figure IV.22, I hope you will agree that it is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \qquad 4.9.1$$

In case this is not clear, try the following argument. At some instant the position vector of P with respect to Σ is **r**. At a time δt later its position vector is $\mathbf{r} + \boldsymbol{\delta}\mathbf{r}$, where $\delta r = r \sin\theta\, \omega\, \delta t$ and $\boldsymbol{\delta}\mathbf{r}$ is at right angles to **r**, and is directed along the small circle whose zenith angle is θ, in the direction of motion of P with respect to Σ. Expressed alternatively, $\boldsymbol{\delta}\mathbf{r} = r\sin\theta\,\omega\,\delta t\,\hat{\boldsymbol{\phi}}$. Draw the vector $\boldsymbol{\delta}\mathbf{r}$ on the figure if it helps. Divide both sides by δt and take the limit as $\delta t \to 0$ to get $\dot{\mathbf{r}}$. The magnitude and direction of $\dot{\mathbf{r}}$ are then expressed by the single vector equation $\dot{\mathbf{r}} = \boldsymbol{\omega}\times\mathbf{r}$, or $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$.

The only thing to look out for is this. In two-dimensional problems we are often used to expressing the relation between linear and angular speed by $\upsilon = r\omega$, and it doesn't matter which way round we write r and ω. When we are doing a three-dimensional problem using vector notation, it is important to remember that it is $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$.and not the other way round.

2. If P is moving with velocity **v'** with respect to Σ', then its velocity **v** with respect to Σ must be

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega}\times\mathbf{r}\,. \qquad 4.9.2$$

This shows that

$$\left(\frac{d}{dt}\right)_{\Sigma} \equiv \left(\frac{d}{dt}\right)_{\Sigma'} + \boldsymbol{\omega}\times \qquad 4.9.3$$

What this equation is intended to convey is that the operation of differentiating with respect to time when referred to the inertial frame Σ has the same result as differentiating with respect to time when referred to the rotating frame Σ', plus the operation $\boldsymbol{\omega}\times$. We shall understand this a little better in the next paragraph.

3. If P is accelerating with respect to Σ', we can apply the operation 4.9.3 to the equation 4.9.2:

$$\mathbf{a} = \left(\frac{d}{dt}\right)_{\Sigma'}(\mathbf{v}' + \boldsymbol{\omega}\times\mathbf{r}) + \boldsymbol{\omega}\times(\mathbf{v}' + \boldsymbol{\omega}\times\mathbf{r})$$

$$= \mathbf{a}' + \boldsymbol{\omega}\times\mathbf{v}' + \boldsymbol{\omega}\times\mathbf{v}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\,.$$

$$\therefore \qquad \mathbf{a} = \mathbf{a}' + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\mathbf{v}'\,. \qquad 4.9.4$$

This, then, answers the third question we posed. All we have to do now is to understand what it means.

To start with, let us return to the case where P is neither moving nor accelerating with respect to Σ'. In that case, equation 4.9.4 is just

$$\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \,, \qquad 4.9.5$$

which we could easily have obtained by applying the operator 4.9.3 to equation 4.9.1. Let us try and understand what this means. In what follows, a "hat" (^) denotes a unit vector.

We have $$\boldsymbol{\omega}\times\mathbf{r} = \omega r \sin\theta\, \hat{\boldsymbol{\phi}}$$

and hence $$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = \omega\hat{\mathbf{z}} \times \omega r \sin\theta\, \hat{\boldsymbol{\phi}} = r\omega^2 \sin\theta\; \hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}}$$

and $\hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}}$ is a unit vector directed towards the *z*-axis. Notice that the point P is moving at angular speed ω in a small circle of radius $r \sin \theta$. The expression $\mathbf{a} = \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = r\omega^2 \sin\theta\; \hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}}$, then, is just the familiar *centripetal acceleration*, of magnitude $r\,\omega^2 \sin\theta$, directed towards the axis of rotation.

We could also think of $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ as a *triple vector product*.

We recall that $$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

so that $$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = (\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} - \omega^2\mathbf{r}.$$

That is $$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = r\omega^2 \cos\theta\,\hat{\mathbf{z}} - r\omega^2\,\hat{\mathbf{r}}.$$

This can be illustrated by the vector diagram shown in figure IV.23. The vectors are drawn in green, in accordance with my convention of red, blue and green for force, velocity and acceleration respectively.

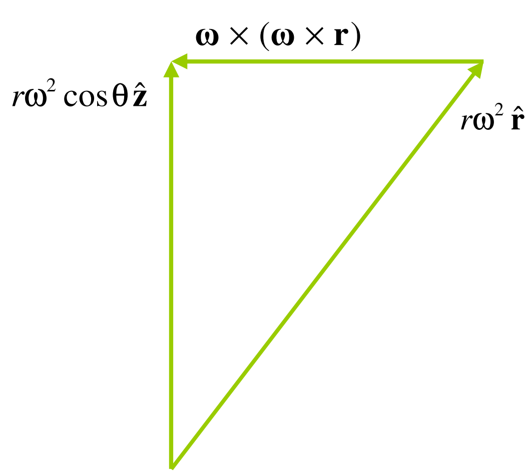


FIGURE IV.23

However, equation 4.9.4 also tells us that, if a particle is *moving* with velocity $\mathbf{v}'$ with respect to Σ', it has an additional acceleration with respect to Σ of $2\boldsymbol{\omega}\times\mathbf{v}'$, which is at right angles to $\mathbf{v}'$ and to $\boldsymbol{\omega}$. This is the *Coriolis acceleration.*

The converse of equation 4.9.4 is

$$\mathbf{a}' = \mathbf{a} + \boldsymbol{\omega}\times(\mathbf{r}\times\boldsymbol{\omega}) + 2\mathbf{v}'\times\boldsymbol{\omega} . \qquad 4.9.6$$

If a particle has a force $\mathbf{F} = m\mathbf{a}$ acting on it with respect to Σ, when referred to Σ' it will appear that the particle is moving under the influence of three forces: the "true force" **F**, the *centrifugal force* acting away from the axis of rotation, and the *Coriolis force*, which acts only when the particle is in motion with respect to Σ', and which is at right angles to both $\mathbf{v}'$ and $\boldsymbol{\omega}$:

$$\mathbf{F}' = \mathbf{F} + m\boldsymbol{\omega}\times(\mathbf{r}\times\boldsymbol{\omega}) + 2m\mathbf{v}'\times\boldsymbol{\omega} . \qquad 4.9.7$$

It is worth now spending a few moments thinking about the direction of the Coriolis force $2m\mathbf{v}'\times\boldsymbol{\omega}$. Earth is spinning on its axis with a period of 24 sidereal hours (23^{h} and 56^{m} of solar time.). The vector $\boldsymbol{\omega}$ is directed upwards through the north pole. Now go to somewhere on Earth at latitude 45° N. Fire a naval shell to the north. To the east. To the south. To the west. Now go to the equator and repeat the experiment. Go to the north pole. There you can fire only due south. Repeat the experiment at 45° south, and at the south pole. Each time, think about the direction of the vector $2m\mathbf{v}'\times\boldsymbol{\omega}$. If your thoughts are to my thoughts, your mind to my mind, you should conclude that the shell veers to the right in the northern hemisphere and to the left in the southern hemisphere, and that the Coriolis force is zero at the equator. As air rushes out of a high pressure system in the northern hemisphere, it will swirl clockwise around the pressure centre. As it rushes in to a low pressure system, it will swirl counterclockwise. The opposite situation will happen in the southern hemisphere.

You can think of the Coriolis force on a naval shell as being a consequence of conservation of angular momentum. Go to 45° N and point your naval gun to the north. Your shell, while waiting in the breech, is moving around Earth's axis at a linear speed of $\omega R \sin 45^\circ$, where R is the radius of Earth. Now fire the shell to the north. By the time it reaches latitude 50° N, it is being carried around Earth's axis in a small circle of radius only $R \sin 40^\circ$. In order for angular momentum to be conserved, its angular speed around the axis must speed up – it will be deviated towards the east.

Now try another thought experiment (*Gedanken Prüfung.).* Go to the equator and build a tall tower. Drop a stone from the top of the tower. Think now about the direction of the vector $2m\mathbf{v}'\times\boldsymbol{\omega}$. I really mean it – think hard. Or again, think about conservation of angular momentum. The stone drops closer to Earth's axis of rotation. It must conserve angular momentum. It falls to the *east* of the tower (*not* to the west!).

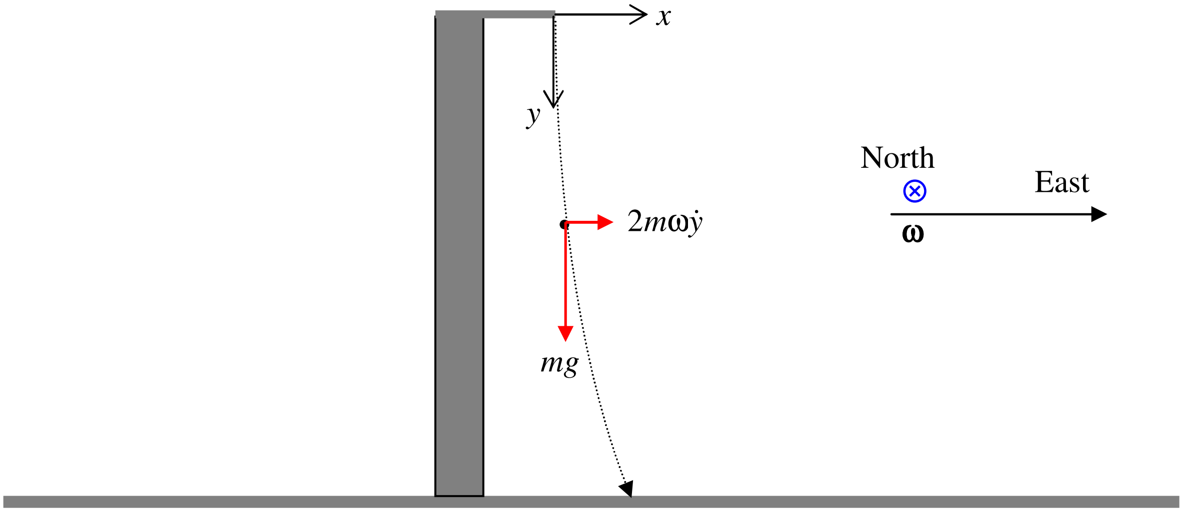


FIGURE IV.24

The two forces on the stone are its weight mg and the Coriolis force. Earth's spin vector **ω** is to the north. The Coriolis force is at right angles to the stone's velocity. If we resolve the stone's velocity into a vertically down component $\dot{y}$ and a horizontal east component $\dot{x}$, the corresponding components of the Coriolis force will be $2m\omega\dot{y}$ to the east and $2m\omega\dot{x}$ upward. However, I'm going to assume that $\dot{x} << \dot{y}$ and the only significant Coriolis force is the eastward component $2m\omega\dot{y}$, which I have drawn. Another way of stating the approximation is to say that the upward component of the Coriolis force is negligible compared with the weight mg of the stone.

After dropping for a time t, the y-coordinate of the stone is found in the usual way from

$$\ddot{y} = g, \qquad \dot{y} = gt, \qquad y = \tfrac{1}{2}gt^2,$$

and the x- coordinate is found from

$$\ddot{x} = 2\omega\dot{y} = 2\omega gt, \qquad \dot{x} = \omega gt^2, \qquad x = \tfrac{1}{3}\omega gt^3.$$

Thus you can find out how far to the east it has fallen after two seconds, or how far to the east it has fallen if the height of the tower is 100 metres. The equation to the trajectory would be the t-eliminant, which is

$$x^2 = \frac{8\omega^2}{9g}y^3. \qquad 4.9.8$$

For Earth, $\omega = 7.292 \times 10^{-5}$ rad s^{-1}, and at the equator $g = 9.780$ m s^{-2}, so that

$$\frac{8\omega^2}{9g} = 4.788 \times 10^{-10} \ \mathrm{m}^{-1} \ .$$

The path is graphed in figure IV.25 for a 100-metre tower. The horizontal scale is exaggerated by a factor of about 6000.

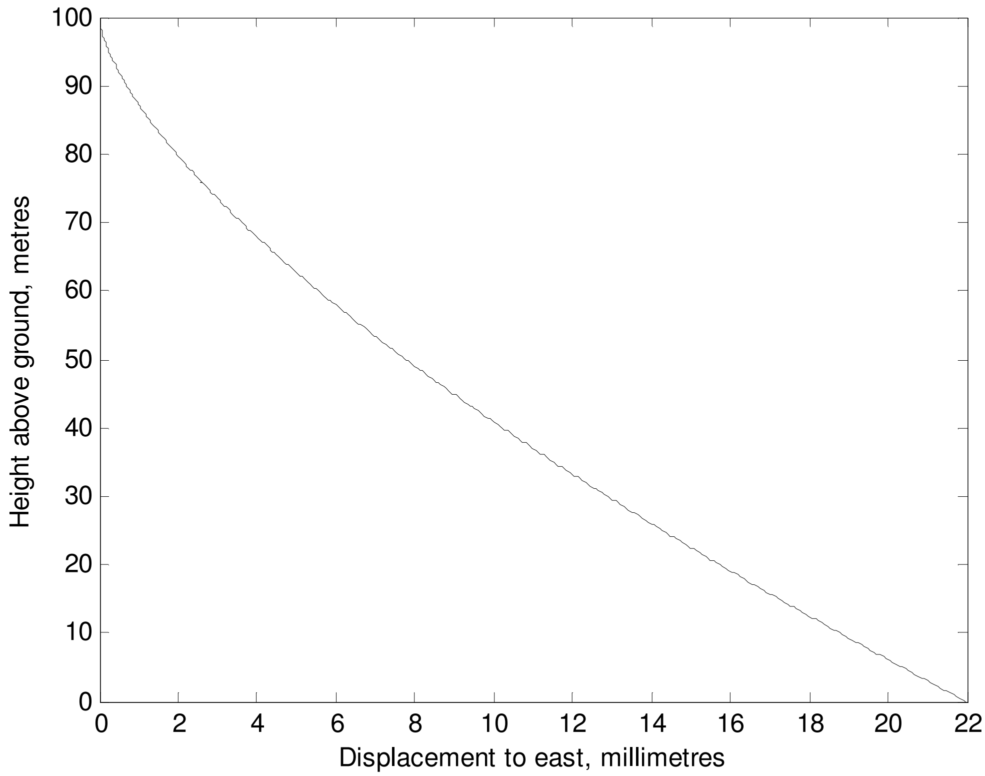


FIGURE IV.25

I once asked myself the question whether a migrating bird could navigate by using the Coriolis force. After all, if it were flying north in the northern hemisphere, it would experience a Coriolis force to its right; might this give it navigational information? I published an article on this in *The Auk* **97**, 99 (1980). Let me know what you think!

You may have noticed the similarity between the equation for the Coriolis force $\mathbf{F} = 2m\mathbf{v}'\times\boldsymbol{\omega}$ and the equation for the Lorentz force on an electric charge moving in a magnetic field: $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$. The analogy can be pursued a bit further. If you rotate a coil in an electric field, a current will flow in the coil. That's electromagnetic induction, and it is the principle of an electric generator. Sometime early in the twentieth century, the American physicist Arthur Compton (that's not Denis Compton, of whom only a few of my readers will have heard, and very few indeed in North America) successfully tried an interesting experiment. He made some toroidal glass tubes, filled with water coloured with $KMnO_4$, so that he could see the water, and he rotated these tubes about a horizontal or vertical diameter, and, lo, the water flowed around the tubes, just as a current flows in a coil when it is rotated in a magnetic field. Imagine a toroidal tube set up in an east-west vertical plane at the equator. The top part of the tube is slightly further from Earth's rotation axis than the bottom part, and consequently the water near the top of tube has more angular momentum per unit mass, around Earth's axis, than the water near the bottom. Now rotate the tube through 180° about its east-west horizontal diameter. The high angular momentum fluid moves closer to Earth's rotation axis, and the low angular momentum fluid moves further from Earth's axis. Therefore, in order to conserve angular momentum, the fluid must flow around the tube. By carrying out a series of such experiments, Compton was able, at least in principle, to measure the speed of Earth's rotation, and even his latitude, without looking out of the window, and indeed without even being aware that there was an external universe out there. You may think that this was a very difficult experiment to do, but you do it yourself every day. There are three mutually orthogonal semicircular canals inside your ear, and, every time you move your head, fluid inside these semicircular canals flows in response to the Coriolis force, and this fluid flow is detected by little nerves, which send a message to your brain to tell you of your movements and to help you to keep your balance. You have a wonderful brain, which is why understanding physics is so easy.

Going back to the Lorentz force, we recall that a moving charge in a magnetic field experiences a force at right angles to its velocity. But what is the origin of a magnetic field? Well, a magnetic field exists, for example, in the interior of a solenoid in which there is a current of moving electrons in the coil windings, and it is these circulating electrons that ultimately cause the Lorentz force on a charge in the interior of the solenoid – just as it is the galaxies in the universe revolving around our stationary Earth which are the ultimate cause of the Coriolis force on a particle moving with respect to our Earth. But there I seem to be getting into deep waters again, so perhaps it is time to move on to something easier.

4.10 *The Top*

We have classified solid bodies technically as symmetric, asymmetric, spherical and linear tops, according to the relative sizes of their principal moments of inertia. In this section, or at least in the title of this section, I mean "top" in the nontechnical sense of the child's toy – that is to say, a symmetric body, pointed at one end, spinning around its axis of symmetry, with the pointed end on the ground or on a table. Technically, it is a "heavy symmetric top with one point fixed."

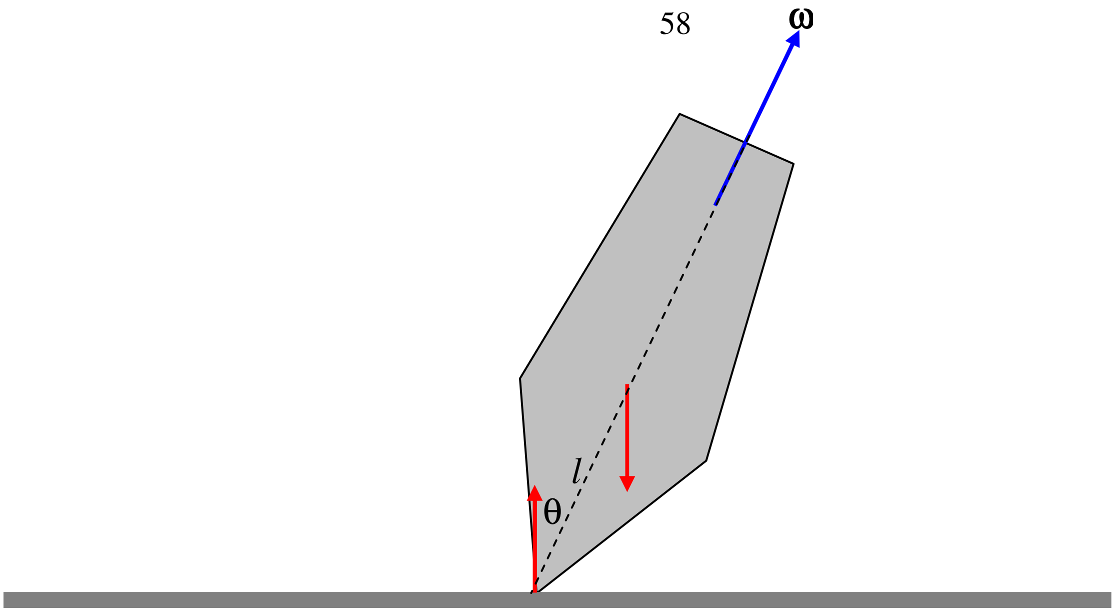


FIGURE IV.26

I have drawn it in figure IV.26, spinning about its symmetry axis, which makes an angle θ with the vertical. The distance between the centre of mass and the point of contact with the table is *l*. It has a *couple* of forces acting on it – its weight and the equal, opposite reaction of the table. In figure IV.27, I replace these two forces by a torque, **τ**, which is of magnitude $Mgl \sin\theta$.

Note that, since there is an *external torque* acting on the system, the *angular momentum vector is not fixed.*

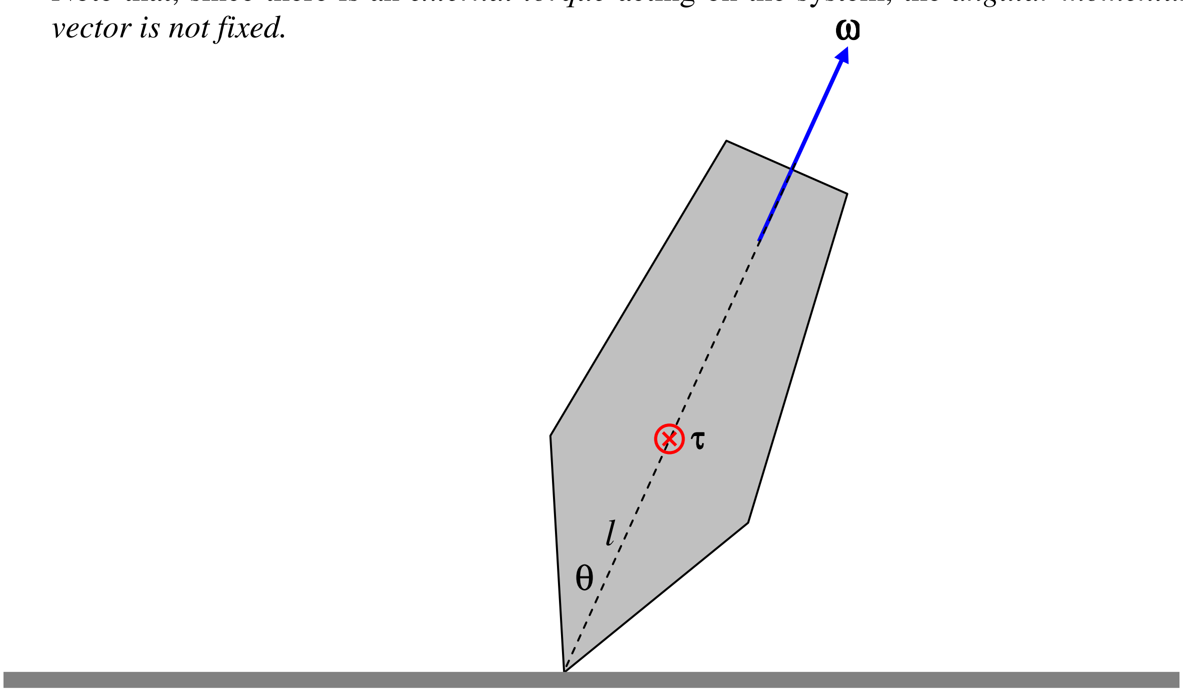


FIGURE IV.27

Before getting too involved with numerous equations, let's spend a little while describing qualitatively the motion of a top, and also describing the various coordinate systems and angles we shall be discussing. First, we shall be making use of a set of *space-fixed coordinates.* We'll let the origin O of the coordinates be at the (fixed) point where the tip of the top touches the table. The axis Oz points vertically up to the zenith. The Ox and Oy axes are in the (horizontal) plane of the table. Their exact orientation is not very important, but let's suppose that Ox points due south, and Oy points due east. Oxyz then constitutes a right-handed set. We'll also make use of a set of *body-fixed* axes, which I'll just refer to for the moment as 1, 2 and 3. The 3-axis is the symmetry axis of the top. The 1- and 2-axes are perpendicular to this. Their exact positions are not very important, but let's suppose that the 31-plane passes through a small ink-dot which you have marked on the side of the top, and that the 123 system constitutes a right-handed set.

We are going to describe the orientation of the top at some instant by means of the three Eulerian angles θ, ϕ and ψ (see figure IV.28). The symmetry axis of the top is represented by the heavy arrow, and it is tipped at an angle θ to the z-axis. I'll refer to a plane normal to the axis of symmetry as the "equator" of the top, and it is inclined at θ to the xy-plane. The ascending node of the equator on the xy-plane has an azimuth ϕ, and ψ is the angular distance of the 1-axis from the node. The azimuth of the symmetry axis of the top is $\phi - 90^{\circ} = \phi + 270^{\circ}$.

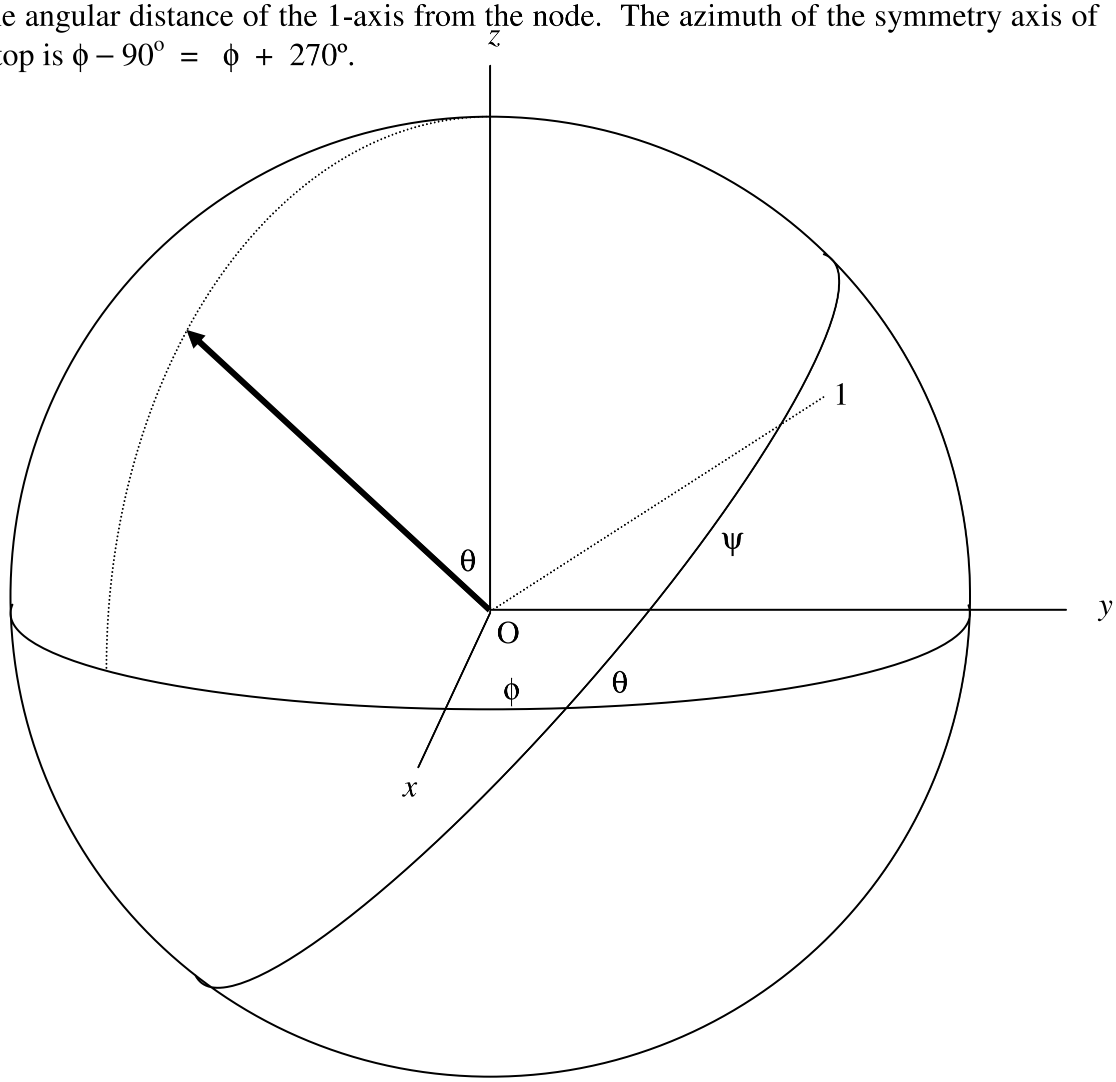


FIGURE IV.28

Now let me anticipate a bit and describe the motion of the top while it is spinning and subject to the torque described above.

The symmetry axis of the top is going to precess around the z-axis, at a rate that will be described as $\dot{\phi}$. Except under some conditions (which I shall eventually describe) this precessional motion is *secular.* That means that ϕ increases all the time – it does not oscillate to and fro. However, the symmetry axis does not remain at a constant angle with the z-axis. It oscillates, or nods, up and down between two limits. This motion is called *nutation* (Latin: *nutare*, to nod). One of our aims will be to try to find the rate of nutation $\dot{\phi}$ and to find the period and amplitude of the nutation.

It may look as though the top is spinning about its axis of symmetry, but this isn't quite so. If the angular velocity vector were exactly along the axis of symmetry, it would stay there, and there would be no precession or nutation, and this cannot be while there is a torque acting on the top. An exception would be if the top were spinning vertically (θ = 0), when there would be no torque acting on it. The top can in fact do that, except that, unless the top is spinning quite fast, this situation is unstable, and the top will tip away from its vertical position at the slightest perturbation. At high spin speeds, however, such motion is stable, and indeed one of our aims must be to determine the least angular speed about the symmetry axis such motion is stable.

However, as mentioned, unless the top is spinning vertically, the vector $\boldsymbol{\omega}$ is not directed along the symmetry axis. We'll call the three components of $\boldsymbol{\omega}$ along the three body-fixed axes ω_1, ω_2 and ω_3, the last of these being the component of $\boldsymbol{\omega}$ along the symmetry axis. One of the things we shall discover when we proceed with the analysis is that ω_3 remains constant throughout the motion. Also, you should be able to distinguish between ω_3 and $\dot{\psi}$. These are not the same, because of the motion of the node. In fact you will probably understand that $\dot{\psi} = \omega_3 - \dot{\phi}\cos\theta$. Indeed, we have already derived the relations between the component of the angular velocity vector and the rate of change of the Eulerian angles – see equations 4.2.1,2 and 3. We shall be making use of these relations in what follows.

To analyse the motion of the top, I am going to make use of Lagrange's equations of motion for a conservative system. If you are familiar with Lagrange's equations, this will be straightforward. If you are not, you might prefer to skip this section until you have become more familiar with Lagrangian mechanics in Chapter 13. However, I introduced Lagrange's equation briefly in section 4.4, in which Lagrange's equation of motion was given as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = P_j . \qquad 4.10.1$$

Here T is the kinetic energy of the system. P_j is the generalized force associated with the generalized coordinate q_j. If the force is a *conservative* force, then P_j can be expressed as the negative of the derivative of a potential energy function:

$$P_j = -\left(\frac{\partial V}{\partial q_j}\right). \qquad 4.10.2$$

Thus we have Lagrange's equation of motion for a system of *conservative forces*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}. \qquad 4.10.3$$

Thus, in solving problems in Lagrangian dynamics, the first line in our calculation is to write down an expression for the kinetic energy. The first line begins: "$T = ...$".

In the present problem, the kinetic energy is

$$T = \tfrac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \tfrac{1}{2}I_3\omega_3^2. \qquad 4.10.4$$

Here the subscripts refer to the principal axes, 3 being the symmetry axis. The Eulerian angles θ and ϕ are zenith distance and azimuth respectively of the symmetry axis with respect to laboratory fixed (space fixed) axes. The Eulerian angle ψ is measured around the symmetry axis. The components of the angular velocity are related to the rates of change of the Eulerian angles by previously derived formulas (equations 4.2.1,2,3), so the kinetic energy can be expressed in terms of $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$:

$$T = \tfrac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \tfrac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2. \qquad 4.10.5$$

The potential energy is

$$V = Mgl\cos\theta + \text{constant}. \qquad 4.10.6$$

Having written down the expressions for the kinetic and potential energies in terms of the Eulerian angles, we are now in a position to apply the Lagrangian equations of motion 4.10.3 for each of the three coordinates. We'll start with the coordinate ϕ. The Lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = -\frac{\partial V}{\partial \phi}. \qquad 4.10.7$$

We see that $\dfrac{\partial T}{\partial \phi}$ and $\dfrac{\partial V}{\partial \phi}$ are each zero, so that $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot\phi}\right) = 0$, or $\dfrac{\partial T}{\partial \dot\phi} = \text{constant}$. This has the dimensions of angular momentum, so I'll call the constant L_1. On evaluating the derivative $\dfrac{\partial T}{\partial \dot\phi}$, we obtain for the Lagrangian equation in φ:

$$I_1 \dot\phi \sin^2\theta + I_3 \dot\phi \cos^2\theta + I_3 \dot\psi \cos\theta = L_1 . \qquad 4.10.8$$

I'll leave the reader to carry out exactly the same procedure with the coordinate ψ. You'll quickly conclude that $\dfrac{\partial T}{\partial \dot\psi} = \text{constant}$, which has the dimensions of angular momentum, so call it L_3, and you will then arrive at the following for the Lagrangian equation in ψ:

$$I_3(\dot\psi + \dot\phi\cos\theta) = L_3 . \qquad 4.10.9$$

But the expression in parentheses is equal to ω_3 (see equation 4.2.3, although we have already used it in equation 4.10.5), so we obtain the result that ω_3, the component of the angular velocity about the symmetry axis, is constant during the motion of the top. It would probably be worth the reader's time at this point to distinguish again carefully in his or her mind the difference between ω_3 and $\dot\psi$.

Eliminate $\dot\psi$ from equations 4.10.8 and 4.10.9:

$$\dot\phi = \frac{L_1 - L_3\cos\theta}{I_1 \sin^2\theta} . \qquad 4.10.10$$

This equation tells us how the rate of precession varies with θ as the top nods or nutates up and down.

We could also eliminate $\dot\phi$ from equations 4.10.8 and 4.10.9:

$$\dot\psi = \frac{L_3}{I_3} - \frac{(L_1 - L_3\cos\theta)\cos\theta}{I_1\sin^2\theta} . \qquad 4.10.11$$

The Lagrangian equation in θ is a little more complicated, but we can obtain a third equation of motion from the constancy of the total energy:

$$\tfrac{1}{2} I_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2} I_3(\dot\psi + \dot\phi\cos\theta)^2 + Mgl\cos\theta = E . \qquad 4.10.12$$

We can eliminate $\dot\phi$ and $\dot\psi$ from this, using equations 4.10.10 and 11, to obtain an equation in θ and the time only. After a little algebra, we obtain

$$\dot{\theta}^2 = A - B\cos\theta - \left(\frac{L_1 - L_3\cos\theta}{I_1\sin\theta}\right)^2, \qquad 4.10.13$$

where $$A = \frac{1}{I_1}\left(2E - \frac{L_3^2}{I_3}\right) \qquad 4.10.14$$

and $$B = \frac{2Mgl}{I_1}. \qquad 4.10.15$$

The turning points in the θ-motion (i.e. the nutation) occur where $\dot{\theta} = 0$. This results (after some algebra! – but quite straightforward all the same) in a cubic equation in $c = \cos\theta$:

$$a_0 + a_1c + a_2c^2 + Bc^3 = 0 \qquad 4.10.16$$

where

$$a_0 = A - \left(\frac{L_1}{I_1}\right)^2 = \frac{2E}{I_1} - \frac{L_3^2}{I_1I_3} - \frac{L_1^2}{I_1^2}, \qquad 4.10.17$$

$$a_1 = \frac{2L_1L_3}{I_1^2} - B = \frac{2L_1L_3}{I_1^2} - \frac{2Mgl}{I_1} \qquad 4.10.18$$

and $$a_2 = -A - \left(\frac{L_3}{I_1}\right)^2 = -\left[\frac{2E}{I_1} - \frac{L_3^2}{I_1I_3} + \left(\frac{L_3}{I_1}\right)^2\right]. \qquad 4.10.19$$

Now equation 4.10.16 is a cubic equation in cos θ and it has either one real root or three real roots, and in the latter case two of them or all three might be equal. We must also bear in mind that θ is real only if cos θ is in the range −1 to +1. We are trying to find the nutation limits, so we are hoping that we will find two and only two real values of θ . (If the tip of the top were poised on top of a point – e.g. if it were poised on top of the Eiffel Tower, rather than on a horizontal table – you could have θ > 90°.)

To try and understand this better, I constructed in my mind a top somewhat similar in shape to the one depicted in figures IV.26 and 27, about 4 cm diameter, 7 cm high, made of brass. For the particular shape and dimensions that I imagined, it worked out to have the following parameters, rounded off to two significant figures:

$M = 0.53$ kg $\quad l = 0.044$ m $\quad I_1 = 1.7 \times 10^{-4}$ kg m^2 $\quad I_3 = 9.8 \times 10^{-5}$ kg m^2

I thought I'd spin the top so that ω_3 (which, as we have seen, remains constant throughout the motion) is 250 rad s^{-1}, and I'd start the top at rest ($\dot{\phi} = \dot{\theta} = 0$) at θ = 30° and then let go. Presumably it would then immediately start to fall, and 30° would then be the upper bound to the nutation. We want to see how far it will fall before nodding upwards again. With $\omega_3 = 250$ rad s^{-1} we find, from equation 4.10.9 that

$L_3 = 2.45 \times 10^{-2}$ J s.

Also, I am assuming that $\dot{\phi} = 0$ when $\theta = 30^\circ$, and equation 4.10.10 tells us that

$L_1 = 2.121\,762 \times 10^{-2}$ J s.

Then with $g = 9.8$ m s^{-2}, we have, from equation 4.10.15,

$B = 2.688\,659 \times 10^3$ s^{-2}.

My initial conditions are that $\dot{\phi}$ and $\dot{\theta}$ are each zero when $\theta = 30^\circ$, and equations 4.10.10 and 4.10.13 between them tell us that $A = B\cos 30^\circ$, so that

$A = 2.328\,447 \times 10^3$ s^{-2}.

From equations 4.12.17, 18 and 19 we now have

$a_0 = -1.324\,898 \times 10^4$ s^{-2}

$a_1 = +3.328\,586 \times 10^4$ s^{-2}

$a_2 = -2.309\,834 \times 10^4$ s^{-2}

and we already have

$B = 2.688\,659 \times 10^3$ s^{-2}.

The "sign rule" for polynomial equations, if you are familiar with it, tells us that there are no negative real roots (i.e. no solutions with $\theta > 90^\circ$), and indeed if we solve the cubic equation 4.10.16 we obtain

$$c = 0.824\,596, \quad 0.866\,025, \quad 6.900\,406.$$

The second of these corresponds to $\theta = 30^\circ$, which we already knew must be a solution. Indeed we could have divided equation 4.10.16 by $c - \cos 30^\circ$ to obtain a quadratic equation for the remaining two roots, but it is perhaps best to solve the cubic equation as it is, in order to verify that cos 30° is indeed a solution, thus providing a check on the arithmetic. The third solution does not give us a real θ (we were rather hoping this would happen). The second solution is the lower limit of the nutation, corresponding to $\theta = 34^\circ$ 27'.

Generally, however, the top will nutate between two values of θ. Let us call these two values α and β, α being the smaller (more vertical) of the two. I'll refer to $\theta = \alpha$ as the "upper bound" of the motion, even though $\alpha < \beta$, since this corresponds to the more vertical position of the top. We have looked a little at the motion in θ; now let's look at the motion in ϕ, starting with equation 4.10.10:

$$\dot{\phi} = \frac{L_1 - L_3 \cos\theta}{I_1 \sin^2\theta}. \qquad 4.10.10$$

If the initial conditions are such that $L_1 > L_3 \cos\alpha$ (and therefore always greater than $L_3 \cos\theta$) $\dot{\phi}$ is always positive. The motion is then something like I try to illustrate in figure IV.29.

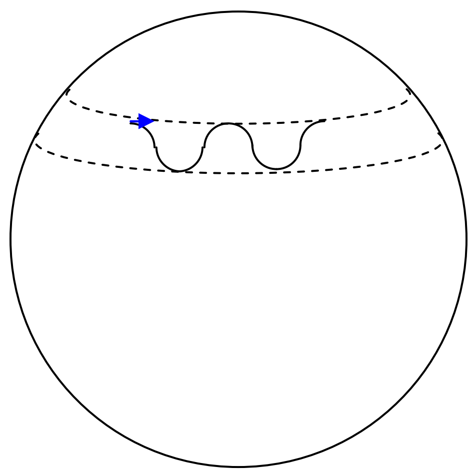


FIGURE IV.29

This motion corresponds to an initial condition in which you give the top an initial push in the forward direction as indicated by the little blue arrow.

If the initial conditions are such that $\cos\alpha > L_1/L_3 > \cos\beta$, the sign of $\dot{\phi}$ is different at the upper and lower bounds. Thus is illustrated in figure IV.30

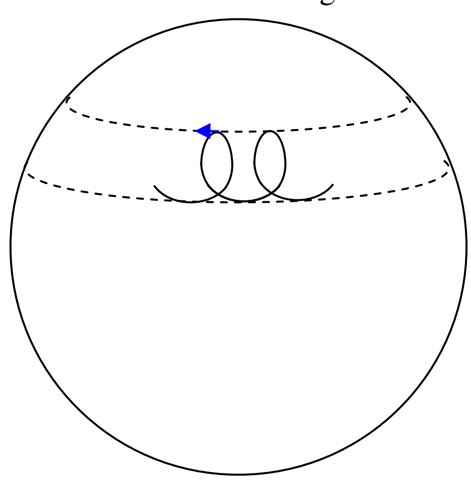


FIGURE IV.30

This motion would arise if you were initially to give a little backward push before letting go of the top, as indicated by the little blue arrow.

If the initial conditions are such that $L_1 = L_3 \cos\alpha$, then $\dot{\theta}$ and $\dot{\phi}$ are each zero at the upper bound of the nutation, and this was the situation in our numerical example. It corresponds to just letting the top drop when you let it go, without giving it either a forward or a backward push. This is illustrated in figure IV.31.

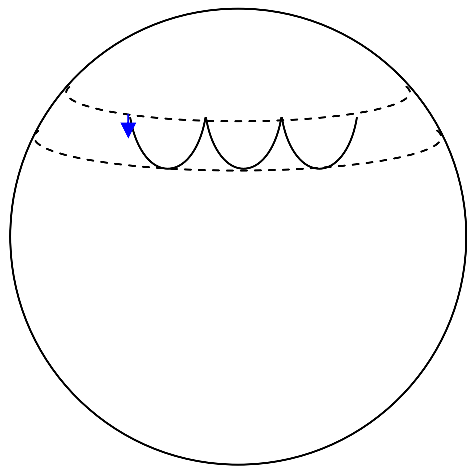


FIGURE IV.31

As we discovered while doing our numerical example, the initial conditions $\dot{\theta} = \dot{\phi} = 0$ when $\theta = \alpha$ lead, in this third type of motion, to

$$L_1 = L_3 \cos\alpha \qquad 4.10.20$$

and

$$A = B\cos\alpha. \qquad 4.10.21$$

In that case equation 4.10.13 becomes

$$\dot{\theta}^2 = B(\cos\alpha - \cos\theta) - \left[\frac{L_3(\cos\alpha - \cos\theta)}{I_1 \sin\theta}\right]^2. \qquad 4.10.22$$

The lower bound to the nutation(i.e. how far the top falls) is found by putting $\theta = \beta$ when $\dot{\theta} = 0$. This gives the following quadratic equation for β:

$$\cos^2\beta - \frac{L_3^2}{I_1^2 B}\cos\beta + \frac{L_3^2\cos\alpha}{I_1^2 B} - 1 = 0. \qquad 4.10.23$$

In our numerical example, this is

$$\cos^2\beta - 7.725\,002\cos\beta + 5.690\,048 = 0, \qquad 4.10.24$$

which, naturally, has the same two solutions as we obtained when we solved the cubic equation, namely 0.824 596 and 6.900 406.

Recalling the definition of B (equation 4.10.15), we see that equation 4.10.23 can be written

$$\cos\alpha - \cos\beta = \frac{2MglI_1}{L_3^2}\sin^2\beta\,, \qquad 4.10.25$$

from which we see that the greater L_3, the smaller the difference between α and β – i.e. the smaller the amplitude of the nutation.

Equation 4.10.12, with the help of equations 4.10.10 and 11, can be written:

$$E - \frac{L_3^2}{2I_3} - \tfrac{1}{2}I_1\dot{\theta}^2 = \frac{1}{2I_1}(L_1\csc\theta - L_3\cot\theta)^2 + Mgl\cos\theta. \qquad 4.10.26$$

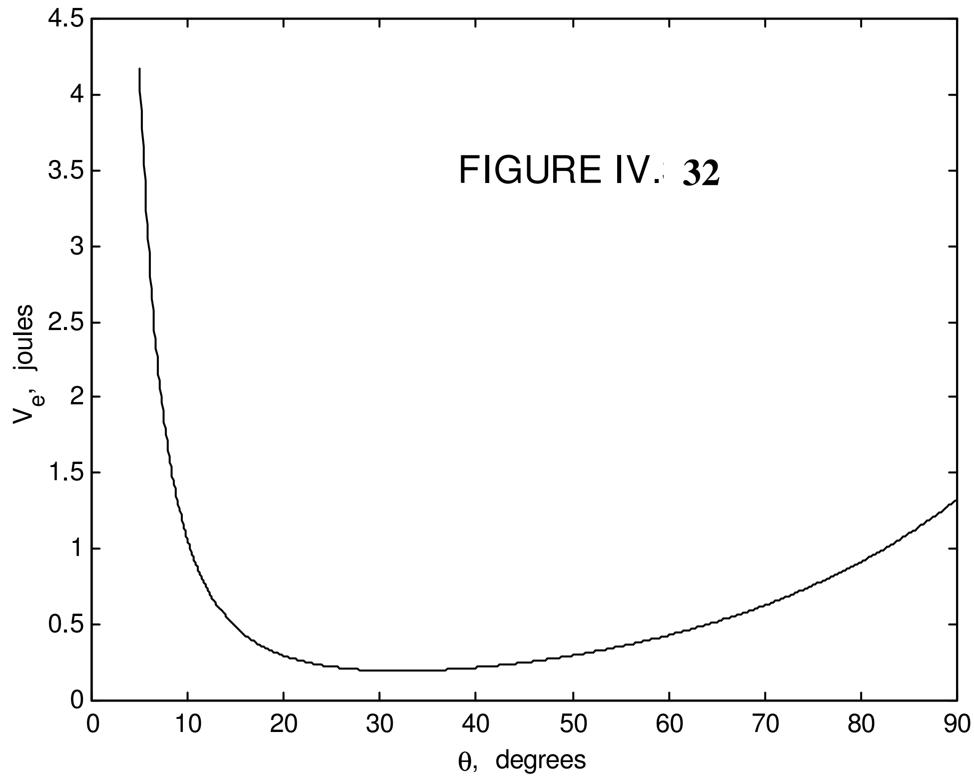
The left hand side is the total energy minus the spin and nutation kinetic energies. Thus the right hand side represents the effective potential energy $V_e(\theta)$ referred to a reference frame that is co-rotating with the precession. The term $Mgl\cos\theta$ needs no explanation. The negative of the derivative of the first term on the right hand side would be the "fictitious" force that "exists" in the corotating reference frame. The effective potential energy $V_e(\theta)$ is given by

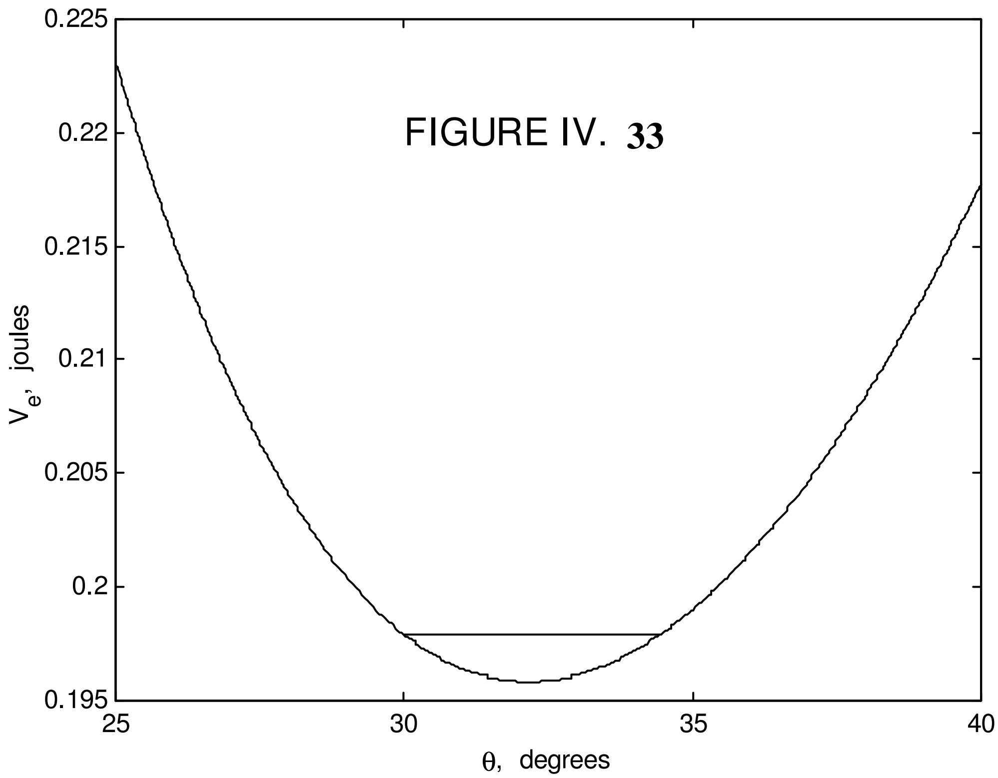
$$\frac{V_e(\theta)}{L_1^2/(2I_1)} = [\csc\theta - (L_3/L_1)\cot\theta]^2 + \frac{2I_1Mgl\cos\theta}{L_1^2}. \qquad 4.10.27$$

I draw $V_e(\theta)$ in figures IV.32 and 33 using the values that we used in our numerical example – that is:

$$V_e(\theta) = 1.324\,081(\csc\theta - 1.154\,701\cot\theta)^2 + 0.228\,536\cos\theta \qquad \text{joules.} \quad 4.10.28$$

Figure 32 is plotted up to 90° (although as mentioned earlier one could go further than this if the top were not spinning on a horizontal table), and Figure 33 is a close look close to the minimum. One can see that if $E - L_3^2/(2I_2) = 0.1979$ the effective potential energy (which cannot go higher than this, and reaches this value only when $)\dot{\theta} = 0$, the nutation limits are between 30° and 34° 24'. For a given L_3, for a larger total energy, the nutation limits are correspondingly wider. But for a given total energy, the larger the component L_3 of the angular momentum is, the lower will be the horizontal line and the narrower the nutation limits. If the top loses energy (e.g. because of air resistance, or friction at the point of contact with the table), the E = constant line will become lower and lower, and the amplitude of the nutation will become less and less. If $E - L_3^2/(2I_3)$ is equal to the minimum value of $V_e(\theta)$ there is only one solution for θ, and there is no nutation. For energy less than this, there is no stationary value of θ and the top falls over.





We can find the rate of true regular precession quite simply as follows – and this is often done in introductory books.

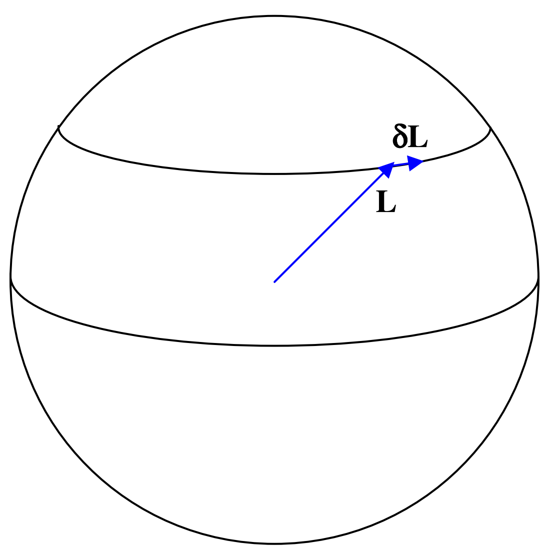


FIGURE IV.34

In figure IV.34, the vector **L** represents the angular momentum at some time, and in a time interval δt later the change in the angular momentum is $\boldsymbol{\delta}\mathbf{L}$. The angular momentum is changing because of the external torque, which is a horizontal vector of magnitude $Mgl\sin\theta$ (remind yourself from figure XV.26 and 27). The rate of change of angular momentum is given by $\dot{\mathbf{L}} = \boldsymbol{\tau}$. In time δt the tip of the vector **L** moves through a "distance" $\tau\,\delta t$. Denote by $\boldsymbol{\Omega}$ precessional angular velocity (the magnitude of which we have hitherto called $\dot{\phi}$). The tip of the angular momentum vector is moving in a small circle of radius $L\sin\theta$. We therefore see that $\tau = \Omega L\sin\theta$. Further, $\boldsymbol{\tau}$ is perpendicular to both $\boldsymbol{\Omega}$ and **L.** Therefore, in vector notation,

$$\boldsymbol{\tau} = \boldsymbol{\Omega}\times\mathbf{L}. \qquad 4.10.29$$

Note that the magnitude of $\boldsymbol{\tau}$ is $Mgl\sin\theta$ and the magnitude of $\boldsymbol{\Omega}\times\mathbf{L}$ is $\Omega L\sin\theta$, so that the rate of precession is

$$\Omega = \frac{Mgl}{L} \qquad 4.10.30$$

and is independent of θ.

One can continue to analyse the motion of a top almost indefinitely, but there are two special cases that are perhaps worth noting and which I shall describe.

<u>Special Case I. $L_1 = L_3$</u>.

In this case, equation 4.10.27 becomes

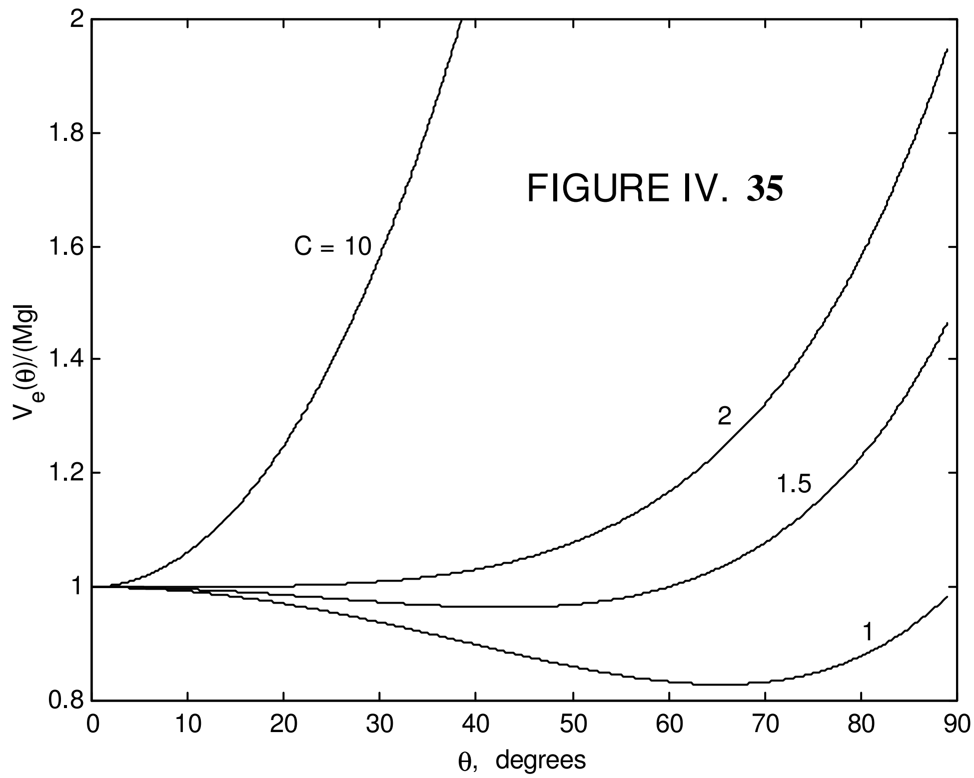
$$\frac{V_{\mathrm{e}}(\theta)}{Mgl} = C(\csc\theta - \cot\theta)^2 + \cos\theta, \qquad 4.10.31$$

where

$$C = \frac{L_1^2}{2MglI_1}. \qquad 4.10.32$$

It may be rather unlikely that $L_1 = L_3$ exactly, but this case is of interest partly because it is exceptional in that $V_{\mathrm{e}}(0)$ does not go to infinity; in fact $V_{\mathrm{e}}(0) = Mgl$ whatever the value of C. Try substituting $\theta = 0$ in equation 4.10.31 and see what you get! The right hand side is indeed 1, but you may have to work a little to get there. The other reason why this case is of interest is that it makes a useful introduction to case II, which is not impossibly unlikely, namely that L_1 is *approximately* equal to L_3, which leads to motion of some interest.

In figure IV.35, I draw $V_{\mathrm{e}}(0)/Mgl$ for several different C.



From the graphs, it looks as though, if $C \geq 2$, there is one equilibrium position, it is at $\theta = 0^{\circ}$ (i.e. the top is vertical), and the equilibrium is stable, If $C < 2$, there are two equilibrium positions: the vertical position is unstable, and the other equilibrium position is stable. Thus if the top is spinning fast (large *C*) it can spin in the vertical position only (a "sleeping top"), but, as the top slows down owing to friction and air resistance, the vertical position will become unstable, and the top will fall down to a positive value of θ . These deductions are correct, for $\frac{dV_{\text{e}}}{d\theta} = 0$ results in

$$2C(1 - \cos\theta)^2 = \sin^4\theta \qquad 4.10.33$$

One solution is $\theta = 0$, and a second differentiation will show that this is stable or unstable according to whether *C* is greater than or less than 2, although the second differentiation is slightly tedious, and it can be avoided. We can also note that $1 - \cos\theta$ is a common factor of the two sides of equation 4.10.33, and it can be divided out to yield a cubic equation in cos θ:

$$2C - 1 - (2C+1)\cos\theta - \cos^2\theta - \cos^3\theta = 0, \qquad 4.10.34$$

which could be solved to find the second equilibrium point – but that again is slightly tedious. A less tedious way might be to take the square root of each side of equation 4.10.33:

$$\sqrt{2C}(1 - \cos\theta) = 1 - \cos^2\theta \qquad 4.10.35$$

and then divide by the common factor $(1 - \cos\theta)$ to obtain

$$\cos\theta = \sqrt{2C} - 1, \qquad 4.10.36$$

which gives a real θ only if $C \geq 2$. Note also, if $C = \frac{1}{2}$, $\theta = 90^\circ$.

<u>Special Case II.</u> $L_1 \approx L_3$.

In other words, L_1 and L_3 are not very different. In figure IX.36 I draw $V_e(0)/Mgl$ for several different C, for $L_3 = 1.01\ L_1$.

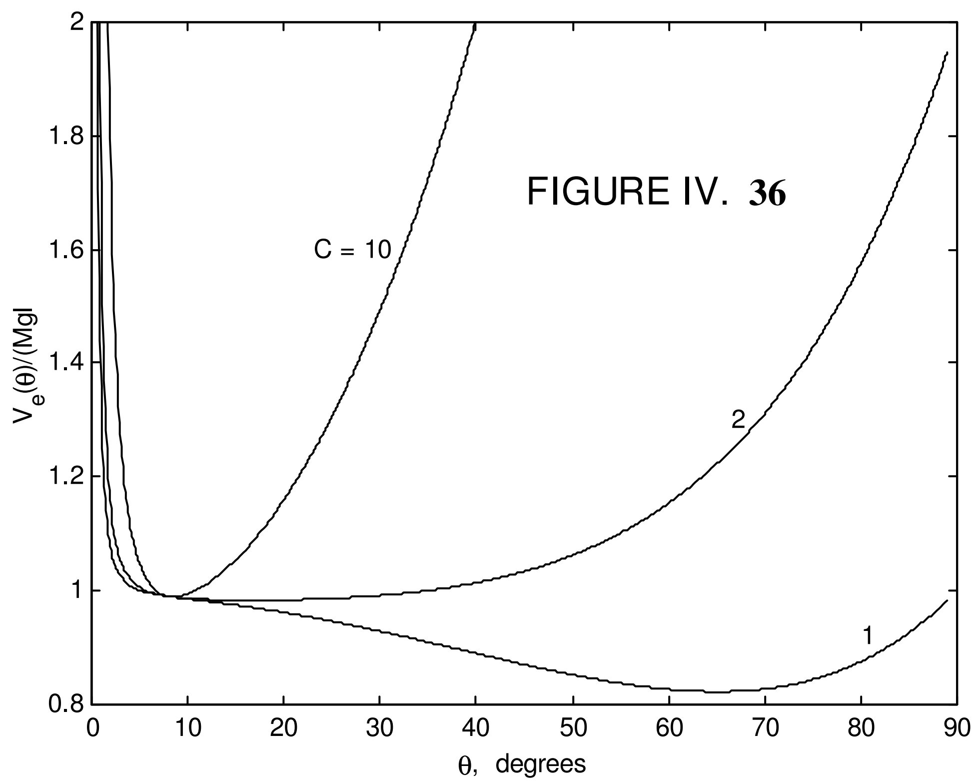


FIGURE IV. 36

We see that for quite a large range of C greater than 2 the stable equilibrium position is close to vertical. Even though the curve for $C = 2$ has a very broad minimum, the actual minimum is a little less than 17°. (I haven't worked out the exact position – I'll leave that to the reader.)

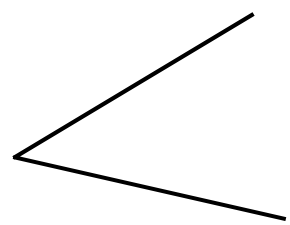
Appendix

In Section 4.4 we raised the question as to whether angle is a dimensioned or a dimensionless quantity, and in Section 4.8 we raised the question as to whether angle is a vector quantity.

I can present two arguments. One of them will prove incontrovertibly that angle is dimensionless. The other will prove, equally incontrovertibly, and equally convincingly that angle has dimensions. Angle, as you know, is defined as the ratio of arc length to radius. It is the ratio of two lengths, and is therefore incontrovertibly dimensionless. Q.E.D. On the other hand, it is necessary to state the units in which angle is expressed. You cannot merely talk about an angle of 1. You must state whether that is 1 degree or 1 degree. Angle therefore has dimensions. Q.E.D. So – you may take your pick. In many contexts, I like to think of angle as a dimensioned quantity, having dimensions Θ. That is to say, not a combination of mass, length and time, but having its own dimensions in its own right. I find I can carry on with dimensional analysis successfully like this.

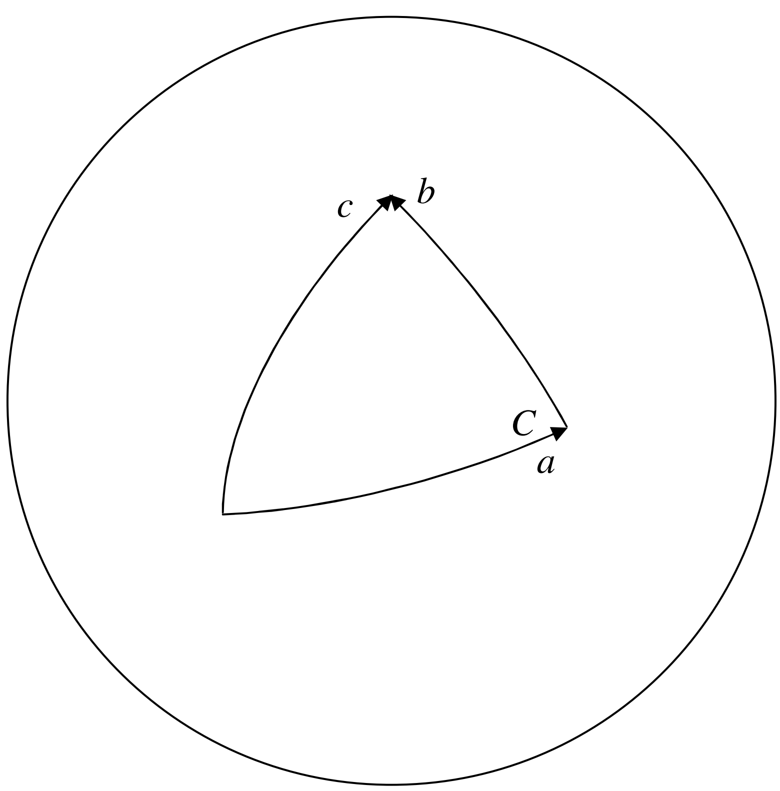
Now for the question: Is angle a vector?

An angle certainly has both magnitude and a direction associated with it. Thus the direction associated with the angle



is at right angles to the plane of the screen, or the paper.

However, this evidently isn't enough for it to be a vector in the sense that we know it.
For example, if you turn through an angle *a*, and then through an angle *b*, you cannot say that the net resultant of these is to turn through an angle *c*, where $c^2 = a^2 + b^2 - 2ab\cos C$.



Thus, although angle has both magnitude and direction, and could be thought of thus far as a vector, angles do not obey the ordinary triangle law of vector addition. For this reason, angles are sometimes called "pseudo-vectors".

In fact, as any astronomy student will tell you, the correct relation between the angles is

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

If the angles *a*, *b*, *c* (not *C*) are very small, then the triangle becomes almost plane. The angles add more and more like the usual plane triangle rule for vector addition. This is probably obvious when thinking about the geometry, but you can also convince yourself of it by expanding the sines and cosines (except for cos *C*) as series, and, to the second order of small quantities $(\cos\theta \approx 1 - \frac{1}{2}\theta^2\ ,\ \sin\theta \approx 1)$, you'll find that the equation $\cos c = \cos a \cos b + \sin a \sin b \cos C$ reduces to $c^2 = a^2 + b^2 - 2ab\cos C.$ For this reason it is sometimes said that an "infinitesimal rotation" can be regarded as a true vector. Also for this reason, the time rate of change of an angle, $\frac{d\theta}{dt}$, that is to say an angular velocity, can quite safely be treated as a true vector, since the numerator and denominator of the derivatives are both infinitesimals.

Thus, although angle has direction associated with it, angle is not a true vector in that angles do not follow the usual rules for vector addition. However, very small angles do approximately follow the addition rules, so that, in the infinitesimal limit, angles can be treated as vectors. And hence angular velocity, being a ratio of infinitesimals ($d\theta$ and dt), can correctly be treated as vectors.