

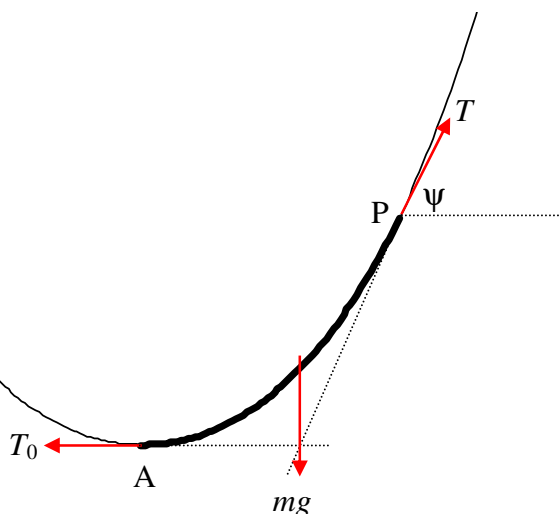
## CHAPTER 18 THE CATENARY

### 18.1 Introduction

If a flexible chain or rope is loosely hung between two fixed points, it hangs in a curve that looks a little like a parabola, but in fact is not quite a parabola; it is a curve called a *catenary*, which is a word derived from the Latin *catena*, a chain.

### 18.2 The Intrinsic Equation to the Catenary

FIGURE XVIII.1



We consider the equilibrium of the portion AP of the chain, A being the lowest point of the chain. See figure XVIII.1 It is in equilibrium under the action of three forces: The horizontal tension  $T_0$  at A; the tension  $T$  at P, which makes an angle  $\psi$  with the horizontal; and the weight of the portion AP. If the mass per unit length of the chain is  $\mu$  and the length of the portion AP is  $s$ , the weight is  $\mu s g$ . It may be noted that these three forces act through a single point.

Clearly, 
$$T_0 = T \cos \psi \tag{18.2.1}$$

and 
$$\mu s g = T \sin \psi, \tag{18.2.2}$$

from which 
$$(\mu s g)^2 + T_0^2 = T^2 \tag{18.2.3}$$

and 
$$\tan \psi = \frac{\mu s g}{T_0}. \tag{18.2.4}$$

Introduce a constant  $a$  having the dimensions of length defined by

$$a = \frac{T_0}{\mu g}. \quad 18.2.5$$

Then equations 18.2.3 and 4 become

$$T = \mu g \sqrt{s^2 + a^2} \quad 18.2.6$$

and

$$s = a \tan \psi. \quad 18.2.7$$

Equation 18.2.7 is the *intrinsic equation* (i.e. the  $s, \psi$  equation) of the catenary.

### 18.3 Equation of the Catenary in Rectangular Coordinates, and Other Simple Relations

The slope at some point is  $y' = \frac{dy}{dx} = \tan \psi = \frac{s}{a}$ , from which  $\frac{ds}{dx} = a \frac{d^2 y}{dx^2}$ . But, from the usual pythagorean relation between intrinsic and rectangular coordinates  $ds = (1 + y'^2)^{1/2} dx$ , this becomes

$$(1 + y'^2)^{1/2} = a \frac{dy'}{dx}. \quad 18.3.1$$

On integration, with the condition that  $y' = 0$  where  $x = 0$ , this becomes

$$y' = \sinh(x/a), \quad 18.3.2$$

and, on further integration,

$$y = a \cosh(x/a) + C. \quad 18.3.3$$

If we fix the origin of coordinates so that the lowest point of the catenary is at a height  $a$  above the  $x$ -axis, this becomes

$$y = a \cosh(x/a). \quad 18.3.4$$

This, then, is the  $x, y$  equation to the catenary. The  $x$ -axis is the *directrix* of this catenary.

The following additional simple relations are easily derived and are left to the reader:

$$s = a \sinh(x/a), \quad 18.3.5$$

$$y^2 = a^2 + s^2, \quad 18.3.6$$

$$y = a \sec \psi. \quad 18.3.7$$

$$x = a \ln(\sec \psi + \tan \psi), \quad 18.3.8$$

$$T = \mu g y. \quad 18.3.9$$

Equations 18.3.7 and 8 may be regarded as parametric equations to the catenary.

If one end of the chain is fixed, and the other is looped over a smooth peg, equation 18.3.9 shows that the loosely hanging vertical portion of the chain just reaches the directrix of the catenary, and the tension at the peg is equal to the weight of the vertical portion.

*Exercise:* By expanding equation 18.3.4 as far as  $x^2$ , show that, near the bottom of the catenary, or for a tightly stretched catenary with a small sag, the curve is approximately a parabola. Actually, it doesn't matter what equation 18.3.4 is – if you expand it as far as  $x^2$ , provided the  $x^2$  term is not zero, you'll get a parabola – so, in order not to let you off so lightly, show that the semi latus rectum of the parabola is  $a$ .

*Exercise:* Expand equation 18.3.5 as far as  $x^3$ .

Now: let  $2s$  = total length of chain,  $2k$  = total span, and  $d$  = sag. Show that for a shallow catenary  $s - k = k^3 / (6a^3)$  and  $k^2 = 2ad$ , and hence that length – span =  $\frac{8}{3} \times \text{sag}^2 / \text{span}$ .

*Question:* A cord is stretched between points on the same horizontal level. How large a force must be applied so that the cord is no longer a catenary, but is accurately a straight line?

*Answer:* There is no force however great  
Can stretch a string however fine  
Into a horizontal line  
That shall be accurately straight.

I am indebted to Hamilton Carter of Texas A & M University for drawing my attention to a note by C. A. Chant in *J. Roy. Astron. Soc. Canada* **33**, 72, (1939), where this doggerel is attributed to the early nineteenth century Cambridge mathematician William Whewell.

And here's something for engineers. We, the general public, expect engineers to built safe bridges for us. The suspension chain of a suspension bridge, though scarcely shallow, is closer to a parabola than to a catenary. There is a reason for this. Discuss.

#### 18.4 Area of a Catenoid

A theorem from the branch of mathematics known as the *calculus of variations* is as follows. Let  $y = y(x)$  with  $y' = dy/dx$ , and let  $f(y, y', x)$  be some function of  $y$ ,  $y'$  and  $x$ . Consider the line integral of  $f$  from A to B along the route  $y = y(x)$ .

$$\int_A^B f(y, y', x) dx \quad 18.4.1$$

In general, and unless  $f$  is a function of  $x$  and  $y$  alone, and not of  $y'$ , the value of this integral will depend on the *route* (i.e.  $y = y(x)$ ) over which this line integral is calculated. The theorem states that the integral is an extremum for a route that satisfies

$$\frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial y} \quad 18.4.2$$

By "extremum" we mean either a minimum or a maximum, or an inflection, though in many – perhaps most – cases of physical interest, it is a minimum. It can be difficult for a newcomer to this theorem to try to grasp exactly what this theorem means, so perhaps the best way to convey its meaning is to start by giving a simple example. Following that, I give an example involving the catenary. There will be another example, involving a famous problem in dynamics, in Chapter 19, and in fact we have already encountered an application of it in Chapter 14 in the discussion of Hamilton's variational principle.

Let us consider, for example, the problem of calculating the distance, measured along some route  $y(x)$  between two points; that is, we want to calculate the arc length  $\int ds$ . From the usual pythagorean relation between  $ds$ ,  $dx$  and  $dy$ , this is  $\int (1+y')^{1/2} dx$ . The variational principle says that this distance – measured along  $y(x)$  – is least for a route  $y(x)$  that satisfies equation 18.4.2, in which in this case  $f = (1 + y')^{1/2}$ .

For this case, we have  $\frac{df}{dy} = 0$  and  $\frac{df}{dy'} = \frac{y'}{(1 + y'^2)^{1/2}}$ . Thus integration of equation 18.4.2 gives

$$y' = c(1 + y'^2)^{1/2}, \quad 18.4.3$$

where  $c$  is the integration constant. If we solve this for  $y'$ , we obtain  $y' = \frac{c}{\sqrt{1 - c^2}}$ , which is just another constant, which I'll write as  $a$ , so that  $y' = a$ . Integrate this to find

$$y = ax + b. \quad 18.4.4$$

This probably seems rather a long way to prove that the shortest distance between two points is a straight line – but that wasn't the point of the exercise. The aim was merely to understand the meaning of the variational principle.

Let's try another example, in which the answer will not be so obvious.

Consider some curve  $y = y(x)$ , and let us rotate the curve through an angle  $\phi$  (which need not necessarily be a full  $2\pi$  radians) about the  $y$ -axis. An element  $ds$  of the curve can be written as

$\sqrt{1 + y'^2} dx$ , and the distance moved by the element  $ds$  (which is at a distance  $x$  from the  $y$ -axis) during the rotation is  $\phi x$ . Thus the area swept out by the curve is

$$A = \phi \int x \sqrt{1 + y'^2} dx. \quad 18.4.5$$

For what shape of curve,  $y = y(x)$ , is this area least? The answer is – a curve that satisfies equation 18.4.2, where  $f = x \sqrt{1 + y'^2}$ . For this function, we have  $\frac{\partial f}{\partial y} = 0$  and  $\frac{\partial f}{\partial y'} = \frac{xy'}{\sqrt{1 + y'^2}}$ .

Therefore the required curve satisfies

$$\frac{xy'}{\sqrt{1 + y'^2}} = a. \quad 18.4.6$$

That is,

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 - a^2}}. \quad 18.4.7$$

On substitution of  $x = a \cosh \theta$ , and looking up everything we have forgotten about hyperbolic functions, and integrating, we obtain

$$y = a \cosh(x/a). \quad 18.4.8$$

Thus the required curve is a catenary.

If a soap bubble is formed between two identical horizontal rings, one beneath the other, it will take up the shape of least area, namely a catenoid.