3 The Cosmic Microwave Background

The Cosmic Microwave Background (CMB) radiation is observed as an all sky radiation field whose peak emission occurs in the mm–wavelength region of the electromagnetic spectrum (confusingly referred to as μ–wave wavelengths). To investigate the "cosmological" nature of the CMB, several effects must be accounted for:

1. An apparent dipole in the cosmological CMB caused by the motion of the local group of galaxies with respect to the CMB rest frame. The motion is ~ 300 kms^{-1} in the direction $l = 264°$, $b = 48°$.

2. Point sources, i.e. bright AGN or sub-mm sources together with SZ 'holes'.

3. Galactic μ–wave emission. This largely arises from dust and synchrotron emission and is removed by fitting the sky emission at wavelengths where dust/synchrotron emission is strong and the CMB contribution is relatively small.

When these effects have been accounted for the CMB is highly isotropic and displays a spectrum that is completely described by a Black Body of temperature $T_{CMB} = 2.73$ K. The isotropy of the CMB and the close approximation to a Black Body constrains key universal properties (which we shall discuss in the following sections):

1. The radiation field is "universal" i.e. homogeneous (note though the case where the radiation could have been emitted from a uniform spherical region).

2. The CMB temperature is $\propto a(t)^{-1}$. Therefore the universe (as measured by the CMB) was hotter in the past.

3. Though T_{CMB} has changed throughout the history of the universe, it has remained a *Black Body*. Therefore, at the epoch of CMB emission, radiation and matter were in thermal equilibrium, i.e. protons and electrons were coupled closely to photons via free–free interactions.

4. The existence of the CMB thus points us to consider universal epochs where matter density did not dominate the behaviour of the Friedmann equation, i.e. the *radiation dominated* epoch.

The above points refer to the **global** or mean properties of the CMB. When viewed on small angular scales ($< 2°$) the CMB displays a structured pattern of temperature variations. These temperature variations are referred to as **CMB anisotropies** and are observed at the level $(\Delta T/T)_{rms} \sim 10^{-5}$. Anisotropies arose from matter density fluctuations in the "baryon-photon fluid" used to describe the behaviour of matter and radiation in the universe prior to the creation of the CMB. The existence of anisotropies in the CMB do not invalidate previous statements that the CMB is isotropic. The pattern of anisotropies is constant in all directions and the universe remains isotropic in a statistical sense. The distribution of anisotropies is described by a damped oscillator equation that includes the effects of gravitating matter, photon pressure and the expansion of the universe.

The scale and distribution of these anisotropies are "frozen" into the CMB radiation during the epoch of CMB creation and they contain information regarding various cosmological parameters (via the location of the so–called "Acoustic" peaks in the temperature power spectrum).

Note: CMB photons have a long free path length. However, one source of local scattering on the line–of–sight to an observer is inverse Compton scattering of low energy CMB photons with high energy electrons in the X–ray emitting intra–cluster gas of galaxy clusters. Photons are scattered to higher energy levels and result in a CMB decrement toward the cluster (when observing in a particular wave band) – the Sunyaev–Zeld'ovich effect.

3.1 Photons, photons everywhere...

The physics of the early universe is dominated by the fact that there are vastly more photons than baryons. It is therefore worthwhile to convince ourselves that this is indeed the case. We can determine the baryon-to-photon ratio of the universe by comparing their respective energy densities. For CMB photons described by a Blackbody distribution the present day energy density is

$$\varepsilon_{\gamma,0} = AT_0^4 = \frac{4\sigma}{c}T_0^4 = 0.262 \text{ MeV m}^{-3}. \tag{1}$$

Taking the mean energy per CMB photon as $E_{\gamma,0} = hc/\lambda_{max,0}$ where $\lambda_{max,0} = 2.898 \times 10^{-3}/T_{CMB}$ we obtain $E_{\gamma,0} = 1.2 \times 10^{-3}$ eV. The number density of CMB photons is therefore

$$n_{\gamma,0} = \frac{\varepsilon_{\gamma,0}}{E_{\gamma,0}} = 2.2 \times 10^8 \text{ m}^{-3}, \tag{2}$$

which corresponds to a dimensionless density parameter $\Omega_\gamma = (5 \times 10^{-5})$[1]. The present day energy density of baryons described by a dimensionless density parameter $\Omega_{b,0} = 0.04$ is

$$\varepsilon_{b,0} = \Omega_{b,0}\rho_{c,0}c^2 = 0.04 \times 5200 \text{ MeV m}^{-3} \approx 210 \text{ MeV m}^{-3}. \tag{3}$$

Taking the mean rest energy of baryons (protons and neutrons) to be $E_b \approx 939$ MeV, the present day number density of baryons is

$$n_{b,0} = \frac{\varepsilon_{b,0}}{E_b} \approx \frac{210 \text{ MeV m}^{-3}}{939 \text{ MeV}} \approx 0.22 \text{ m}^{-3}. \tag{4}$$

[1]Note that $\Omega_{rel} = \Omega_\gamma + \Omega_\nu = (5 + 3.4) \times 10^{-5} = 8.4 \times 10^{-5}$, where Ω_{rel} refers to the dimensionless mass-energy density from all relativistic particles and Ω_ν refers to the contribution from neutrinos.

Finally, defining the baryon-to-photon ration as η, we have

$$\eta = \frac{n_{b,0}}{n_{\gamma,0}} \approx \frac{0.22 \text{ m}^{-3}}{2.2 \times 10^8 \text{ m}^{-3}} \approx 10^{-9}. \tag{5}$$

Note that as the **number density** of both baryons and photons scale as a^{-3}, the value of η is fixed for all time.

3.2 Dependence of the CMB temperature upon the scale factor

To consider the properties of the CMB as a function of the scale factor we first define the radiation brightness (spectral intensity) of the CMB as $i(\nu, t)$ which has units of **energy / area / time / frequency / solid angle** (e.g. W m^{-2} Hz^{-1} steradian^{-1}). For a Black Body spectrum of temperature, T, one may write

$$i(\nu, t) = \frac{2h\nu^3}{c^2}\left(e^{\frac{h\nu}{kT}} - 1\right)^{-1}, \tag{6}$$

(Planck's Law) where k is the Boltzmann constant. The temperature of the CMB is defined via Wien's Law, which states that $\lambda_{max}T = 2.898 \times 10^{-3}$m K. Consider the time variation of $i(\nu, t)$ in an expanding universe, i.e. we seek to characterise an equation of the form,

$$i(\nu + \delta\nu, t + \delta t) - i(\nu, t) = ? \tag{7}$$

Within the time interval δt each photon suffers a cosmological redshift of the form $\nu \propto a^{-1}$, i.e.

$$\begin{aligned} \frac{d\nu}{dt} &\propto -a^{-2}\,\frac{da}{dt} \\ \delta\nu &= -\nu\,\frac{\dot{a}}{a}\,\delta t. \end{aligned} \tag{8}$$

However, this relation affects both **energy** and **frequency** and thus redshift effects do not affect $i(\nu, t)$. One may consider a similar relation for the number density of photons recalling that $n_\gamma \propto a^{-3}$, i.e.

$$\begin{aligned} \frac{dn_\gamma}{dt} &\propto -3a^{-4}\,\frac{da}{dt} \\ \frac{\delta n_\gamma}{n_\gamma} &= -3\,\frac{\dot{a}}{a}\,\delta t. \end{aligned} \tag{9}$$

This is the only cosmological factor to affect $i(\nu, t)$. Noting that $n_\gamma \propto i$ we can re–write Equation 7 (in the absence of any sources or sinks of radiation) as,

$$\begin{aligned} i(\nu + \delta\nu, t + \delta t) - i(\nu, t) &= -3i\,\frac{\dot{a}}{a}\,\delta t \\ \frac{di(\nu, t)}{dt} &= -3i\,\frac{\dot{a}}{a} \end{aligned} \tag{10}$$

which has the solution

$$\frac{d}{dt}[a(t)^3\, i(\nu,t)] = 0 \tag{11}$$

and provides a basic conservation equation.

One may now consider the effects of this conservation equation upon a Black Body spectrum. The spectral intensity at any epoch t_e is $i(\nu_e, t_e)$. At some later time t the brightness must satisfy,

$$i(\nu,t) = \left(\frac{a(t_e)}{a(t)}\right)^3 i(\nu_e, t_e) \tag{12}$$

Substituting this relation into the initial Black Body spectrum indicates that the spectral intensity at some time t must satisfy

$$\begin{aligned} i(\nu,t) &= \frac{2h\nu_e^3}{c^2}\left(\frac{a(t_e)}{a(t)}\right)^3 \left(e^{\frac{h\nu_e}{kT_e}} - 1\right)^{-1} \\ &= \frac{2h\nu^3}{c^2}\left(e^{\frac{h\nu}{kT(t)}} - 1\right)^{-1}, \text{ where} \\ T(t) &= T_e\,\frac{a(t_e)}{a(t)}. \end{aligned} \tag{13}$$

This result indicates that at all times since the epoch of emission the CMB radiation was described by a Black Body radiation law characterised by a temperature $T \propto a^{-1}$. A Black Body spectrum arises when matter (in this case protons and electrons) and radiation are in thermal equilibrium, i.e. photon emission and absorption proceed at the same rate. This points to an epoch when the interactions between matter and radiation proceeded much more rapidly compared to today. The dependence of the Blackbody temperature upon the scale factor also indicates that the universe was hotter and smaller in past. Unsurprisingly, these last two statements are linked. The dependence of radiation temperature upon the scale factor may also be obtained applying the Fluid equation to a radiation dominated universe and by noting that $P = \rho c^2/3$, i.e.

$$\begin{aligned} \dot\rho + 3H\left(\rho + \frac{P}{c^2}\right) &= 0 \\ \dot\rho + 3H(\rho + \rho/3) &= 0 \\ \dot\rho + 4\rho H &= 0 \\ \frac{1}{\rho}\frac{d\rho}{dt} &= -\frac{4}{a}\frac{da}{dt} \end{aligned}$$

$$\ln \rho = -4 \ln a$$

$$\rho \propto a^{-4}$$

$$\rho = AT^4/c^2 \Rightarrow T \propto a^{-1}. \quad (14)$$

Finally, noting the definition of redshift in terms of the scale factor, the radiation temperature of the universe at some epoch defined by a redshift z may therefore be expressed as

$$T = (1+z)\,2.7\,\mathrm{K} \quad (15)$$

Extension of these relations to a time $t \to 0$ implies that the universe existed in some extremely hot dense early state. This is the basic justification of the Hot Big Bang model within which the physics of the early universe is discussed. In fact, this equation provides us with a thermal history of the universe as a function of redshift (or time if one assumes a specific cosmological model). Consideration the radiation temperature of the universe as a function of epoch – either measured in Kelvin or MeV – provides the link between universal epoch and the dominant physical processes occurring at that time. When we apply this analysis to the CMB we discover that the process of CMB creation can be split into four distinct physical process occurring in a well defined order:

- **Matter and radiation equality.**
- **Recombination of electrons and protons.**
- **Decoupling of matter and radiation.**
- **Last scattering.**

3.3 Matter versus radiation dominance

We have demonstrated in Lecture 1 that the matter density of the universe follows a relation of the form $\rho \propto a^{-3}$ and that for a spatially flat universe (nominally EdS), insertion of this relation into the Friedmann equation generates a relation of the form $a \propto t^{2/3}$. We will now pursue a similar discussion for a universe dominated by relativistic particles[2]. The energy density of a radiation field described by a Planck function may be expressed as $\varepsilon = AT^4$. The effective mass density associated with the CMB is therefore $\rho_{rad} = \varepsilon/c^2 = AT^4/c^2$, i.e. based upon our previous discussion we

[2]Relativistic particles in this context refers to photons and neutrinos. Both obey the same equation of state and therefore both display $\rho \propto a^{-4}$. The contribution of photons and neutrinos to the total mass density of the universe are $\Omega_\gamma = 5 \times 10^{-5}$ and $\Omega_\nu = 3.4 \times 10^{-5}$ respectively (see Ryden Table 6.2). We refer to both types of particle as "radiation" in the following text.

may write $\rho_{rad} \propto a^{-4}$. Recalling that $\rho_{matter} \propto a^{-3}$ we note that, though the universe is matter dominated in the present epoch[3], that there must exist some earlier time t where $\rho_{rad} \gg \rho_{matter}$. Returning to the Friedmann equation one may consider the evolution of the scale factor for the case $\rho_{rad} \gg \rho_{matter}$ in an EdS universe (a conveniently simple case) to obtain a relation of the form

$$\begin{aligned}
(\dot{a})^2 - \frac{8\pi G}{3}\,\rho\, a^2 &= 0 \\
(\dot{a}a)^2 &= K \quad \text{inserting } \rho \propto \mathrm{a}^{-4} \\
\frac{1}{2}\,a^2 &= \sqrt{K}t \\
a &\propto t^{1/2} \qquad (16)
\end{aligned}$$

Comparing this result to $a \propto t^{2/3}$ during the matter dominated epoch, we note that the universe was expanding less rapidly during radiation dominance. It is therefore natural to ask, "when did the the universe change from a radiation to a matter dominated state?". To answer this equation, we assume that at some past time t_{rm}, matter and radiation were an equilibrium expressed via $\rho_{rad}(t_{rm}) = \rho_{matter}(t_{rm})$. We may therefore write

$$\begin{aligned}
\rho_{rad,0}\left(\frac{a_0}{a_{rm}}\right)^4 &= \rho_{matter,0}\left(\frac{a_0}{a_{rm}}\right)^3 \\
\frac{\rho_{matter,0}}{\rho_{rad,0}} &= 1 + z_{rm} \qquad (17)
\end{aligned}$$

Inserting the corresponding present day values for the density of matter and radiation given in Section 3.1 (and taking $\Omega_{m,0} = 0.3$), one obtains a value of $z_{rm} \approx 3570$. Employing Equation 15. indicates that at this epoch the CMB displayed a temperature $T_{rm} \sim 10,000$K. Therefore, when considering the epoch of recombination, we encounter a number of fundamental changes in the properties of the universe:

- The universe changes from a radiation– to a matter–dominated state: $\rho_m > \rho_r$.
- The expansion rate of the universe **increases**.
- There is no longer sufficient energy in the high–energy tail of the photon distribution to maintain protons and electrons in a plasma state – they **recombine** to form hydrogen.
- Baryonic matter and radiation **decouple**, i.e. the rate of interaction drops to the order of the Hubble time or greater. The radiation from this epoch travels to us unimpeded from this epoch. For this reason, early texts refer to the CMB as "fossil radiation".

[3] We conveniently ignore the effect of Λ for the present

3.4 The creation of the CMB

We have seen that at a redshift $z = 3750$ the universe changed from a radiation dominated state to a matter dominated state. The temperature of the radiation field at this time was 10,000 K. The energy of a photon of wavelength λ_{max} at this epoch was 4.3 eV – less than the energy required to convert a hydrogen atom into a free electron and proton. However, given the large number of photons compared to each baryon and the form of the Planck function, there existed sufficient higher energy photons to maintain the baryonic contents of the universe in an ionised state. Under these conditions the dominant interactions consisted of photon-electron interactions via Thomson scattering and electron-proton interactions via Coulomb scattering. These reactions linked the massive protons to the high–energy photons in what is known as a **photon–baryon fluid** and maintained thermodynamic equilibrium between photons and baryons.

The creation of the CMB involves two linked processes – **decoupling** and **recombination**. To illustrate the importance of each process we will first demonstrate that decoupling alone is not responsible for the creation of the CMB. We will then demonstrate that the introduction of recombination to the process of decoupling produces a consistent answer.

3.4.1 Decoupling

The epoch of decoupling can be defined as the epoch after which the rate of photon-electron interactions decreases below the expansion rate of the universe. Put another way, decoupling occurs when the universe expands faster than a photon can find an electron to scatter from.

The length between successive photon-electron scattering events is the mean free path for Thomson scattering, i.e.

$$l = \frac{1}{n_e \sigma_e}, \tag{18}$$

where n_e is the proper electron density and $\sigma_e = 6.65 \times 10^{-29}$ m^2 is the Thomson scattering cross section. The rate of photon-electron scattering is then simply the inverse time between successive interactions, i.e.

$$\Gamma = \frac{c}{l} = n_e \sigma_e c. \tag{19}$$

As the total number of electrons is constant, the proper electron density at some epoch is related to the present day density by the relation

$$n_e(t) = n_e(t_0)(1+z)^3, \tag{20}$$

and the scattering rate as a function of redshift is

$$\Gamma(z) = n_{e,0} \sigma_e c (1+z)^3. \tag{21}$$

Decoupling occurs when $\Gamma(z) = H(z)$ and the problem reduces to determining the redshift evolution of the Hubble parameter in the appropriate model universe. Rearranging the Friedmann equation for a flat universe gives

$$\begin{aligned} \frac{H(z)^2}{H_0^2} &= \Omega_{m,0}(1+z)^3 \quad \text{in a matter-dominated universe} \\ &= \Omega_{r,0}(1+z)^4 \quad \text{in a radiation-dominated universe.} \end{aligned} \tag{22}$$

If we look at the epoch $z = z_{rm} = 3570$ – the changeover between matter and radiation dominance – we may write $H(z) = H_0\sqrt{\Omega_{r,0}}(1+z)^2$ and therefore the decoupling condition is

$$\begin{aligned} \Gamma(z_{rm}) &= H(z_{rm}) \\ n_{e,0}\sigma_e c(1+z_{rm})^3 &= H_0\sqrt{\Omega_{r,0}}(1+z_{rm})^2. \end{aligned} \tag{23}$$

Inserting values for the lhs of the equation we obtain

$$\Gamma(z_{rm}) = (0.22 \text{ m}^{-3})(6.65 \times 10^{-29} \text{ m}^2)(3 \times 10^8 \text{ ms}^{-1})(3571)^3 = 2 \times 10^{-10} \text{ s}^{-1} \tag{24}$$

The number of scatterings per unit time is $n_\gamma\Gamma$ and is approximately equal to one interaction per second. Inserting numbers for the rhs of the equation we obtain

$$H(z_{rm}) = (2.27 \times 10^{-18} \text{ s}^{-1})(\sqrt{5 \times 10^{-5}})(3571)^2 = 2 \times 10^{-13} \text{ s}^{-1}. \tag{25}$$

As the quantity H^{-1} is the time taken for the scale for the universe to double at any epoch we note that at the epoch of radiation-matter equality the universe takes approximately 160,000 years to double in size. We note that $\Gamma(z_{rm}) \gg H(z_{rm})$ and that decoupling is not an issue in a radiation dominated universe.

To obtain the actual redshift we at which decoupling occurs – z_d – we set the condition

$$\begin{aligned} \Gamma(z_d) &= H(z_d) \\ n_{e,0}\sigma_e c(1+z_d)^3 &= H_0\sqrt{\Omega_{m,0}}(1+z_d)^{3/2} \\ (1+z_d)^{3/2} &= \frac{H_0\sqrt{\Omega_{m,0}}}{n_{e,0}\sigma_e c} \\ z_d &= \left(\frac{H_0\sqrt{\Omega_{m,0}}}{n_{e,0}\sigma_e c}\right)^{2/3} + 1 = 42. \end{aligned} \tag{26}$$

At a redshift $z = 42$ the temperature of the CMB is of order 120K and the energy per photon is several orders of magnitude lower than the ionisation potential of Hydrogen. Under these conditions the baryonic contents of the universe would exist as neutral Hydrogen gas. As we shall see in the next section, if we break the assumption $n_e = n_{e,0}(1+z)^3$ = constant (bearing in mind that **free electrons** are required for Thomson scattering), we greatly reduce the scattering rate and push decoupling to a higher redshift.

3.4.2 Recombination

Free electrons may combine with protons to form atomic Hydrogen via the reaction

$$p + e^- \rightleftharpoons H + \gamma. \tag{27}$$

This reaction is reversed if the incoming photon possess an energy exceeding the ionisation potential of Hydrogen $Q_H = 13.6$ eV. To describe the statistical balance between each of these two states (free electrons and protons versus atomic Hydrogen) we require a particular case of Saha's ionisation equation relating the number density of electrons (n_e), protons ($n_p = n_e$) and Hydrogen atoms (n_H) at a given radiation temperature T, i.e.

$$\frac{n_e^2}{n_H} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{Q}{kT}\right), \tag{28}$$

where m_e is the electron mass and k is Boltzmann's constant. Expressing the ionized fraction as $x = n_e/n_b$ and employing $n_H = n_b - n_e$, one may write

$$\frac{X^2}{1-X} = \frac{1}{n_b}\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{Q}{kT}\right). \tag{29}$$

As $n_b \propto \Omega_b$ (and is known) the recombination epoch may be reduced to a dependence upon the radiation temperature. Setting the condition for recombination as $X = 0.5$ we note from Figure 1 that $T_{recomb} = 4000$K and that

$$z_{recomb} = \left(\frac{4000}{2.7} - 1\right) = 1480. \tag{30}$$

It is straightforward to demonstrate that $\Gamma(z_{recomb}) \gg H(z_{recomb})$ and that recombination precedes decoupling. To obtain a more precise estimate of the decoupling redshift we note that assuming $\Omega_{b,0} = 0.04$ the scattering rate becomes

$$\Gamma(z) = 4.4 \times 10^{-21} X(z)(1+z)^3 \text{ s}^{-1}. \tag{31}$$

In a similar manner, taking $H_0 = 70$ kms^{-1}Mpc^{-1} and $\Omega_{m,0} = 0.3$, the Hubble parameter becomes

$$H(z) = 1.24 \times 10^{-18} (1+z)^{3/2} \text{ s}^{-1}. \tag{32}$$

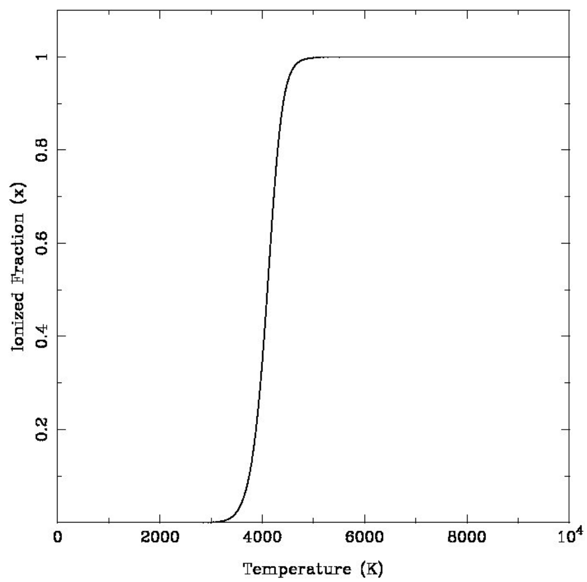

Figure 1: The ionised fraction of Hydrogen as a function of radiation temperature.

By setting $\Gamma(z_d) = H(z_d)$ and solving for z_d we obtain the relation

$$1 + z_d = \frac{43}{X(z_d)^{2/3}}. \tag{33}$$

This equation may be solved using Newton-Raphson or graphically and one obtains $z_d \sim 1200$. As noted in Ryden, the Saha equation assumes that the process given by Equation 27 is in equilibrium. However, as decoupling commences the photo-ionisation reaction move rapidly away from equilibrium. A more detailed non-equilibrium analysis of decoupling under these conditions reveals that $z_d \sim 1100$.

3.4.3 Last scattering

Recombination was not instantaneous. Alternatively, the surface of last scattering was not infinitely thin. The source of ionizing photons during recombination was within the Maxwellian tail of the CMB Black Body spectrum. The finite length of this tail implies that a finite amount of time (measured via universal expansion) was required to pass from the state $n_\gamma(kT > 13.6\text{eV}) \simeq n_e$ to $n_\gamma(kT > 13.6\text{eV}) \sim 0$. The width of the last scattering surface is $\Delta z \simeq 90$. This may also be expressed via a plot of the probability of last scattering versus redshift. The main observation effect of the finite width of the last scattering surface is to blur the two-dimensional project of features in the CMB comparable to the width ($\theta \sim 0.1^\circ$).

3.4.4 Timeline of CMB creation

Event	redshift	temperature (K)	time (years)
radiation-matter equality	3570	9730	47,000
recombination	1370	3740	240,000
decoupling	1100	3000	350,000
last scattering	1100	3000	350,000

Table 1: Timeline of CMB creation

3.5 CMB anisotropies

3.5.1 Measuring CMB anisotropies

The observed pattern of CMB temperature variations on the sky can be thought of as a signal of varying amplitude on the surface of a sphere. We denote the CMB temperature in the sky direction $\hat{\mathbf{n}}$ as $T(\hat{\mathbf{n}})$. The CMB variations over the sky are considered as a Gaussian random field. The dimensionless temperature variation is

$$\Delta(\hat{\mathbf{n}}) \equiv \frac{T(\hat{\mathbf{n}})}{\langle T(\hat{\mathbf{n}}) \rangle} - 1, \tag{34}$$

which is the equivalent of the term $\Delta T/T$. The **angular correlation function** of CMB temperature variations is defined as

$$c(\theta) = \langle \Delta(\hat{\mathbf{n}})\ \Delta(\hat{\mathbf{n}'}) \rangle, \tag{35}$$

where $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}'} = \cos\theta$. Rather like Fourier analysis permits a signal in linear space to be decomposed into a linear series of sine and cosine terms, spherical harmonics permit a signal on the surface of a sphere to be decomposed into a series of linear spherical harmonics, i.e.

$$\Delta = \frac{\Delta T}{T} = \sum_{l,m} a_l^m Y_l^m(\theta, \phi), \tag{36}$$

where a_l^m is the coefficient describing the contribution of the term $Y_l^m(\theta, \phi)$ – see "spherical harmonics" the web for the more detailed form of $Y_l^m(\theta, \phi)$. Each spherical harmonic mode is often referred to by the l mode. To visualise each mode, note that $l = 1$ refers to the mean CMB level and $l = 2$ refers to the dipole signal (both removed in the anisotropy analysis). The square of the coefficients describing the terms $l > 2$ define the **CMB angular power spectrum**, i.e.

$$C_l = \langle |a_l^m|^2 \rangle. \tag{37}$$

Inflationary models of the universe state that density/temperature fluctuations on all scales are independent. This is akin to saying that the set of a_l^m are independent. In this case, the set C_l provides a complete statistical description of the temperature variations. The **angular correlation function** (what is actually measured) is related to the **angular power spectrum** (the useful theoretical quantity) by the relationship

$$c(\theta) = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)C_l P_l(\cos\theta). \tag{38}$$

Therefore, given the map of CMB temperature variations at $l > 2$, one fits for the individual coefficients a_l^m and generates the power spectrum from the resulting terms. One can immediately see that the greatest temperature variations (essentially the squared deviation from the mean) occur at well defined angular scales (note that $\theta \sim 1/l$).

3.5.2 The physics of CMB anisotropies

To understand the distribution of temperature variations in the CMB, we must consider the interaction between protons and electrons (a plasma) and photons prior to the epoch of decoupling. Anisotropies in the CMB largely arise from matter density fluctuations. Although at early times photons dominate dominate the total density, small dark matter fluctuations grow with time. As baryons and linked photons fall into a given potential, the photon density increases, with a corresponding increase in the radiation pressure. The outward force due to radiation pressure eventually overcomes the inward force gravity and causes the baryons and photons to expand once again. Photons that have fallen into a potential well generate compression peaks that appear as cooler regions in CMB temperature maps. This occurs as the photons lose energy via gravitational redshift in the process of travelling to the observer. Photons located in underdense regions of the universe at the time of decoupling generate rarefaction peaks that appear as hotter regions in CMB temperature maps. This occurs as the photons are blueshifted relative to an average photon on the way to the observer. Note that the power spectrum has units of absolute temperature variation so both hot and cool spots generate a positive signal.

Each potential does not exist in isolation – the density field sets up an oscillating medium. What is important is that the density field displays fluctuations on all scales. This leads to a spectrum of oscillations on (effectively) all wavelength/frequency scales.

The first peak in the CMB is akin to the “blast wave” of material pushed outwards due to radiation pressure. Travelling at the sound speed of the photon baryon fluid, the scale of this feature is set by the sound horizon at the age of the universe at last scattering. Potentials on a length scale equal to approximately half of the sound horizon have just enough time to expand once and compress once again before recombination to form a matter + radiation enhancement and thus a temperature decrement. Potentials on this scale generate the second, fourth, etc. acoustic peaks in the CMB.

The wavenumber of the 1st peak, k_1, is defined as π divided by the distance the pressure wave (sound) can travel before recombination (the sound horizon), i.e. $k_1 = \pi/d_s$ and $\lambda_1 = 2d_s$. Remember that c_s is the sound speed and is $c_s \simeq c/\sqrt{3}$ for a relativistic medium.

- **Aside: computing the angular scale of the first peak**

  As the sound speed for a relativistic medium is $c_s \simeq c/\sqrt{3}$, we can compute the sound horizon at recombination employing a modified version of the light horizon. Recalling that $a \propto t^{2/3}$ for a matter dominated universe, the sound horizon reduces to $r_{SH} = (3/\sqrt{3})\ ct_0$. Assuming the age of the universe at recombination to be $t_0 \sim 350,000$ years, the sound horizon is approximately 3.8×10^{21}m. The angular diameter distance to the redshift of last scattering is $d_A(z_{ls} = 1100) = 13$Mpc for a low-density, Λ–dominated universe. Dividing the physical scale corresponding to the first peak by the angular diameter distance to the last scattering surface generates an angular scale of a few degrees. The observed value is approximately one degree – so, to first order, the above simple calculation is instructive.

Potentials with half the wavelength (twice the wavenumber) of this scale have enough time to rarify and compress once, forming a temperature enhancement (remember that CMB power is measured as the square, i.e. absolute value, of a given fluctuation). The third peak represents rarefaction–compression–rarefaction, etc. Intermediate modes are also oscillating but are caught closer to the zero position (no compression/rarefaction). Thus the harmonic series of k_1 dominates.

These spatial inhomogeneities are observed as angular inhomogeneities as our growing horizon encompasses an increasing volume of the universe and an increasingly complex (yet correlated) spatial fluctuation pattern.

Observation of a harmonic series of peaks in the power spectrum provides strong observational confirmation of theory of a primordial density field which expanded during an early inflationary phase. Inflation predicts a scale–independent density field whereas theories such as cosmic defects (strings, monopoles) have a hard time explaining this harmonic series.

The **1st peak** is located at a characteristic scale proportional to sound speed. This implies a characteristic length scale which, when converted to an angular scale, provides a constraint upon flatness. What one actually does is to calculate the physical transverse scale based upon the physics of the photon–baryon fluid and having observed the angular scale, one may estimate the angular diameter distance to the CMB. The redshift of the CMB is measured knowing the radiation and baryon densities and by following the discussion in Section 3.

The relative height of the **2nd peak** compared to the 1st constrains the baryon density. Additional baryons enhance compression peaks (even) w.r.t. rarefaction peaks (odd) as, for a given potential, the photons experience a tighter coupling to the baryons. Additional baryons also damp higher order peaks on increasingly larger scales.

The properties of the **3rd peak** constrain the dark matter density. The radiation density is fixed by observation of the CMB itself. Peaks one and two constrain the total matter plus baryon density. The combination of all these leaves the DM density. The ratio of total matter density to radiation density determines when the universe was radiation dominated. When the universe was radiation dominated the effects of radiation pressure were much greater than the effects of the potentials resulting in larger amplitude fluctuations at scales that were oscillating at this epoch (smaller scales). Therefore, the relative height of peak 3 to peaks 1 and 2 constrains the total DM density.

Higher peaks - the appearance of higher peaks are increasingly affected by photon diffusion, i.e. the distance covered by the random motion of photons during the time take for recombination to occur blurs the acoustic signal on angular scales corresponding to that length – hot and cold photons mix (actually they are exponentially damped). More baryons lowers the random walk distance thus reducing the damping effect. Increasing the age of the universe (depends upon DM density) increases the damping scale.

Estimating cosmological parameters via observation of CMB anisotropies is complicated by statistical and instrumental limitations and secondary physics: For each angular mode l there are only $2(l + 1)$ independent modes on the sky – cosmic variance. Any given survey will only cover some fraction of the sky f_{sky} leading to reduced statistical power – sample variance. Inference of cosmological parameters from the CMB must take into account the finite beam size (resolution) of the instrument used to perform the observations. The beam size effectively smoothes the CMB on scales equal to the instrumental resolution. This is why COBE (resolution $\sim 7^\circ$) was only able to set the scale of the initial conditions, i.e. the rough amplitude of the temperature fluctuations, and not the acoustic peaks.

Many additional physical processes affect the CMB power spectrum. The most important being the process of universal Re-ionisation and gravitational waves. Re–ionisation acts to blur temperature features on scales corresponding to the horizon at that epoch.

Future CMB missions (e.g. continuing WMAP and launching Planck in 2007) aim to extend the SNR and resolution of higher order peaks and thus further refine the current (exceptional) constraints upon cosmological parameters.