CHAPTER 3
PLANE AND SPHERICAL TRIGONOMETRY

3.1 *Introduction*

It is assumed in this chapter that readers are familiar with the usual elementary formulas encountered in introductory trigonometry. We start the chapter with a brief review of the solution of a plane triangle. While most of this will be familiar to readers, it is suggested that it be not skipped over entirely, because the examples in it contain some cautionary notes concerning hidden pitfalls.

This is followed by a quick review of spherical coordinates and direction cosines in three-dimensional geometry. The formulas for the velocity and acceleration components in two-dimensional polar coordinates and three-dimensional spherical coordinates are developed in section 3.4.

Section 3.5 deals with the trigonometric formulas for solving spherical triangles. This is a fairly long section, and it will be essential reading for those who are contemplating making a start on celestial mechanics.

Sections 3.6 and 3.7 deal with the rotation of axes in two and three dimensions, including Eulerian angles and the rotation matrix of direction cosines.

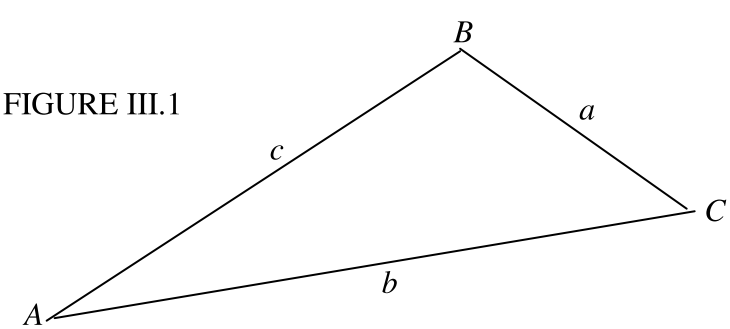
Finally, in section 3.8, a number of commonly encountered trigonometric formulas are gathered for reference.

3.2 *Plane Triangles.*

This section is to serve as a brief reminder of how to solve a plane triangle. While there may be a temptation to pass rapidly over this section, it does contain a warning that will become even more pertinent in the section on spherical triangles.

Conventionally, a plane triangle is described by its three angles *A, B, C* and three sides *a, b, c,* with *a* being opposite to *A, b* opposite to *B*, and *c* opposite to *C*. See figure III.1.

FIGURE III.1



It is assumed that the reader is familiar with the sine and cosine formulas for the solution of the triangle:

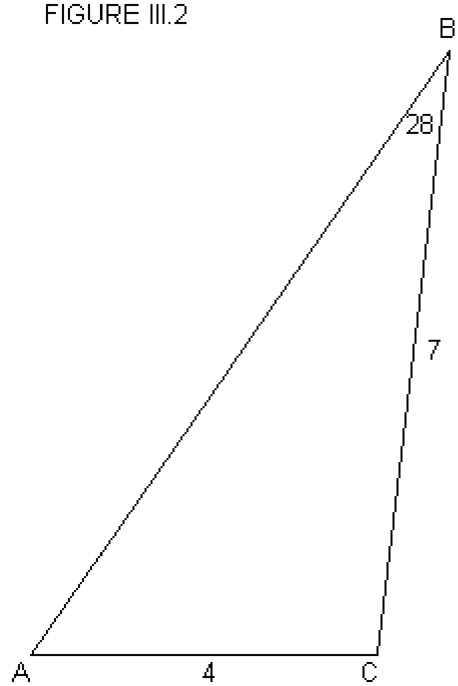
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad 3.2.1$$

and $$a^2 = b^2 + c^2 - 2bc\cos A, \qquad 3.2.2$$

and understands that the art of solving a triangle involves recognition as to which formula is appropriate under which circumstances. Two quick examples - each with a warning - will suffice.

Example: A plane triangle has sides a = 7 inches, b = 4 inches and angle B = 28°. Find the angle A.

FIGURE III.2



See figure III.2.

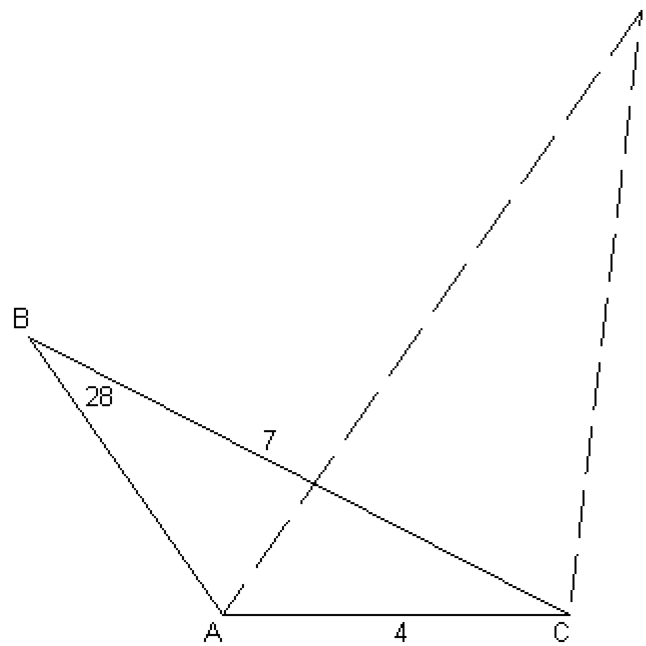
We use the sine formula, to obtain

$$\sin A = \frac{7\sin 28^\circ}{4} = 0.821575$$

$$A = 55^\circ\ 14'.6$$

The pitfall is that there are two values of A between 0° and 180° that satisfy $\sin A = 0.821575$, namely 55° 14'.6 and 124° 45'.4. Figure III.3 shows that, given the original data, either of these is a valid solution.

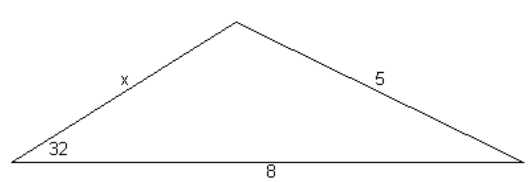
FIGURE III.3



The lesson to be learned from this is that all inverse trigonometric functions ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$) have two solutions between 0° and 360°. The function $\sin^{-1}$ is particularly troublesome since, for positive arguments, it has two solutions between 0° and 180°. The reader must always be on guard for "quadrant problems" (i.e. determining which quadrant the desired solution belongs to) and is warned that, unless particular care is taken in programming calculators or computers, quadrant problems are among the most frequent problems in trigonometry, and especially in spherical astronomy.

Example: Find x in the triangle illustrated in figure III.4.

FIGURE III.4



Application of the cosine rule results in

$$25 = x^2 + 64 - 16x \cos 32^\circ$$

Solution of the quadratic equation yields

$$x = 4.133 \text{ or } 9.435$$

This illustrates that the problem of "two solutions" is not confined to angles alone. Figure III.4 is drawn to scale for one of the solutions; the reader should draw the second solution to see how it is that two solutions are possible.

The reader is now invited to try the following "guaranteed all different" problems by hand calculator. Some may have two real solutions. Some may have none. The reader should draw the triangles accurately, especially those that have two solutions or no solutions. It is important to develop a clear geometric understanding of trigonometric problems, and not merely to rely on the automatic calculations of a machine. Developing these critical skills now will pay dividends in the more complex real problems encountered in celestial mechanics and orbital computation.

PROBLEMS

1. $a = 6 \quad b = 4 \quad c = 7 \quad C = ?$

2. $a = 5 \quad b = 3 \quad C = 43^\circ \quad c = ?$

3. $a = 7 \quad b = 9 \quad C = 110^\circ \quad B = ?$

4. $a = 4 \quad b = 5 \quad A = 29^\circ \quad c = ?$

5. $a = 5 \quad b = 7 \quad A = 37^\circ \quad B = ?$

6. $a = 8 \quad b = 5 \quad A = 54^\circ \quad C = ?$

7. $A = 64^\circ \quad B = 37^\circ \quad a/c = ? \quad b/c = ?$

8. $a = 3 \quad b = 8 \quad c = 4 \quad C = ?$

9. $a = 4 \quad b = 11 \quad A = 26^\circ \quad c = ?$

The reader is now further invited to write a computer program (in whatever language is most familiar) for solving each of the above problems for arbitrary values of the data. Lengths should be read in input and printed in output to four significant figures. Angles should be read in input and printed in output in degrees, minutes and tenths of a minute (e.g. 47° 12'.9). Output should show two solutions if there are two, and should print "NO SOLUTION" if there are none. This exercise will familiarize the reader with the manipulation of angles, especially inverse trigonometric functions in whatever computing language is used, and will be rewarded in future more advanced applications.

Solutions to problems.

1. $C = 86^\circ\ 25'.0$
2. $c = 3.473$
3. $B = 40^\circ\ 00'.1$
4. $c = 7.555$ or 1.191
5. $B = 57^\circ\ 24'.6$ or $122^\circ\ 35'.4$
6. $C = 95^\circ\ 37'.6$
7. $a/c = 0.9165$ $b/c = 0.6131$
8. No real solution
9. No real solution

The area of a plane triangle is $\frac{1}{2}\times$ base $\times$ height, and it is easy to see from this that

$$\text{Area} \;= \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ca\sin B = \tfrac{1}{2}ab\sin C \qquad 3.2.3$$

By making use $\sin^2 A = 1 - \cos^2 A$ and $\cos A = \left(b^2 + c^2 - a^2\right)/(2bc),$ we can express this entirely in terms of the lengths of the sides:

$$\text{Area} \;= \sqrt{s(s-a)(s-b)(s-c)}, \qquad 3.2.4$$

where s is the semi-perimeter $\frac{1}{2}(a + b + c)$.

3.3 *Cylindrical and Spherical Coordinates*

It is assumed that the reader is at least somewhat familiar with cylindrical coordinates (ρ, ϕ, z) and spherical coordinates (r, θ, ϕ) in three dimensions, and I offer only a brief summary here. Figure III.5 illustrates the following relations between them and the rectangular coordinates (x, y, z).

$$x = \rho\cos\phi = r\sin\theta\cos\phi \qquad 3.3.1$$

$$y = \rho\sin\phi = r\sin\theta\sin\phi \qquad 3.3.2$$

$$z = r\cos\theta \qquad 3.3.3$$

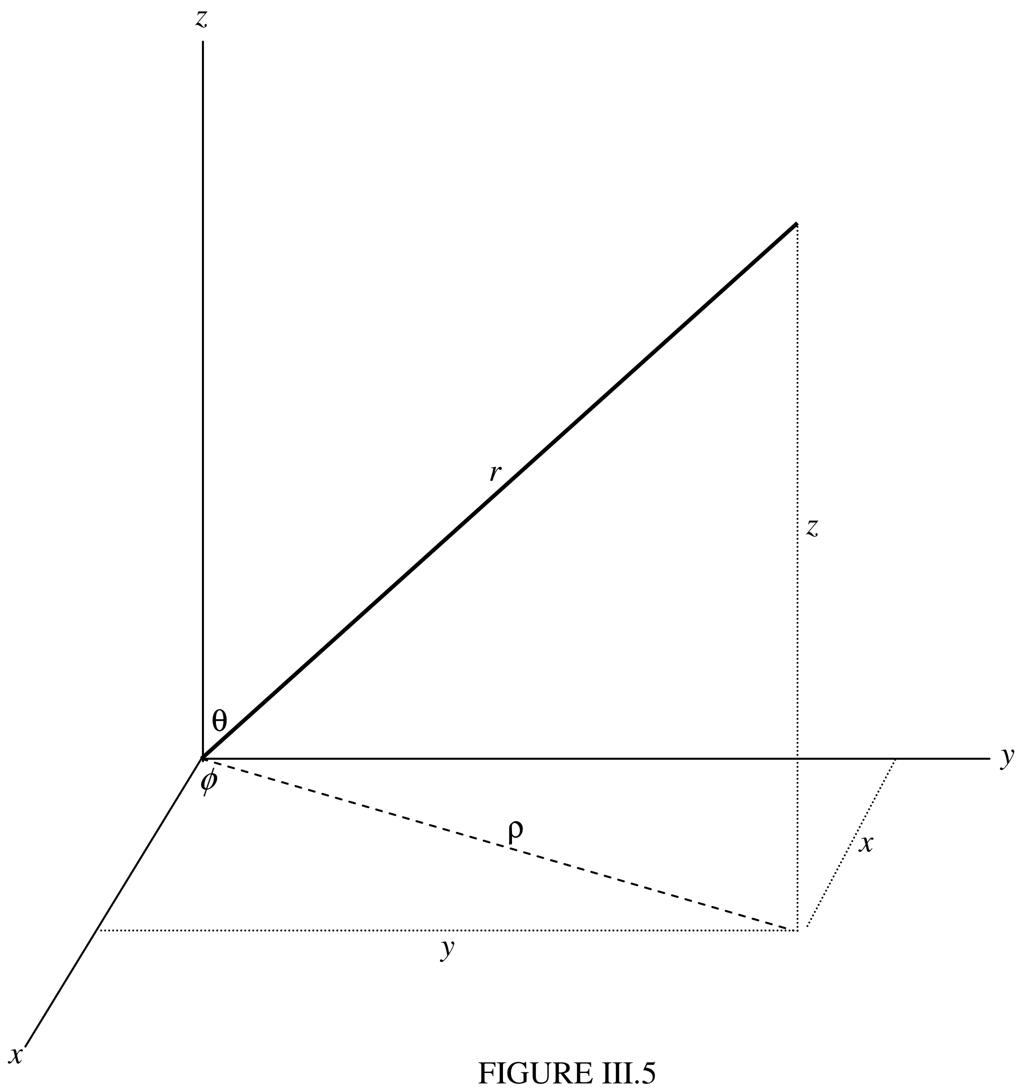


FIGURE III.5

The inverse relations between spherical and rectangular coordinates are

$$r = \sqrt{x^2 + y^2 + z^2} \tag{3.3.4}$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \tag{3.3.5}$$

$$\phi = \cos^{-1} \frac{x}{\sqrt{x^2 + y^2}} = \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}} \tag{3.3.6}$$

The coordinates r, θ and ϕ are called, respectively, the "radial", "polar" or "meridional", and "azimuthal" coordinates respectively.

Note that r is essentially positive (the symbol $\sqrt{\ }$ denotes the positive or absolute value of the square root). The angle θ is necessarily between 0° and 180° and therefore there is no quadrant ambiguity in the evaluation of θ. The angle ϕ, however, can be between 0° and 360°. Therefore, in order to determine ϕ uniquely, both of the above formulas for ϕ must be evaluated, or the signs of x and y must be inspected. It does not suffice to calculate ϕ from $\phi = \tan^{-1}(y/x)$ alone. The reader, however, should be aware that some computer languages and some hand calculator functions will inspect the signs of x and y for you and will return ϕ in its correct quadrant. For example, in FORTRAN, the function ATAN2(X,Y) (or DATAN2(X,Y) in double precision) will return ϕ uniquely in its correct quadrant (though perhaps as a negative angle, in which case 360° should be added to the outputted angle) provided the arguments X and Y are inputted with their correct signs. This can save an immense amount of trouble in programming, and the reader should become familiar with this function.

Direction cosines

The direction to a point in three dimensional space relative to the origin can be described, as we have seen, by the two angles θ and ϕ. Another way of describing the direction to a point, or the orientation of a vector, is to give the angles α, β, γ that the vector makes with the x-, y- and z-axes, respectively (see figure III.5). The angle γ is the same as the angle θ.

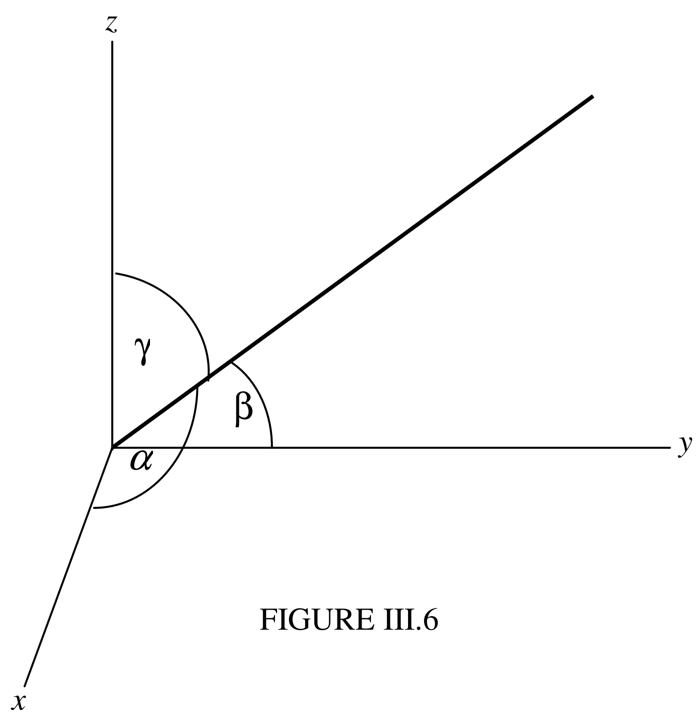


FIGURE III.6

More commonly one quotes the cosines of these three angles. These are called the *direction cosines*, and are often denoted by (*l,m,n*). It should not take long for the reader to be convinced that the relation between the direction cosines and the angles θ and φ are

$$l = \cos\alpha = \sin\theta\cos\phi \qquad 3.3.7$$

$$m = \cos\beta = \sin\theta\sin\phi \qquad 3.3.8$$

$$n = \cos\gamma = \cos\theta \qquad 3.3.9$$

These are not independent, and are related by

$$l^2 + m^2 + n^2 = 1. \qquad 3.3.10$$

A set of numbers that are multiples of the direction cosines - i.e. are proportional to them - are called *direction ratios*.

Latitude and Longitude.

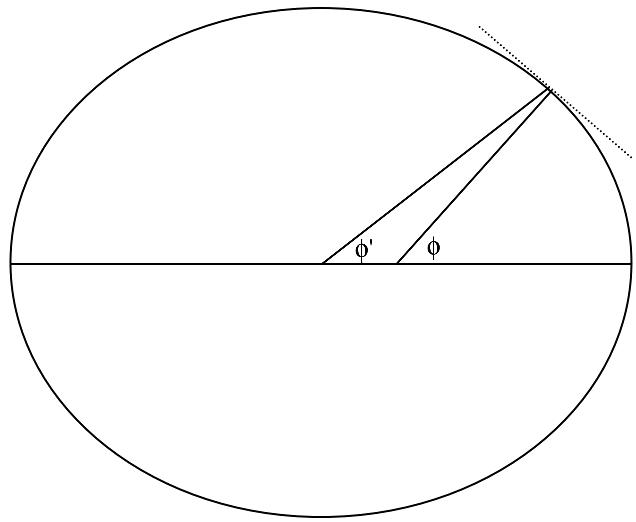
The figure of the Earth is not perfectly spherical, for it is slightly flattened at the poles. For the present, however, our aim is to become familiar with spherical coordinates and with the geometry of the sphere, so we shall suppose the Earth to be spherical. In that case, the position of any town on Earth can be expressed by two coordinates, the latitude ϕ, measured north or south of the equator, and the longitude λ, measured eastwards or westwards from the meridian through Greenwich. These symbols, ϕ for latitude and λ for longitude, are unfortunate, but are often used in this context. In terms of the symbols θ , ϕ for spherical coordinates that we have used hitherto, the east longitude would correspond to ϕ and the latitude to $90^\circ - \theta$.

A plane that intersects a sphere does so in a circle. If that plane passes through the centre of the sphere (so that the centre of the circle is also the centre of the sphere), the circle is called a *great circle*. All the meridians (the circles of fixed longitude that pass through the north and south poles) including the one that passes through Greenwich, are great circles, and so is the equator. Planes that do not pass through the centre of the sphere (such as parallels of latitude) are *small circles*. The radius of a parallel of latitude is equal to the radius of the sphere times the cosine of the latitude.

We have used the example of latitude and longitude on a spherical Earth in order to illustrate the concepts of great and small circles. Although it is not essential to pursue it in the present context, we mention in passing that the true figure of the Earth at mean sea level is a *geoid* - which merely means the shape of the Earth. To a good approximation, the geoid is an oblate spheroid (i.e. an ellipse rotated about its minor axis) with semi major axis a = 6378.140 km and semi minor axis c = 6356.755 km. The ratio $(a-c)/a$ is called the *geometric ellipticity* of the Earth and it has the value 1/298.3. The mean radius of the Earth, in the sense of the radius of a sphere having the same volume as the actual geoid, is $\sqrt[3]{a^2c}$ = 6371.00 km.

It is necessary in precise geodesy to distinguish between the geographic or geodetic latitude ϕ of a point on the Earth's surface and its geocentric latitude ϕ' Their definitions evident from figure III.7. In this figure, the ellipticity of the Earth is greatly exaggerated; in reality it would scarcely be discernible. The angle ϕ is the angle between a plumb-bob and the equator. This differs from ϕ' partly because the gravitational field of a spheroid is not the same as that of an equal point mass at the centre, and partly because the plumb bob is pulled away from the Earth's rotation axis by centrifugal force.

FIGURE III.7



The relationship between ϕ and ϕ' is

$$\phi - \phi' = 692'.74 \sin 2\phi - 1'.16 \sin 4\phi.$$

3.4 *Velocity and Acceleration Components.*

i. Two-dimensional polar coordinates

Sometimes the symbols r and θ are used for two-dimensional polar coordinates, but in this section I use (ρ , ϕ) for consistency with the (r, θ, ϕ) of three-dimensional spherical coordinates. In what follows I am setting vectors in **boldface**. If you make a print-out, you should be aware that some printers apparently do not print Greek letter symbols in boldface, even though they appear in boldface on screen. You should be on the look-out for this. Symbols with ^ above them are intended as *unit* vectors, so you will know that they should be in boldface even if your printer does not recognize this. If in doubt, look at what appears on the screen.

FIGURE III.8

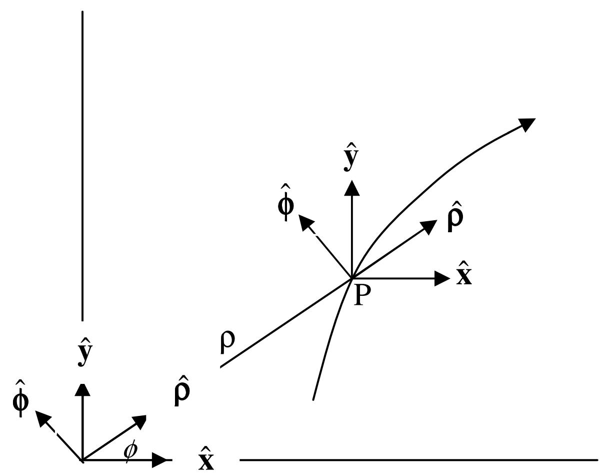


Figure III.8 shows a point P moving along a curve such that its polar coordinates are changing at rates $\dot{\rho}$ and $\dot{\phi}$. The drawing also shows fixed unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ parallel to the x- and y-axes, as well as unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ in the radial and transverse directions. We shall find expressions for the rate at which the unit radial and transverse vectors are changing with time. (Being unit vectors, their magnitudes do not change, but their directions do.)

We have
$$\hat{\boldsymbol{\rho}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}} \qquad 3.4.1$$

and
$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}. \qquad 3.4.2$$

$\therefore$
$$\dot{\hat{\boldsymbol{\rho}}} = -\sin\phi\,\dot{\phi}\,\hat{\mathbf{x}} + \cos\phi\,\dot{\phi}\,\hat{\mathbf{y}} = \dot{\phi}\left(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}\right) \qquad 3.4.3$$

$\therefore$
$$\dot{\hat{\boldsymbol{\rho}}} = \dot{\phi}\hat{\boldsymbol{\phi}}. \qquad 3.4.4$$

In a similar manner, by differentiating equation 3.4.2. with respect to time and then making use of equation 3.4.1, we find

$$\dot{\hat{\boldsymbol{\phi}}} = -\dot{\phi}\,\hat{\boldsymbol{\rho}} \qquad 3.4.5$$

Equations 3.4.4 and 3.4.5 give the rate of change of the radial and transverse unit vectors. It is worthwhile to think carefully about what these two equations mean.

The position vector of the point P can be represented by the expression $\boldsymbol{\rho} = \rho\hat{\boldsymbol{\rho}}$. The velocity of P is found by differentiating this with respect to time:

$$\mathbf{v} = \dot{\boldsymbol{\rho}} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\,\dot{\hat{\boldsymbol{\rho}}} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\,\hat{\boldsymbol{\phi}}. \qquad 3.4.6$$

The radial and transverse components of velocity are therefore $\dot{\rho}$ and $\rho\,\dot{\phi}$ respectively.

The acceleration is found by differentiation of equation 3.4.6, and we have to differentiate the products of two and of three quantities that vary with time:

$$\begin{aligned} \mathbf{a} = \dot{\mathbf{v}} &= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\dot{\hat{\boldsymbol{\rho}}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} + \rho\dot{\phi}\dot{\hat{\boldsymbol{\phi}}} \\ &= \ddot{\rho}\hat{\boldsymbol{\rho}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{\rho}\dot{\phi}\hat{\boldsymbol{\phi}} + \rho\ddot{\phi}\hat{\boldsymbol{\phi}} - \rho\dot{\phi}^2\hat{\boldsymbol{\rho}} \\ &= \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\boldsymbol{\phi}}. \end{aligned} \qquad 3.4.7$$

The radial and transverse components of acceleration are therefore $(\ddot{\rho} - \rho\dot{\phi}^2)$ and $(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})$ respectively.

ii. Three-dimensional spherical coordinates

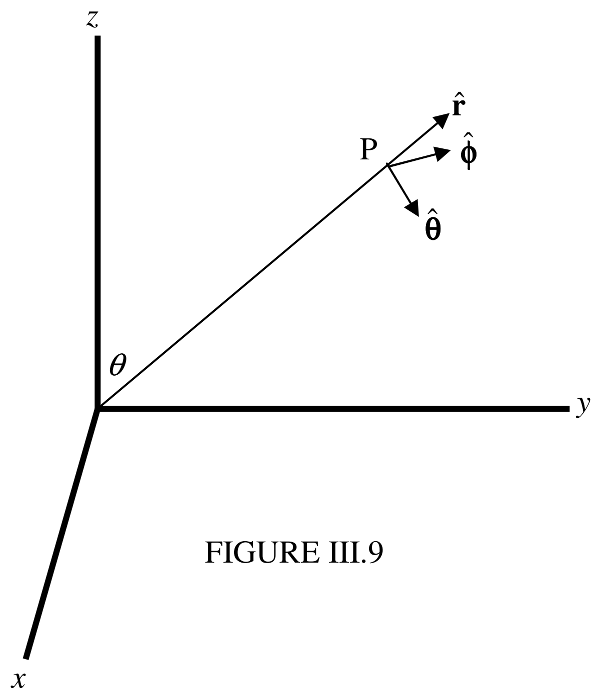


FIGURE III.9

In figure III.9, P is a point moving along a curve such that its spherical coordinates are changing at rates $\dot{r}$, $\dot{\theta}$, $\dot{\phi}$. We want to find out how fast the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ in the radial, meridional and azimuthal directions are changing.

We have
$$\hat{\mathbf{r}} = \sin\theta \cos\phi\, \hat{\mathbf{x}} + \sin\theta \sin\phi\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}} \tag{3.4.8}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta \cos\phi\, \hat{\mathbf{x}} + \cos\theta \sin\phi\, \hat{\mathbf{y}} - \sin\theta\, \hat{\mathbf{z}} \tag{3.4.9}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\, \hat{\mathbf{x}} + \cos\phi\, \hat{\mathbf{y}} \tag{3.4.10}$$

$$\therefore \quad \dot{\hat{\mathbf{r}}} = (\cos\theta\,\dot{\theta}\cos\phi - \sin\theta\sin\phi\,\dot{\phi})\hat{\mathbf{x}} + (\cos\theta\,\dot{\theta}\sin\phi + \sin\theta\cos\phi\,\dot{\phi})\hat{\mathbf{y}} - \sin\theta\,\dot{\theta}\,\hat{\mathbf{z}}. \tag{3.4.11}$$

We see, by comparing this with equations 3.4.9 and 3.4.10 that

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\,\hat{\boldsymbol{\theta}} + \sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}} \tag{3.4.12}$$

By similar arguments we find that

$$\dot{\hat{\boldsymbol{\theta}}} = \cos\theta\ \dot{\phi}\ \hat{\boldsymbol{\phi}} - \dot{\theta}\hat{\mathbf{r}} \qquad 3.4.13$$

and $$\dot{\hat{\boldsymbol{\phi}}} = -\sin\theta\ \dot{\phi}\hat{\mathbf{r}} - \cos\theta\ \dot{\phi}\ \hat{\boldsymbol{\theta}} \qquad 3.4.14$$

These are the rates of change of the unit radial, meridional and azimuthal vectors.

The position vector of the point P can be represented by the expression $\mathbf{r} = r\ \hat{\mathbf{r}}$. The velocity of P is found by differentiating this with respect to time:

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\,\dot{\hat{\mathbf{r}}} = \dot{r}\,\hat{\mathbf{r}} + r\,(\dot{\theta}\,\hat{\boldsymbol{\theta}} + \sin\theta\ \dot{\phi}\ \hat{\boldsymbol{\phi}})$$

$$= \dot{r}\,\hat{\mathbf{r}} + r\ \dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\ \dot{\phi}\ \hat{\boldsymbol{\phi}} \qquad 3.4.15$$

The radial, meridional and azimuthal components of velocity are therefore $\dot{r}$, $r\dot{\theta}$, and $r\sin\theta\ \dot{\phi}$ respectively.

The acceleration is found by differentiation of equation 3.4.15. [It might not be out of place here for a quick hint about differentiation. Most readers will know how to differentiate a product of two functions. If you want to differentiate a product of several functions, for example four functions, *a, b*, *c* and *d*, the procedure is $(abcd)' = a'bcd + ab'cd + abc'd + abcd'$. In the last term of equation 3.4.15, all four quantities vary with time, and we are about to differentiate the product.]

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r}\,\hat{\mathbf{r}} + \dot{r}(\dot{\theta}\,\hat{\boldsymbol{\theta}} + \sin\theta\ \dot{\phi}\,\hat{\boldsymbol{\phi}}) + \dot{r}\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\ddot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\theta}(\cos\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}} - \dot{\theta}\hat{\mathbf{r}})$$

$$+ \dot{r}\sin\theta\dot{\phi}\ \hat{\boldsymbol{\phi}} + r\ \cos\theta\ \dot{\theta}\dot{\phi}\,\hat{\boldsymbol{\phi}} + r\sin\theta\ \ddot{\phi}\,\hat{\boldsymbol{\phi}} + r\sin\theta\ \dot{\phi}(-\sin\theta\,\dot{\phi}\,\hat{\mathbf{r}} - \cos\theta\,\dot{\phi}\ \hat{\boldsymbol{\theta}}) \qquad 3.4.16$$

On gathering together the coefficients of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$, we find that the components of acceleration are:

Radial: $\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2$

Meridional: $r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2$

Azimuthal: $2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\,\ddot{\phi}$

3.5 *Spherical Triangles.*

As with plane triangles, we denote the three angles by *A, B, C* and the sides opposite to them by *a, b, c*. We are fortunate in that we have four formulas at our disposal for the solution of a spherical triangle, and, as with plane triangles, the art of solving a spherical triangle entails understanding which formula is appropriate under given circumstances. Each formula contains four elements (sides and angles), three of which, in a given problem, are assumed to be known, and the fourth is to be determined.

Three important points are to be noted before we write down the formulas.

1. The formulas are valid only for triangles in which the three sides are arcs of great circles. They will not do, for example, for a triangle in which one side is a parallel of latitude.

2. The sides of a spherical triangle, as well as the angles, are all expressed in angular measure (degrees and minutes) and not in linear measure (metres or kilometres). A side of 50° means that the side is an arc of a great circle subtending an angle of 50° at the centre of the sphere.

3. The sum of the three angles of a spherical triangle add up to more than 180° .

In this section are now given the four formulas without proof, the derivations being given in a later section. The four formulas may be referred to as the sine formula, the cosine formula, the polar cosine formula, and the cotangent formula. Beneath each formula is shown a spherical triangle in which the four elements contained in the formula are highlighted.

The sine formula:
$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} \left(= \frac{\sin c}{\sin C} \right) \qquad 3.5.1$$

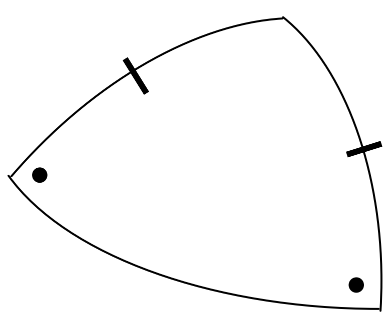


FIGURE III.10

The cosine formula: $\cos a = \cos b \cos c + \sin b \sin c \cos A$ 3.5.2

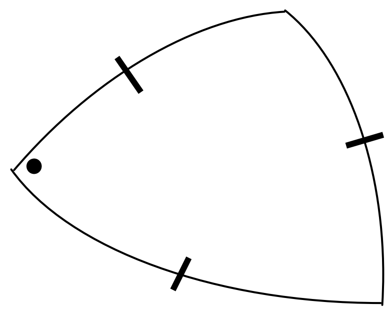


FIGURE III.11

The polar cosine formula:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \qquad 3.5.3$$

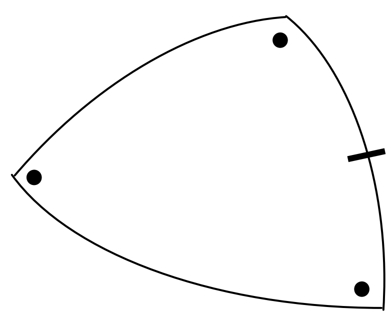


FIGURE III.12

The cotangent formula:

$$\cos b \cos A = \sin b \cot c - \sin A \cot C \qquad 3.5.4$$

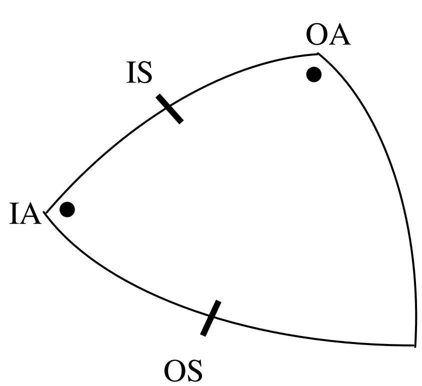


FIGURE III.13

The cotangent formula is a particularly useful and frequently needed formula, and it is unfortunate that it is not only difficult to commit to memory but, even with the formula written out in front of one, it is often difficult to decide which is *b*, which is *A* and so on. However, it should be noted from the drawing that the four elements, side-angle-side-angle, lie adjacent to each other in the triangle, and they may be referred to as outer side (OS), inner angle (IA), inner side (IS) and outer angle (OA) respectively. Many people find that the formula is much easier to use when written in the form

$$\cos(\mathrm{IS})\cos(\mathrm{IA}) = \sin(\mathrm{IS})\cot(\mathrm{OS}) - \sin(\mathrm{IA})\cot(\mathrm{OA}) \qquad 3.5.5$$

The reader will shortly be offered a goodly number of examples in the use of these formulas. However, during the course of using the formulas, it will be found that there is frequent need to solve deceptively simple trigonometric equations of the type

$$4.737\sin\theta + 3.286\cos\theta = 5.296 \qquad 3.5.6$$

After perhaps a brief pause, one of several methods may present themselves to the reader - but not all methods are equally satisfactory. I am going to suggest four possible ways of solving this equation. The first method is one that may occur very quickly to the reader as being perhaps rather obvious - but there is a cautionary tale attached to it. While the method may seem very obvious, a difficulty does arise, and the reader would be advised to prefer one of the less obvious methods. There are, incidentally, two solutions to the equation between 0° and 360°. They are 31° 58'.6 and 78° 31'.5.

Method i.

The obvious method is to isolate $\cos\theta$:

$$\cos\theta = 1.611\,686 - 1.441\,570\sin\theta.$$

Although the constants in the problem were given to four significant figures, do not be tempted to round off intermediate calculations to four. It is a common fault to round off intermediate calculations prematurely. The rounding-off can be done at the end.

Square both sides, and write the left hand side, $\cos^2\theta$, as $1 - \sin^2\theta$. We now have a quadratic equation in $\sin\theta$:

$$3.078\,125\sin^2\theta - 4.646\,717\sin\theta + 1.597\,532 = 0.$$

The two solutions for $\sin\theta$ are 0.529 579 and 0.908 014 and the four values of θ that satisfy these values of $\sin\theta$ are 31° 58'.6, 148° 01'.4, 78° 31'.5 and 101° 28'.5.

Only two of these angles are solutions of the original equation. The fatal move was to square both sides of the original equation, so that we have found solutions not only to

$$\cos\theta = 1.611\,686 - 1.441\,570\sin\theta$$

but also to the different equation

$$-\cos\theta = 1.611\,686 - 1.441\,570\sin\theta.$$

This generation of extra solutions always occurs whenever we square an equation. For this reason, method (i), however tempting, should be avoided, particularly when programming a computer to carry out a computation automatically and uncritically.

If in doubt whether you have obtained a correct solution, substitute your solution in the original equation. You should always do this with any equation of any sort, anyway.

Method ii.

This method makes use of the identities

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2},$$

where $t = \tan\frac{1}{2}\theta$.

When applied to the original equation, this results in the quadratic equation in t:

$$8.582t^2 - 9.474t + 2.010 = 0$$

with solutions $t = 0.286528$ and $t = 0.817410$.

The only values of θ between 0° and 360° that satisfy these are the two correct solutions $31^\circ\ 58'.6$ and $78^\circ\ 31'.5$.

It is left as an exercise to show, using this method algebraically, that the solutions to the equation

$$a\sin\theta + b\cos\theta = c$$

are given by

$$\tan\tfrac{1}{2}\theta = \frac{a \pm \sqrt{a^2+b^2-c^2}}{b+c}.$$

This shows that there are no real solutions if $a^2 + b^2 < c^2$, one real solution if $a^2 + b^2 = c^2$, and two real solutions if $a^2 + b^2 > c^2$.

Method iii.

We divide the original equation

$$4.737 \sin\theta + 3.286 \cos\theta = 5.296$$

by the "hypotenuse" of 4.737 and 3.286; that is, by $\sqrt{(4.737^2 + 3.286^2)} = 5.765151$.

Thus $$0.821\,661 \sin\theta \quad + \quad 0.569\,976 \cos\theta \quad = \quad 0.918\,623$$

Now let 0.821 661 = cos α and 0.569 976 = sin α (which we can, since these numbers now satisfy $\sin^2\alpha + \cos^2\alpha = 1$) so that $\alpha = 34^\circ\ 44'.91$.

We have $$\cos\alpha \sin\theta \quad + \quad \sin\alpha \cos\theta \quad = \quad 0.918\,623$$

or $$\sin(\theta + \alpha) = 0.918623$$

from which $$\theta + \alpha \quad = \quad 66^\circ\ 43'.54 \quad \text{or} \quad 113^\circ\ 16'.46$$

Therefore $$\theta \quad = \quad 31^\circ\ 58'.6 \quad \text{or} \quad 78^\circ\ 31'.5$$

Method iv.

Methods ii and iii give explicit solutions, so there is perhaps no need to use numerical methods. Nevertheless, the reader might like to solve, by Newton-Raphson iteration, the equation

$$f(\theta) \quad = \quad a \sin\theta + b \cos\theta - c \quad = \quad 0,$$

for which $$f'(\theta) = a \cos\theta - b \sin\theta.$$

Using the values of *a, b* and *c* from the example above and using the Newton-Raphson algorithm, we find with a first guess of 45° the following iterations, working in radians:

0.785 398
0.417 841
0.541 499
0.557 797
0.558 104
0.558 104 = $31^\circ 58'.6$

The reader should verify this calculation, and, using a different first guess, show that Newton-Raphson iteration quickly leads to $78^\circ\ 31'.5$.

Having now cleared that small hurdle, the reader is invited to solve the spherical triangle problems below. Although these twelve problems look like pointless repetitive work, they are in fact all different. Some have two solutions between 0° and 360° ; others have just one. After solving each problem, the reader should sketch each triangle - especially those that have two solutions - in order to see how the two-fold ambiguities arise. The reader should also write a computer program that will solve all twelve types of problem at the bidding of the user. Answers should be given in degrees, minutes and tenths of a minute, and should be correct to that precision. For example, the answer to one of the problems is 47° 37'.3. An answer of 47° 37'.2 or 47° 37'.4 should be regarded as wrong. In celestial mechanics, there is no place for answers that are "nearly right". An answer is either right or it is wrong. (This does not mean, of course, that an angle can be measured with no error at all; but the answer to a calculation given to a tenth of an arcminute should be correct to a tenth of an arcminute.)

PROBLEMS

(All angles and sides in degrees.)

10. $a = 64$ $b = 33$ $c = 37$ $C = ?$
11. $a = 39$ $b = 48$ $C = 74$ $c = ?$
12. $a = 16$ $b = 37$ $C = 42$ $B = ?$
13. $a = 21$ $b = 43$ $A = 29$ $c = ?$
14. $a = 67$ $b = 54$ $A = 39$ $B = ?$
15. $a = 49$ $b = 59$ $A = 14$ $C = ?$
16. $A = 24$ $B = 72$ $c = 19$ $a = ?$
17. $A = 79$ $B = 84$ $c = 12$ $C = ?$
18. $A = 62$ $B = 49$ $a = 44$ $b = ?$
19. $\mathrm{A} = 59$ $B = 32$ $a = 62$ $c = ?$
20. $A = 47$ $B = 57$ $a = 22$ $C = ?$
21. $A = 79$ $B = 62$ $C = 48$ $\mathrm{c} = ?$

Solutions to problems.

10. 28° 18'.2
11. 49° 32'.4
12. 117° 31'.0
13. 30° 46'.7 or 47° 37'.3
14. 33° 34'.8
15. 3° 18'.1 or 162° 03'.9
16. 7° 38'.2
17. 20°46'.6
18. 36° 25'.5
19. 76° 27'.7
20. 80° 55'.7 or 169° 05'.2
21. 28° 54'.6

Derivation of the formulas.

Before moving on to further problems and applications of the formulas, it is time to derive the four formulas which, until now, have just been given without proof. We start with the cosine formula. There is no loss of generality in choosing rectangular axes such that the point A of the spherical triangle ABC is on the *z*-axis and the point B and hence the side *c* are in the *zx*-plane. The sphere is assumed to be of unit radius.

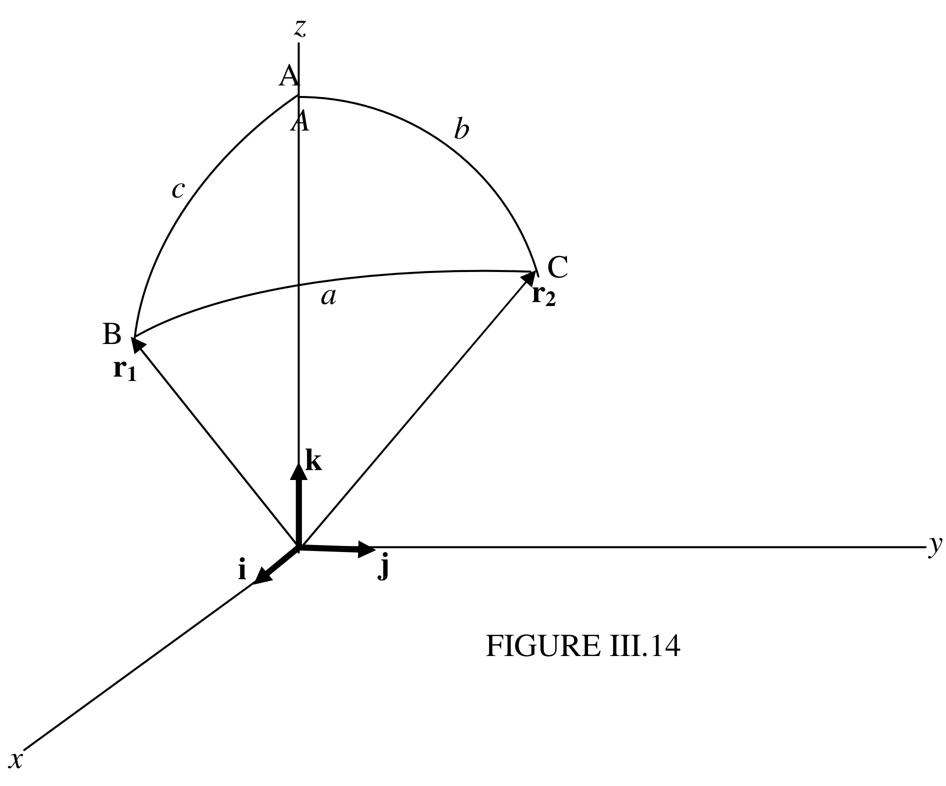


FIGURE III.14

If $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors directed along the x–, y– and z–axes respectively, inspection of the figure will show that the position vectors of the points B and C with respect to the centre of the sphere are

$$\mathbf{r}_1 = \mathbf{i} \sin c + \mathbf{k} \cos c \qquad 3.5.7$$

and

$$\mathbf{r}_2 = \mathbf{i} \sin b \cos A + \mathbf{j} \sin b \sin A + \mathbf{k} \cos b \qquad 3.5.8$$

respectively.

The scalar product of these vectors (each of magnitude unity) is just the cosine of the angle between them, namely cos a, from which we obtain immediately

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad 3.5.9$$

To obtain the sine formula, we isolate cos A from this equation, square both sides, and write $1 - \sin^2 A$ for $\cos^2 A$. Thus,

$$(\sin b \sin c \cos A)^2 = (\cos a - \cos b \cos c)^2, \qquad 3.5.10$$

and when we have carried out these operations we obtain

$$\sin^2 A = \frac{\sin^2 b \sin^2 c - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}. \qquad 3.5.11$$

In the numerator, write $1 - \cos^2 b$ for $\sin^2 b$ and $1 - \cos^2 c$ for $\sin^2 c$, and divide both sides by $\sin^2 a$. This results in

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}.. \qquad 3.5.12$$

At this stage the reader may feel that we are becoming bogged down in heavier and heavier algebra and getting nowhere. But, after a careful look at equation 3.5.12, it may be noted with some delight that the next line is:

Therefore

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \qquad 3.5.13$$

The derivation of the polar cosine formula may also bring a small moment of delight. In figure III.15, A'B'C' is a spherical triangle. ABC is also a spherical triangle, called the polar triangle to A'B'C'. It is formed in the following way. The side BC is an arc of a great circle 90° from A'; that is, BC is part of the equator of which A' is pole. Likewise CA is 90° from B' and AB is 90°

from C'. In the drawing, the side B'C' of the small triangle has been extended to meet the sides AB and CA of the large triangle. It will be evident from the drawing that the angle *A* of the large

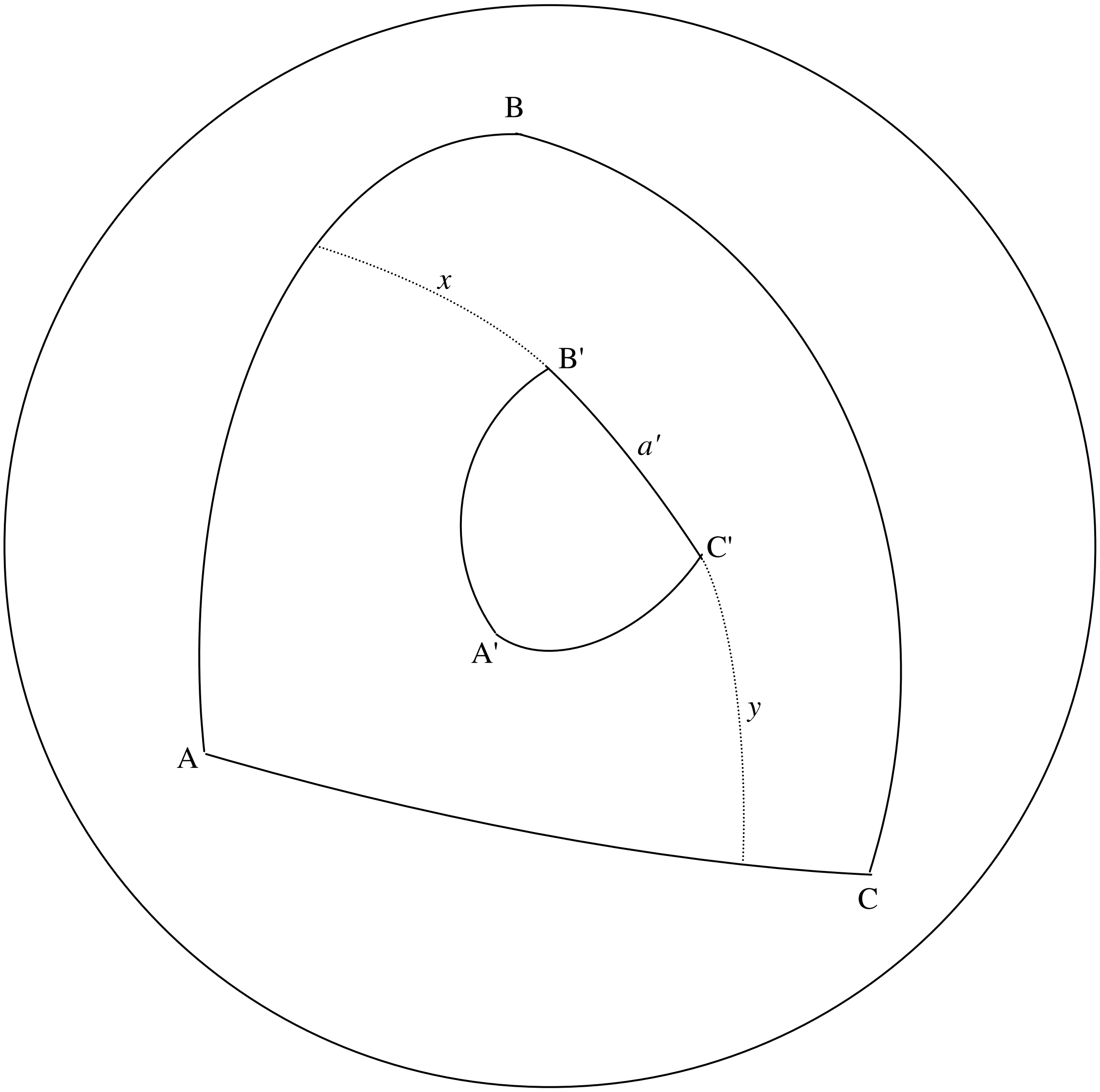


FIGURE III.15

triangle is equal to $x + a' + y$. Further, from the way in which the triangle ABC was formed, $x + a'$ and $a' + y$ are each equal to 90° . From these relations, we see that

$$A + A = [(x + a') + y] + [x + (a' + y)]$$

or $$2A = 180^\circ + x + y = 180^\circ + A - a'$$

Therefore $$A = 180^\circ - a'$$

In a similar manner, $$B = 180^\circ - b' \text{ and } C = 180^\circ - c'$$

Now, suppose $f(A', B', C', a', b', c') = 0$ is any relation between the sides and angles of the triangle A'B'C'. We may replace a' by $180^\circ - A$, b' by $180^\circ - B$, and so on, and this will result in a relation between A, B, C, a, b and c ; that is, it will result in a relation between the sides and angles of the triangle ABC.

For example, the equation

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A' \qquad 3.5.14$$

is valid for the triangle A'B'C'. By making these substitutions, we find the following formula valid for triangle ABC:

$$-\cos A = \cos B \cos C - \sin B \sin C \cos a, \qquad 3.5.15$$

which is the polar cosine formula.

The reader will doubtless like to try starting from the sine and cotangent formulas for the triangle A'B'C' and deduce corresponding polar formulas for the triangle ABC, though this, unfortunately, may give rise to some anticlimactic disappointment.

I know of no particularly interesting derivation of the cotangent formula, and I leave it to the reader to work through the rather pedestrian algebra. Start from

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

and $$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

Eliminate $\cos c$ (but retain $\sin c$) from these equations, and write $1 - \sin^2 b$ for $\cos^2 b$. Finally substitute $\dfrac{\sin c \sin A}{\sin C}$ for $\sin a$, and, after some tidying up, the cotangent formula should result.

Examples.

At this stage, we have had some practice in solving the four spherical triangle formulas, and we have derived them. In this section we encounter examples in which the problem is not merely to solve a triangle, but to gain some experience in setting up a problem and deciding which triangle has to be solved.

1. The coordinates of the Dominion Astrophysical Observatory, near Victoria, British Columbia, are

Latitude 48° 31'.3 N   Longitude 123° 25'.0 W

and the coordinates of the David Dunlap Observatory, near Toronto, Ontario, are

Latitude 43° 51'.8 N   Longitude 79° 25'.3 W

How far is Toronto from Victoria, and what is the azimuth of Toronto relative to Victoria?

The triangle to be drawn and solved is the triangle PVT, where P is the Earth's north pole, V is Victoria, and T is Toronto. On figure III.16 are marked the colatitudes of the two cities and the difference between their longitudes.

The great circle distance ω between the two observatories is easily given by the cosine formula:

$$\cos \omega = \cos 41^\circ\, 28'.7 \cos 46^\circ\, 08'.2 + \sin 41^\circ\, 28'.7 \sin 46^\circ\, 08'.2 \cos 43^\circ\, 59'.7$$

From this, we find $\omega = 30^0\, 22'.7$ or 0.53021 radians. The radius of the Earth is 6371 km, so the distance between the observatories is 3378 km or 2099 miles.

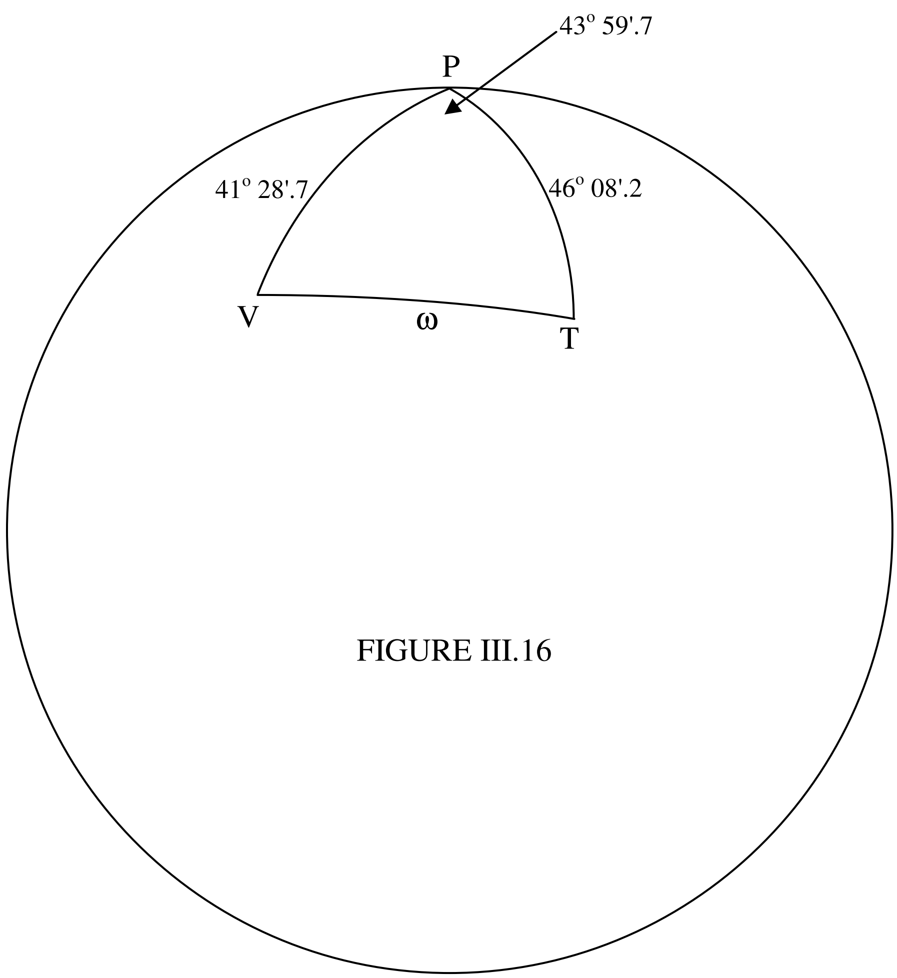


FIGURE III.16

Now that we have found ω, we can find the azimuth, which is the angle V, from the sine formula:

$$\sin V \quad = \quad \frac{\sin 46^o 08'.2 \sin 43^o 59'.7}{\sin 30^o 22'.7} \quad = \quad 0.990\,275$$

and hence $\qquad V = \; 82^\circ\, 00'.3$

But we should now remember that $\sin^{-1}$ 0.990 275 has two values between 0° and 180°, namely 82° 00'.3 and 97° 59'.7.

Usually it is obvious from inspection of a drawing which of the two values of $\sin^{-1}$ is the required one. Unfortunately, in this case, both values are close to 90°, and it may not be immediately obvious which of the two values we require. However, it will be noticed that Toronto has a more southerly latitude than Victoria, and this should easily resolve the ambiguity.

We could, of course, have found the azimuth V by using the cotangent formula, without having to calculate ω first. Thus

$$\cos 41^\circ\, 28'.7 \cos 43^\circ 59'.7 = \sin 41^\circ\, 28'.7 \cot 46^\circ\, 08'.2 \; - \; \sin 43^\circ\, 59'.7 \cot V$$

There is only one solution for V between 0° and 180°, and it is the correct one, namely 82° 00'.3. A good drawing will show the reader why the correct solution was the acute rather than the obtuse angle (in our drawing the angle was made to be close to 90^0 in order not to bias the reader one way or the other), but in any case all readers, especially those who were trapped into choosing the obtuse angle, *should take careful note of the difficulties that can be caused by the ambiguity of the function* $\sin^{-1}$. *Indeed it is the strong advice of the author never to use the sine formula, in spite of the ease of memorizing it. The cotangent formula is more difficult to commit to memory, but it is far more useful and not so prone to quadrant mistakes.*

2. Consider two points, A and B, at latitude 20° N, longitude 25° E, and latitude 72° N, longitude 44° E. Where are the poles of the great circle passing through these two points? We shall present **three methods** of doing the problem. First, **(a)**, by solving spherical triangles. Second, **(b)**, suggested to me by Achintya Pal, using the methods of algebraic coordinate geometry. And third, **(c)**, suggested to me by J. Viswanathan.

(a) Let us call the colatitude and longitude of the first point (θ_1, ϕ_1) and of the second point (θ_2, ϕ_2). We shall consider the question answered if we can find the coordinates (θ_0, ϕ_0) of the poles Q and Q' of the great circle passing through the two points. In figure III.17, P is the north pole of the Earth, A and B are the two points in question, and Q is one of the two poles of the great circle joining A and B. The figure also shows the triangle PQA. We'll suppose that the origin for longitudes ("Greenwich") is behind the plane of the paper. The east longitudes of Q, A and B are, respectively, ϕ_0, ϕ_1, ϕ_2; and their colatitudes are $\theta_0, \theta_1, \theta_2$.

$$0 = \cos\theta_0 \cos\theta_1 + \sin\theta_0 \sin\theta_1 \cos(\phi_1 - \phi_0), \qquad 3.5.16$$

from which

$$\tan\theta_0 = -\frac{1}{\tan\theta_1\cos(\phi_1 - \phi_0)}. \qquad 3.5.17$$

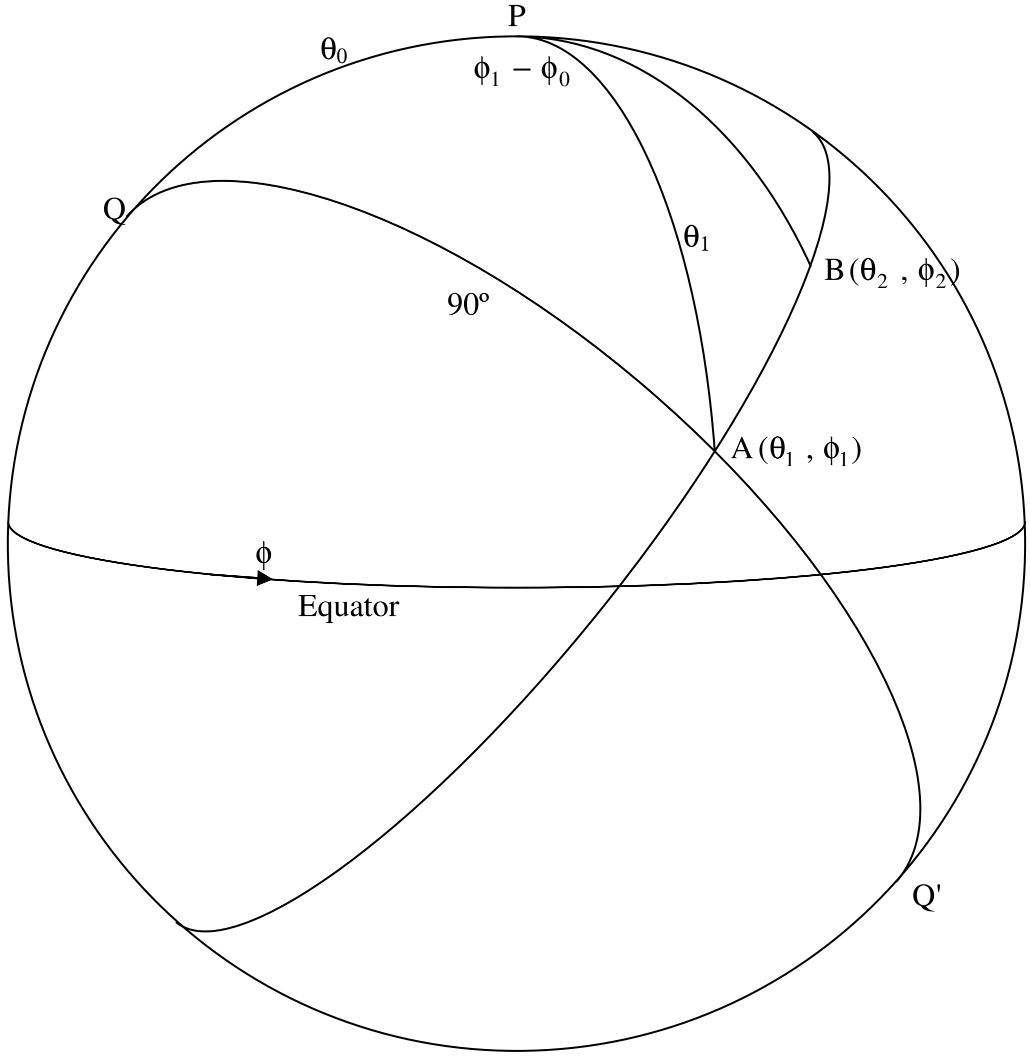


FIGURE III.17

Similarly from triangle PQB we would obtain

$$\tan\theta_0 = -\frac{1}{\tan\theta_2\cos(\phi_2 - \phi_0)}. \qquad 3.5.18$$

These are two equations in θ_0 and ϕ_0, so the problem is in principle solved. Equate the right-hand sides of the two equations, expand the terms cos $(\phi_1 - \phi_0)$ and cos $(\phi_2 - \phi_0)$, gather the terms in sin ϕ_0 and cos ϕ_0, eventually to obtain

$$\tan\phi_0 = \frac{\tan\theta_1\cos\phi_1 - \tan\theta_2\cos\phi_2}{\tan\theta_2\sin\phi_2 - \tan\theta_1\sin\phi_1}. \qquad 3.5.19$$

If we substitute the angles given in the original problem, we obtain

$$\tan\phi_0 \;=\; \frac{\tan 70^\circ \cos 25^\circ \;-\; \tan 18^\circ \cos 44^\circ}{\tan 18^\circ \sin 44^\circ \;-\; \tan 70^\circ \sin 25^\circ} \;=\; -2.412\ 091\ 0$$

from which $\phi_0 = 112^\circ\ 31'.1$ or $292^\circ\ 31'.1$

Note that we get two values for ϕ_0 differing by 180°, as expected.

We then use either of the equations for tan θ_0 to obtain θ_0 (It is good practice to use both of them as a check on the arithmetic.) The north polar distance, or colatitude, must be between 0° and 180°, so there is no ambiguity of quadrant.

With $\phi_0 = 112^\circ\ 31'.1$, we obtain $\theta_0 = 96^\circ\ 47'.1$, i.e. latitude $6^\circ\ 47'.1$ S.
and with $\phi_0 = 292^\circ\ 31'.1$, we obtain $\theta_0 = 83^\circ\ 12'.9$, i.e. latitude $6^\circ\ 47'.1$ N.

and these are the coordinates of the two poles of the great circle passing through A and B. The reader is strongly urged actually to carry out these computations numerically in order to be quite sure that the quadrants are correct and unambiguous. Indeed, dealing with the quadrant problem may be regarded as the most important part of the exercise.

(b) Pal's method. We arrived at equation 3.5.17 and 3.5 18 by solving two spherical triangles by the methods of spherical trigonometry. The second method, suggested, as mentioned above, by Achintya Pal, uses the methods of algebraic coordinate geometry in three dimensions to arrive at the same equations. We refer coordinates to axes O*xyz*. O is the centre of the Earth, taken to be of unit radius. OP is the *z*-axis. The O*x* and O*y* axes are not drawn in figure III.17, but the *x*-axis may be taken to be directed somewhere to the rear of the drawing (away from the reader), and the *y*-axis somewhere in the front of the drawing, both being, of course, in the plane of the equator.

Let us write the equation to the plane containing A and B in the form

$$lx + my + nz = 0 \qquad 3.5\ 20$$

Here (l, m, n) are the direction cosines of the normal to the plane AB, and are given by

$$l = \sin\theta_0 \cos\phi_0 \qquad m = \sin\theta_0 \sin\phi_0 \qquad n = \cos\theta_0 \qquad 3.5.21\text{a,b,c}$$

The (x, y, z) coordinates of the point A are

$$x = \sin\theta_1 \cos\phi_1 \qquad y = \sin\theta_1 \sin\phi_1 \qquad x = \cos\theta_1 \qquad 3.5.22\text{a,b,c}$$

On substitution of equations 3.5.21a,b,c and 3.5.22a,b,c into equation 3.5.20 we obtain:

$$\sin\theta_0\cos\phi_0\sin\theta_1\cos\phi_1 + \sin\theta_0\sin\phi_0\sin\theta_1\sin\phi_1 + \cos\theta_0\cos\theta_1 = 0 \qquad 3.5.23$$

After some very modest algebraic manipulation (e.g., start by dividing by $\sin\theta_1\cos\theta_0$) we very soon arrive again at equation 3.5.17, and in a similar manner at equation 3.5.18.

As a bonus we note that *any* point having spherical coordinates (θ, ϕ) lying on the great circle whole pole is at (θ_0, ϕ_0) satisfies the equation

$$\cot\theta = -\tan\theta_0\cos(\phi - \phi) \qquad 3.5.24$$

This equation may be regarded as the (θ, ϕ) equation to the great circle AB, and it answers the problem converse to the one originally posed: What is the equation to the great circle whose pole is at (θ_0, ϕ_0)?

(c) Viswanathan's method.

Let (θ_1, ϕ_1) be the colatitude and longitude of A, and $\mathbf{r_1} = (x_1, y_2, z_1)$ be its Cartesian coordinates.
Let (θ_2, ϕ_2) be the colatitude and longitude of B, and $\mathbf{r_2} = (x_2, y_2, z_2)$ be its Cartesian coordinates.

Then

$$\begin{aligned}(x_1, y_1, z_1) &= (\sin\theta_1\cos\phi_1, \sin\theta_1\sin\phi_1, \cos\theta_1)\\ (x_2, y_2, z_2) &= (\sin\theta_2\cos\phi_2, \sin\theta_2\sin\phi_2, \cos\theta_2)\end{aligned}$$

[The angle ω between $\mathbf{r_1}$ and $\mathbf{r_2}$ is given by $\cos\omega = x_1x_2 + y_1y_2 + z_1z_2$. This result is not needed here. It is pointed out here only to show that the same method can be used in the first example - to find the distance between the observatories in Victoria and Toronto.]

Let $(l, m, 1)$ be the direction ratios of the line passing through the origin that is perpendicular to the plane containing the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$. Then we have:

$$\begin{aligned}lx_1 + my_1 &= z_1\\ lx_2 + my_2 &= z_2.\end{aligned}$$

Thus $lx_1 + my_1 = z_1$ is the equation to the plane passing through the origin and containing $\mathbf{r_1}$ and $\mathbf{r_2}$; i.e. the great circle passing through A and B lies at the intersection of the unit sphere with this plane. In our particular numerical example, the solution of the above two equations gives $l = -3.21852823$ and $m = +7.763383$.

The unit vector **p** this is one of the poles being sought is obtained by normalizing the direction ratios by $\sqrt{l^2 + m^2 + 1}$. Let (θ_0, ϕ_0) denote the colatitude and longitude of **p**. Then

$$p = \left(\frac{l}{\sqrt{l^2 + m^2 + 1}}, \frac{m}{\sqrt{l^2 + m^2 + 1}}, \frac{1}{\sqrt{l^2 + m^2 + 1}} \right)$$

$$= (\sin\theta_0 \cos\phi_0 \ , \ \sin\theta_0 \sin\phi_0, \cos\theta_0).$$

From the equality of the third component, we obtain $83^\circ\ 12'.9$ (i.e. latitude $6^\circ\ 47'.1$ N). From the first and second components, we have $\tan\phi_0 = \frac{m}{l}$ and hence

$\phi_0 = 112^\circ\ 31'.1$ E. The other pole is at latitude $6^\circ\ 47'.1$ S, longitude $292^\circ\ 31'.1$ E

3. Here is a challenging exercise and an important one in meteor astronomy. Two shower meteors are seen, diverging from a common radiant. One starts at right ascension 6 hours, declination +65 degrees, and finishes at right ascension 1 hour, declination +75 degrees. The second starts at right ascension 5 h, declination +35 degrees, and finishes at right ascension 3 hours, declination +15 degrees. Where is the radiant?

The assiduous student will make a good drawing of the celestial sphere, illustrating the situation as accurately as possible. The calculation will require some imaginative manipulation of spherical triangles. After arriving at what you believe to be the correct answer, look at your drawing to see whether it is reasonable. The next step might be to develop a general trigonometrical expression for the answer in terms of the original data, or to program the calculation for a computer, so that it is henceforth available for any similar calculation. Or one can go yet further, and write a computer program that will give a least-squares solution for the radiant for many more than two meteors in the shower. I find for the answer to the above problem that the radiant is at right ascension 7.26 hours and declination +43.8 degrees.

Uniqueness of Solutions

The reader who has by now worked through a variety of problems in the solution of a triangle will have noticed that, given three elements of a triangle, sometimes there is a unique solution, whereas sometimes there are two possible triangles that satisfy the original data. Yet again, it may sometimes be found that there is no possible solution, meaning that there is no possible triangle that satisfies the given data, which must therefore be presumed incorrect. I am very much indebted to Alan Johnstone for lengthy discussions on this problem, and indeed for pointing out that some of the "solutions" given in an earlier version of these notes were in fact invalid (and have now been corrected). I believe the following criteria determine how many valid solutions there are for a given triplet of data, for plane triangles and for spherical triangles.

We may be given three elements of a triangle,

Thus

- **i.** Three sides: a, b, c,
- **ii.** Two sides and the included angle: b, c, A.
- **iii.** Two sides and a nonincluded angle: a, b, A.
- **iv.** Two angles and a common side: a, B, C.
- **v.** Two angles and another side: A, B, a.
- **vi.** Three angles: A, B, C.

Question:

Which of these give a unique solution, and which admit of two solutions? And which are impossible triangles? I believe the answers are as follows:

Plane Triangles

i. Let $d = a + b - c, \qquad e = b + c - a, \qquad f = c + a - b$

For a valid triangle, d, e, and f must all be positive. If so, there is a unique solution.

ii. There is a unique solution.

iii. If $a > b$ there is a unique solution.

If $a = b$, there is a unique solution if $A < 90°$. Otherwise there is no valid triangle.

If $a < b$ there are zero, one or two solutions, according as to whether

$\sin A > \dfrac{a}{b}, \quad \sin A = \dfrac{a}{b} \quad \text{or} \quad \sin A < \dfrac{a}{b}$.

iv. There is a unique solution.

v. There is a unique solution.

vi. There is a unique solution except that only the relative lengths of the sides are determined.

Spherical Triangles

i. Let $d = a + b - c, \qquad e = b + c - a, \qquad f = c + a - b$

For a valid triangle, d, e, and f must all be positive. If so, there is a unique solution.

ii. There is a unique solution.

iii. If $\sin A > \dfrac{\sin a}{\sin b}$,there is no real solution.

If $A = a = b = 90^\circ$, then $B = 90^\circ$, and c and C are equal but indeterminate.

Otherwise:

If $a > b$ there is a unique solution.

If $a = b$, there is a unique solution if $A < 90^\circ$. Otherwise there is no real solution.

If $a < b$ there are one or two solutions, according as to whether

$$\sin A = \frac{\sin a}{\sin b} \quad \text{or} \quad \sin A < \frac{\sin a}{\sin b}.$$

iv. There is a unique solution.

v. If $\sin a > \dfrac{\sin A}{\sin B}$, there is no real solution.

If $A = B = a = 90^\circ$, then $b = 90^\circ$, and c and C are equal but indeterminate.

Otherwise:

If $A > B$ there is a unique solution.

If $A = B$, there is a unique solution if $a < 90^\circ$. Otherwise there is no real solution.

If $A < B$ there are one or two solutions, according as to whether

$$\sin a = \frac{\sin A}{\sin B} \quad \text{or} \quad \sin a < \frac{\sin A}{\sin B}.$$

3.6 *Rotation of Axes, Two Dimensions*

In this section we consider the following problem. Consider two sets of orthogonal axes, Ox, Oy, and Ox', Oy', such that one set makes an angle θ with respect to the other. See figure (a) below. A point P can be described either by its coordinates (x,y) with respect to one "basis set" Ox, Oy, or by its coordinates with respect to the other basis set Ox', Oy'. The question is, what is the relation between the coordinates (x,y) and the coordinates (x',y')? See figure III.18.

We see that OA = x, AP = y, ON = x', PN = y', OM = $x\cos\theta$, MN = $y\sin\theta$,

$$\therefore \qquad x' = x\cos\theta + y\sin\theta. \qquad 3.6.1$$

Also MA = NB = $x\sin\theta$, PB = $y\cos\theta$,

$$\therefore \qquad y' = -x\sin\theta + y\cos\theta. \qquad 3.6.2$$

These two relations can be written in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \qquad 3.6.3$$

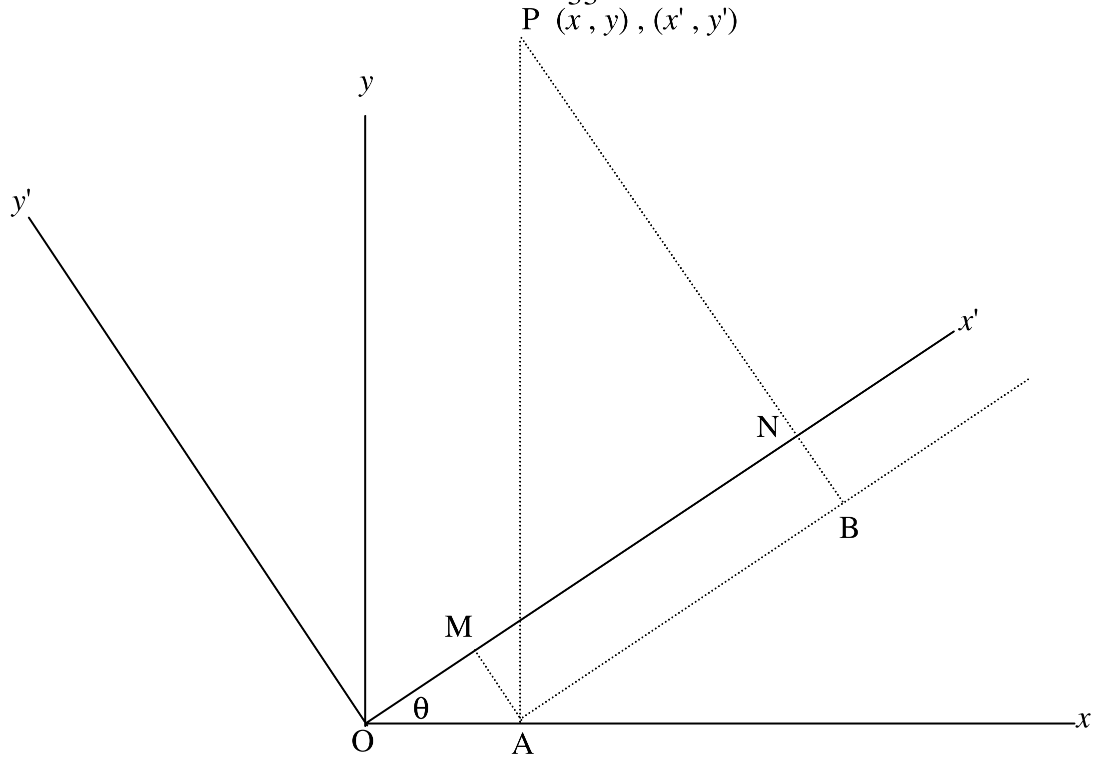


FIGURE III.18

There are several ways of obtaining the converse relations; that is, equations for x and y in terms of x' and y'. One way would be to design drawings similar to (b) and (c) that show the converse relations clearly, and the reader is encouraged to do this. Another way is merely to solve the above two equations (which can be regarded as two simultaneous equations in x and y) for x and y. Less tedious is to interchange the primed and unprimed symbols and change the sign of θ. Perhaps the quickest of all is to recognize that the determinant of the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is unity and therefore the matrix is an orthogonal matrix. One important property of an orthogonal matrix $\mathbf{M}$ is that its reciprocal $\mathbf{M}^{-1}$ is equal to its transpose $\tilde{\mathbf{M}}$ (formed by transposing the rows and columns). Therefore the converse relation that we seek is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}. \qquad 3.6.4$$

The reader might like to try all four methods to ensure that they all arrive at the same result.

3.7 *Rotation of Axes, Three Dimensions. Eulerian Angles*

We now consider two sets of orthogonal axes Ox, Oy, Oz and Ox', Oy', Oz' in three-dimensional space and inclined to each other. A point in space can be described by its coordinates (x,y,z) with respect to one basis set or (x',y',z') with respect to the other. What is the relation between the coordinates (x,y,z) and the coordinates (x',y',z')?

We first need to describe exactly how the primed axes are inclined with respect to the unprimed axes. In the figure below are shown the axes Ox, Oy and Oz. Also shown are the axes Ox' and Oz'; the axis Oy' is directed behind the plane of the paper and is not drawn. The orientation of the primed axes with respect to the unprimed axes is described by three angles θ , ϕ and ψ, known as the *Eulerian angles*, and they are shown in figure III.19.

The precise definitions of the three angles can be understood by three consecutive rotations, illustrated in figures III.20,21,22.

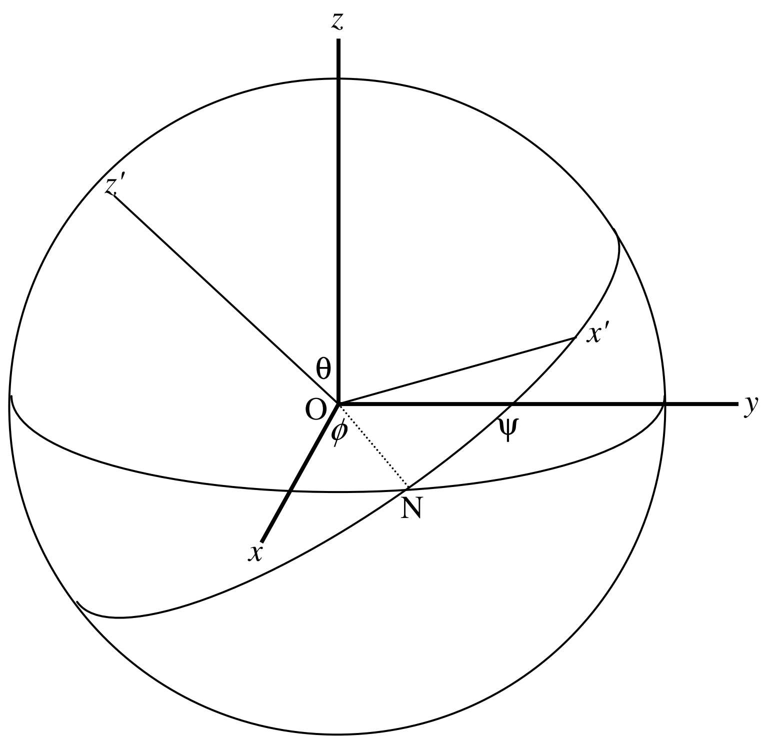


FIGURE III.19

First, a rotation through ϕ counterclockwise around the Oz axis to form a set of intermediate axes Ox_1 , Oy_1 , Oz_1 , as shown in figure III.20. The Oz and Oz_1 axes are identical. Part (b) shows the rotation as seen when looking directly down the Oz (or Oz_1) axis.

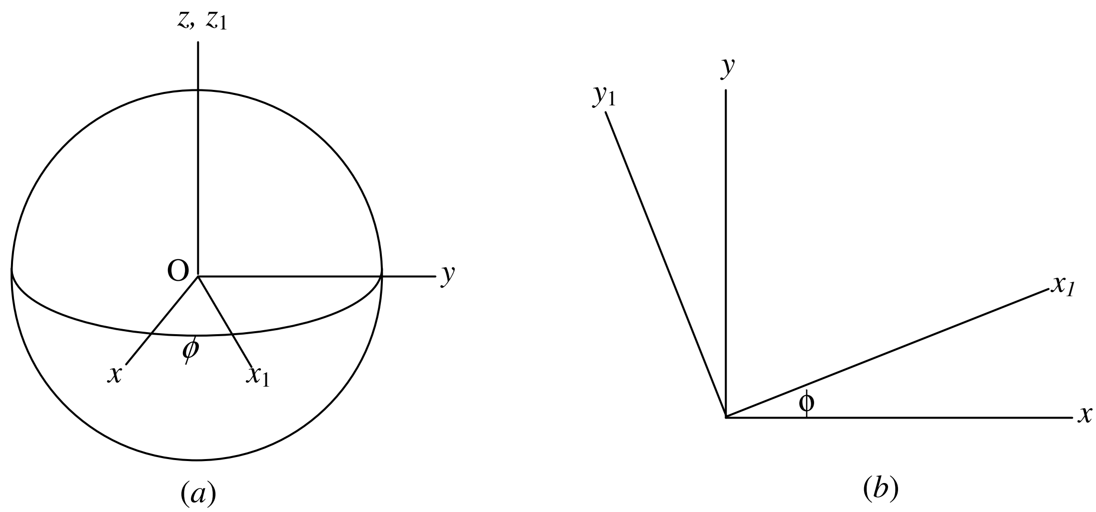


FIGURE III.20

The relation between the (x, y, z) and (x_1, y_1 , z_1) coordinates is

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \qquad 3.7.1$$

Next, a rotation through θ counterclockwise around the Ox_1 axis to form a set of axes Ox_2 , Oy_2 , Oz_2 . The Ox_1 and Ox_2 axes are identical (Figure III.21). Part (b) of the figure shows the rotation as seen when looking directly towards the origin along the Ox_1 (or Ox_2) axis.

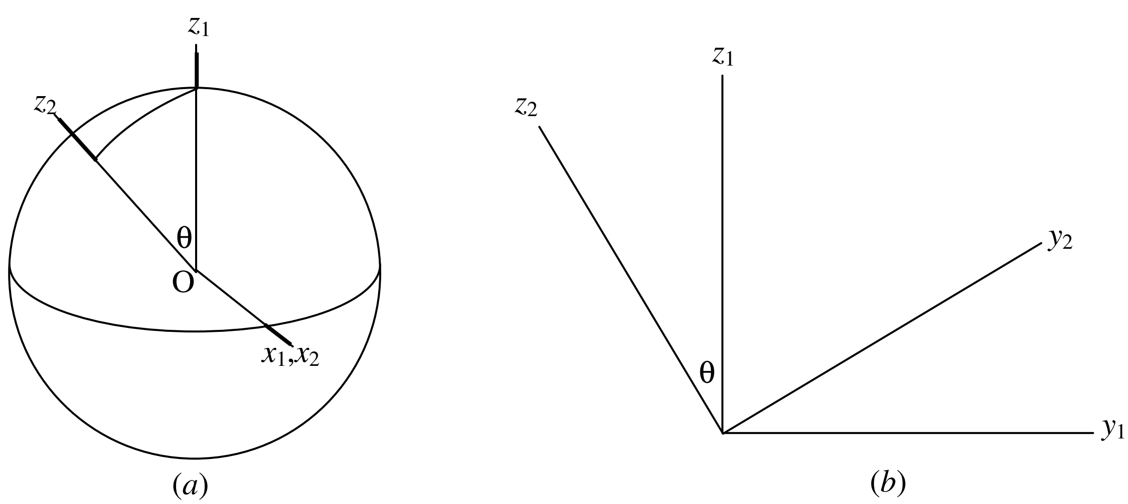


FIGURE III.21

The relation between the (x_1, y_1, z_1) and (x_2, y_2, z_2) coordinates is

$$\begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}. \qquad 3.7.2$$

Lastly, a rotation through ψ counterclockwise around the Oz_2 axis to form the set of axes Ox', Oy' Oz' (figure III.22). The Oz_2 and Oz' axes are identical. Part (b) of the figure shows the rotation as seen when looking directly down the Oz_2 (or Oz') axis.

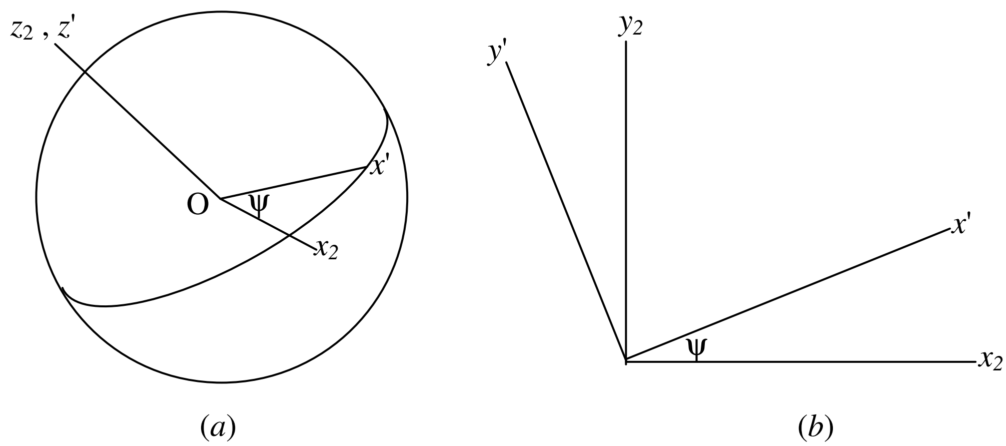


FIGURE III.22

The relation between the (x_2, y_2, z_2) and (x', y', z') coordinates is

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 1 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}. \qquad 3.7.3$$

Thus we have for the relations between (x',y',z') and (x,y,z)

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \qquad 3.7.4$$

On multiplication of these matrices, we obtain

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \qquad 3.7.5$$

The inverse of this may be found, as in the two-dimensional case, either by solving these three equations for x, y and z (which would be rather tedious); or by interchanging the primed and unprimed quantities and reversing the order and signs of all operations (replace ψ with $-\phi$, θ with $-\theta$, and ϕ with $-\psi$) which is less tedious; or by recognizing that the determinant of the matrix is unity and therefore its reciprocal is its transpose, which is hardly tedious at all. The reader should verify that the determinant of the matrix is unity by multiplying it out and making use of trigonometric identities. The reason that the determinant must be unity, however, and that the rotation matrix must be orthogonal, is that rotation of axes cannot change the magnitude of a vector.

Each element of the matrix is the cosine of the angle between an axis in one basis set and an axis in the other basis set. For example, the second element in the first row is the cosine of the angles between Ox' and Oy. The first element of the third row is the cosine of the angles between Oz' and Ox. The matrix can be referred to as the matrix of direction cosines between the axes of one basis set and the axes of the other basis set, and the relations between the coordinates can be written

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad 3.7.6$$

$$\mathbf{R'} = \mathbf{CR}. \qquad 3.7.7$$

You will note the similarity of the forms of the direction cosines to the cosine formula for the solution of a spherical triangle, and indeed the direction cosines can all be derived by drawing and solving the relevant spherical triangles. You might (or might not!)enjoy trying to do this.

The matrix $\mathbf{C}$ of the direction cosines is orthogonal, and the properties of an orthogonal matrix are as follows. The reader should verify this using the formulas for the direction cosines in terms of the Eulerian angles. The properties also apply, of course, although more trivially, to the rotation matrix in two dimensions.

(a) det $\mathbf{C} = \pm 1$
(det $\mathbf{C} = -1$ implies that the two basis sets are of opposite chirality or "handedness"; that is, if one basis set is right-handed, the other is left-handed.)

(b) The sum of the squares of the elements in any row or any column is unity. This merely means that the magnitudes of unit orthogonal vectors are indeed unity.

(c) The sum of the products of corresponding elements in any two rows or any two columns is zero. This is merely a reflection of the fact that the scalar or dot product of any two unit orthogonal vectors is zero.

(d) Every element is equal to its own cofactor. This a reflection of the fact that the vector or cross product of any two unit orthogonal vectors in cyclic order is equal to the third.

(e) $\mathbf{C}^{-1} = \tilde{\mathbf{C}}$, or the reciprocal of an orthogonal matrix is equal to its transpose.

The first four properties above can be (and should be) used in a numerical case to verify that the matrix is indeed orthogonal, and they can be used for detecting and for correcting mistakes.

For example, the following matrix is supposed to be orthogonal, but there are, in fact, two mistakes in it. Using properties (b) and (c) above, locate and correct the mistakes. (It will become clear when you do this why verification of property (b) alone is not sufficient.) When you have corrected the matrix, see if you can find the Eulerian angles θ, ϕ and ψ without ambiguity of quadrant. As a hint, start at the bottom right hand side of the matrix and note, from the way in which the Eulerian angles are set up, that θ must be between 0° and 180°, so that there is no ambiguity of quadrant. The other two angles, however, can lie between 0^0 and 360^0 and must be determined by examining the signs of their sines and cosines. When you have calculated the Eulerian angles, a further useful exercise would be to prepare a drawing showing the orientation of the primed axes with respect to the unprimed axes.

$$\begin{pmatrix} +0.075\ 284\ 882\ 7 & -0.518\ 674\ 468\ 2 & +0.851\ 650\ 739\ 6 \\ -0.553\ 110\ 473\ 2 & -0.732\ 363\ 000\ 8 & +0.397\ 131\ 261\ 9 \\ -0.829\ 699\ 337\ 5 & +0.442\ 158\ 963\ 2 & -0.342\ 020\ 143\ 3 \end{pmatrix}$$

Note, as a matter of good computational practice, that the numbers are written in groups of three separated by half spaces after the decimal point, all numbers, positive and negative, are signed, and leading zeroes are not omitted.

3.8 *Trigonometric Formulas*

I gather here merely for reference a set of commonly-used trigonometric formulas. It is a matter of personal preference whether to commit them to memory. It is probably fair to remark that anyone who is regularly engaged in problems in celestial mechanics or related disciplines will be familiar with most of them, at least from frequent use, whether or not any conscious effort was made to memorize them. At the very least, the reader should be aware of their existence, even if he or she has to look to recall the exact formula.

$$\frac{\sin A}{\cos A} = \tan A$$

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \cot^2 A = \csc^2 A$$

$$1 + \tan^2 A = \sec^2 A$$

$$\sec A \csc A = \tan A + \cot A$$

$$\sec^2 A \csc^2 A = \sec^2 A + \csc^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin \tfrac{1}{2} A = \sqrt{\frac{1 - \cos A}{2}}$$

$$\cos \tfrac{1}{2} A = \sqrt{\frac{1 + \cos A}{2}}$$

$$\tan \tfrac{1}{2} A = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \frac{1 - \cos A}{\sin A} = \frac{\sin A}{A + \cos A} = \csc A - \cot A$$

$$\sin A + \sin B = 2\sin\tfrac{1}{2}S\cos\tfrac{1}{2}D,$$

where $S = A + B$ and $D = A - B$

$$\sin A - \sin B = 2\cos\tfrac{1}{2}S\sin\tfrac{1}{2}D$$

$$\cos A + \cos B = 2\cos\tfrac{1}{2}S\cos\tfrac{1}{2}D$$

$$\cos A - \cos B = -2\sin\tfrac{1}{2}S\sin\tfrac{1}{2}D$$

$$\sin A\sin B = \tfrac{1}{2}(\cos D - \cos S)$$

$$\cos A\cos B = \tfrac{1}{2}(\cos D + \cos S)$$

$$\sin A\cos B = \tfrac{1}{2}(\sin S + \sin D)$$

$$\sin A = \frac{T}{\sqrt{1+T^2}} = \frac{2t}{1+t^2},$$

where $T = \tan A$ and $t = \tan\tfrac{1}{2}A$

$$\cos A = \frac{1}{\sqrt{1+T^2}} = \frac{1-t^2}{1+t^2}$$

$$\tan A = T = \frac{2t}{1-t^2}$$

$$s = \sin A, \quad c = \cos A$$

$\cos A = c$

$\cos 2A = 2c^2 - 1$

$\cos 3A = 4c^3 - 3c$

$\cos 4A = 8c^4 - 8c^2 + 1$

$\cos 5A = 16c^5 - 20c^3 + 5c$

$\cos 6A = 32c^6 - 48c^4 + 18c^2 - 1$

$\cos 7A = 64c^7 - 112c^5 + 56c^3 - 7c$

$\cos 8A = 128c^8 - 256c^6 + 160c^4 - 32c^2 + 1$

$\sin A = s$

$\sin 2A = 2cs$

$\sin 3A = 3s - 4s^3$

$\sin 4A = 4c(s - 2s^3)$

$\sin 5A = 5s - 20s^3 + 16s^5$

$\sin 6A = 2c(3s - 16s^3 + 16s^5)$

$\sin 7A = 7s - 56s^3 + 112s^5 - 64s^7$

$\sin 8A = 8c(s - 10s^3 + 24s^5 - 16s^7)$

$\tan A = T$

$$\tan 2A = \frac{2T}{1 - T^2}$$

$$\tan 3A = \frac{T(3 - T^2)}{1 - 3T^2}$$

$$\tan 4A = \frac{4T(1 - T^2)}{1 - 6T^2}$$

$$\tan 5A = \frac{T(5 - 10T^2 + T^4)}{1 - 10T^2 + 5T^4}$$

$$\tan nA = \frac{T\left(\binom{n}{1} - \binom{n}{3}T^2 + \binom{n}{5}T^4 - \ldots\right)}{1 - \binom{n}{2}T^2 + \binom{n}{4}T^4 - \ldots} \quad , \quad \binom{n}{r} = \text{binomial coefficient}$$

$\cos^2 A = \frac{1}{2}(\cos 2A + 1)$

$\cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$

$\cos^4 A = \frac{1}{8}(\cos 4A + 4\cos 2A + 3)$

$\cos^5 A = \frac{1}{16}(\cos 5A + 5\cos 3A + 10\cos A)$

$\cos^6 A = \frac{1}{32}(\cos 6A + 6\cos 4A + 15\cos 2A + 10)$

$\cos^7 A = \frac{1}{64}(\cos 7A + 7\cos 5A + 21\cos 3A + 35\cos A)$

$\cos^8 A = \frac{1}{128}(\cos 8A + 8\cos 6A + 28\cos 4A + 56\cos 2A + 35)$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$
$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$
$\sin^4 A = \frac{1}{8}(\cos 4A - 4\cos 2A + 3)$
$\sin^5 A = \frac{1}{16}(\sin 5A - 5\sin 3A + 10\sin A)$
$\sin^6 A = \frac{1}{32}(10 - 15\cos 2A + 6\cos 4A - \cos 6A)$
$\sin^7 A = \frac{1}{64}(35\sin A - 21\sin 3A + 7\sin 5A - \sin 7A)$
$\sin^8 A = \frac{1}{128}(\cos 8A - 8\cos 6A + 28\cos 4A - 56\cos 2A + 35)$

$$\sin A = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \cdots$$

$$\cos A = 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - \cdots$$

$$\int_0^{\pi/2} \sin^m \theta \cos^n \theta d\theta = \frac{(m-1)!!(n-1)!!X}{(m+n)!!},$$ where $X = \pi/2$ if m and n are both even, and $X = 1$ otherwise.

$e^{ni\theta} = e^{in\theta}$ (de Moivre's theorem - the only one you need know. All others can be deduced from it.)

Plane triangles:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a\cos B + b\cos A = c$$

$$s = \tfrac{1}{2}(a + b + c)$$

$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$
$$\cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$$

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Spherical triangles

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

$$\cos(\mathrm{IS})\cos(\mathrm{IA}) = \sin(\mathrm{IS})\cot(\mathrm{OS}) - \sin(\mathrm{IA})\cot(\mathrm{OA})$$