CHAPTER 6
THE CELESTIAL SPHERE

6.1. *Introduction*

If you look up in the sky, it appears as if you are at the centre of a vast crystal sphere with the stars fixed on its surface. This sphere is the *celestial sphere*. It has no particular radius; we record positions of the stars merely by specifying angles. We see only half of the sphere; the remaining half is hidden below the *horizon*. In this section we describe the several coordinate systems that are used to describe the positions of stars and other bodies on the celestial sphere, and how to convert between one system and another. In particular, we describe *altazimuth*, *equatorial* and *ecliptic coordinates* and the relations between them. The relation between ecliptic and equatorial coordinates varies with time owing to the *precession of the equinoxes* and *nutation*, which are also described in this chapter.

6.2. *Altazimuth Coordinates.*

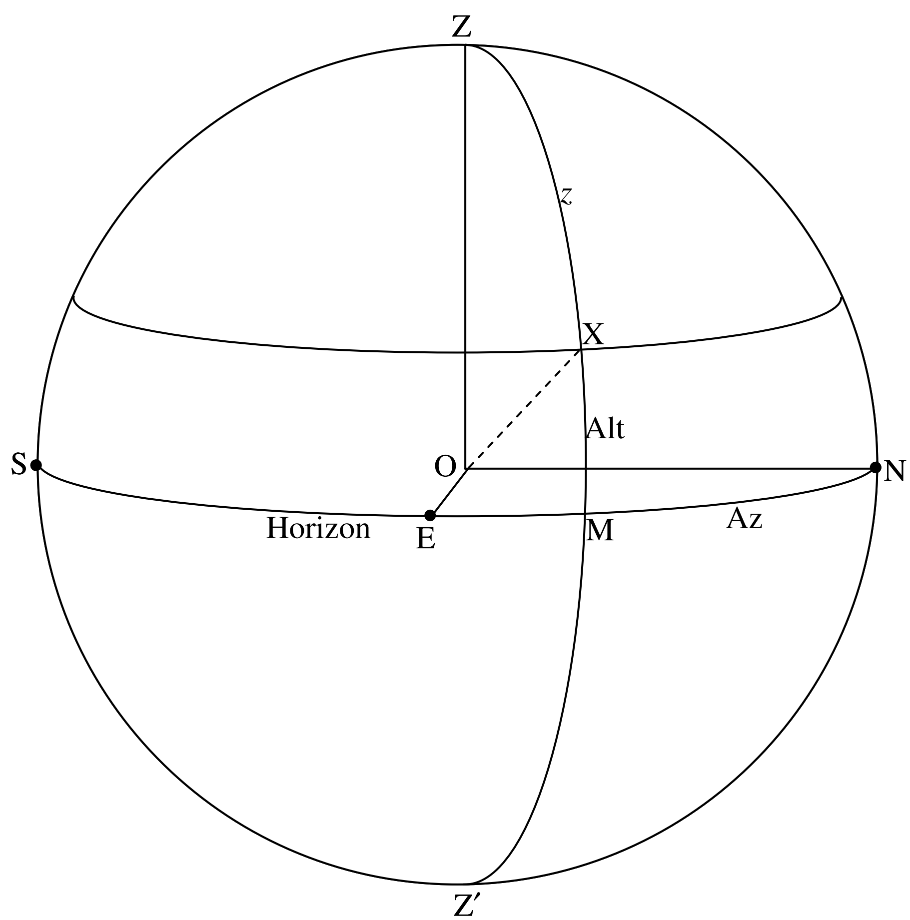


FIGURE VI.1

In figure VI.1 we see the celestial sphere with the observer O at its centre. The point immediately overhead, Z, is the *zenith*. The point directly underneath, Z′, is the *nadir*. The points marked N, E, S are the *north*, *east* and *south points of the horizon*. The west point of the horizon is behind the plane of the paper (or of your computer screen) and is not drawn. The great circle NESW is, of course, the *horizon*.

Any great circle passing through Z and Z′ is called a *vertical circle*. The vertical circle passing through S and N, the south and north points of the horizon, is the *meridian*. The vertical circle passing through the east and west points of the horizon (which I have not drawn) is the *prime vertical*. X is the position of a star on the celestial sphere, and I have drawn the vertical circle ZXMZ′ passing through the star. The angle MX is the *altitude* of the star (also referred to in some contexts as its "elevation"). The complement of its altitude, the angle z, is the *zenith distance* (also called, not unreasonably, the "zenith angle").

A small circle of constant altitude – i.e. a small circle parallel to the horizon – has the curious name of an *almucantar*, and I have drawn the almucantar through the star X. An almucantar can also be called a parallel of altitude.

The angle NM that I have denoted by Az on figure VI.1 is called the *azimuth* (or "bearing") of the star. As drawn on the figure, it is measured eastwards from the north point of the horizon. This is perhaps the most common convention for observers in the northern hemisphere. However, for stars that are west of the meridian, it may often be convenient to express azimuth as measured westwards from the north point. I don't know what the custom is of astronomers who live in the southern hemisphere, but it would not surprise me if often they express azimuth as measured from the south point of their horizon. In any case, it is important not to assume that there is some universal convention that will be understood by everybody, and it is *essential* when quoting the azimuth of a star to add a phrase such as "measured from the north point eastwards". If you merely write "an azimuth of 32 degrees", it is almost certain that you will be either misunderstood or not understood at all.

In the altazimuth system of coordinates, the position of a star is uniquely specified by its azimuth and either its altitude or its zenith distance.

Of course the altitude and azimuth of a star are changing continuously all the time, and they are also different for all observers at different geographical locations.

6.3. *Equatorial Coordinates.*

If you live in the northern hemisphere and if you face north, you will observe that the entire celestial sphere is rotating slowly counterclockwise about a point in the sky close to the star Polaris (α Ursae Minoris). The point P about which the sky appears to rotate is the *North Celestial Pole*. If you live in the southern hemisphere and if you face south you will see the entire sky rotating clockwise about a point Q, the *South Celestial Pole.* There is no bright star near the south celestial pole; the star σ Octantis is close to the south celestial pole, but it is only just visible to the unaided eye provided you are dark adapted and if you have a clear sky free of light pollution. The great circle that is 90° from either pole is the *celestial equator*, and it is the projection of Earth's equator on to the celestial sphere.

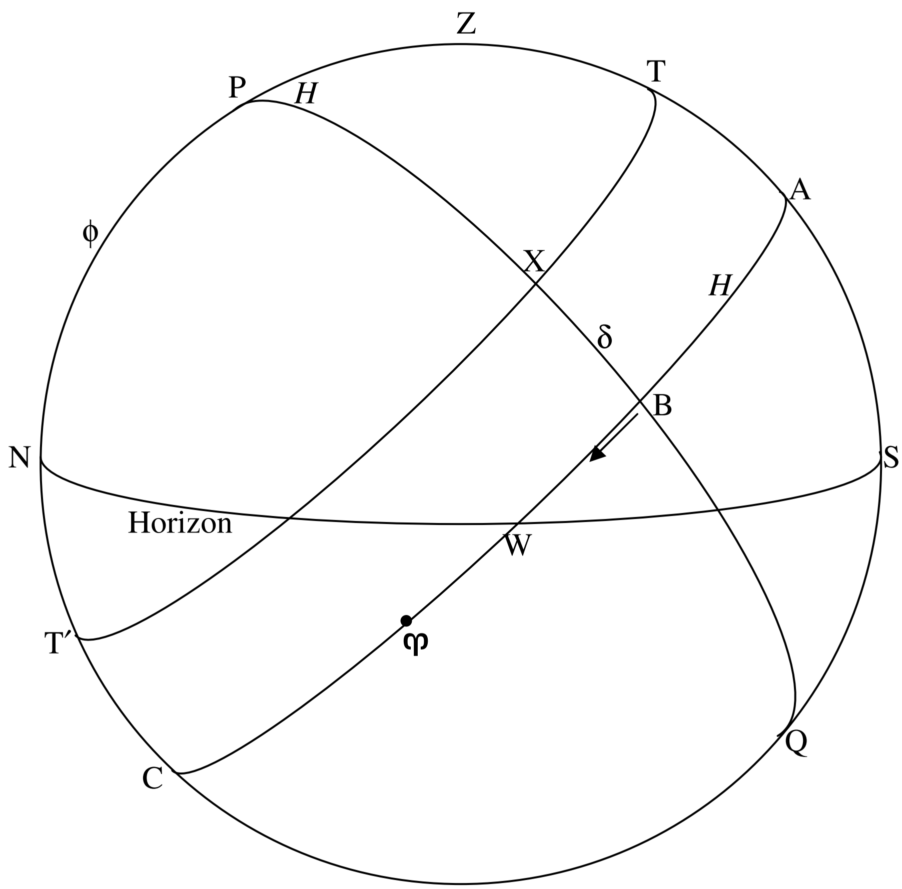


FIGURE VI.2

In figure VI.2 I have drawn the celestial sphere from the opposite side from the drawing of figure VI.1, so that, this time, you can see the west point of the horizon, but not the east point. The celestial equator is the great circle ABW♈C.

You might possibly have noticed that, in section 2, I had not properly defined the north point of the horizon other than by saying that it was the point marked N in figure VI.1. We see now that the north and south points of the horizon are the points where the vertical circle that passes through the celestial poles (i.e. the meridian) meets the horizon.

The altitude ϕ of the north celestial pole is equal to the geographical north latitude of the observer. Thus for an observer at Earth's north pole, the north celestial pole is at the zenith, and for an observer at Earth's equator, the north celestial pole is on the horizon.

You will see that a star such as X transits across the meridian twice. *Lower meridian transit* occurs at the point T′, when the star is north of the observer and is directly below the north celestial pole. For the star X of figure VI.2, lower meridian transit is also below the horizon, and it cannot be seen. The star reaches its highest point in the sky (i.e. it *culminates*) at upper meridian transit.

The first quantitative astronomical observation I ever did was to see how long the celestial sphere takes to rotate through 360°. This is best done by timing the interval between two consecutive upper meridian transits of a star. It will be found that this interval is $23^h\ 56^m\ 04^s.099$ of mean solar time, although of course it requires more than a casual observation to determine the interval to that precision. The rotation of the celestial sphere is, of course, a reflection of the rotation of Earth on its axis. In other words, this interval is the sidereal (i.e. relative to the stars) rotation period of Earth.

We are now in a position to describe the position of a star on the celestial sphere in *equatorial coordinates.* The angle δ in figure VI.2 is called the *declination* of the star. It is usually expressed in degrees, arcminutes and arcseconds, from 0° to $+90^\circ$ for stars on or north of the equator, and from 0° to -90° for stars on or south of the equator. When quoting the declination of a star, the *sign* of the declination must *always* be given.

When the star X in figure VI.2 is at lower meridian transit, it is below the horizon and is not visible. However, if the declination of a star is greater than $90^\circ - \phi$, the star will not reach the horizon and it will never set. Such stars are called *circumpolar* stars.

The second coordinate is the angle *H* in figure VI.2. It is measured westward from the meridian. It will immediately be noticed that, while the declination of a star does not change through the night, its hour angle continuously increases, and also the hour angle of a star at any given time depends on the geographical longitude of the observer. While hour angle could be expressed in either radians or degrees, it is customary to express the hour angle in hours, minutes and seconds of time. Thus hour angle goes from 0^h to 24^h. When a star has an hour angle of, for example, 3^h, it means that it is three sidereal hours since it transited (upper transit) the meridian. Conversion factors are

$$1^h = 15^\circ \qquad 1^m = 15' \qquad 1^s = 15'' \qquad 1^\circ = 4^m \qquad 1' = 4^s.$$

(The reader may have noticed that I have just used the term "sidereal hours". For the moment, just read this as "hours" – but a little later on we shall say what we mean by "sidereal" hours, and you may then want to come back and re-read this.)

While it is useful to know the hour angle of a star at a particular time for a particular observer, we still need a coordinate that is fixed on the celestial sphere. To do this, we refer to a point on the celestial equator, which I shall define more precisely later on, denoted on figure VI.2 by the symbol ♈. This is the astrological symbol for the sign Aries, and it was originally in the constellation Aries, although at the present time it is in the constellation Pisces. In spite of its present location, it is still called the *First Point of Aries*. The angle measured eastward from ♈ to the point B is called the *right ascension* of the star X, and is denoted by the symbol α. This does not change (at least not very much – but we shall deal with small refinements later) during the night or from night to night. Thus we can describe the position of a star on the celestial sphere by the two coordinates δ, its declination, and α, its right ascension, and since its right ascension does not change (at least not very much), we can list the right ascensions as well as the declinations of the stars in our catalogues. The right ascension of the First Point of Aries is, of course, 0^h.

I have hinted in the last paragraph that the right ascension of a star, although it doesn't change "very much" during a night, does change quite perceptibly over a year. We shall have to return to this point later. I have not as yet precisely defined where the point ♈ is or how it is defined, but we shall later learn that it is not quite fixed on the equator, but it moves slightly in a manner that I shall have to describe in due course. Thus the entire system of equatorial coordinates, and the right ascensions and declinations of the stars, depends on where this mysterious First Point of Aries is. For that reason, it is always necessary to state the epoch to which right ascensions and declinations are referred. For much of the twentieth century, equatorial coordinates were referred to the epoch 1950.0 (strictly it was B1950.0, but I shall have to postpone explaining the meaning of the prefix B). At present catalogues and atlases refer right ascensions and declinations to the epoch J2000.0, where again I shall have to defer an explanation of the prefix J. While there is evidently some further explanation yet to come, suffice it to say at this point that, when giving the right ascension and declination of any object, it is *essential* that the epoch also be given. The First Point of Aries moves very, very slowly westward relative to the stars, so that the right ascensions of all the stars are increasing at a rate of about $0^s.008$ per day. This does not amount to much for day-to-day purposes, but it does emphasize why it is always necessary to state the epoch to which right ascensions and declinations of stars are quoted. It also means that, if you were able to observe two consecutive upper transits of ♈ across the meridian, the interval would be $0^s.008$ shorter than the sidereal rotation period of Earth. It would be, in fact, $23^h\ 56^m\ 04^s.091$. This interval between two consecutive upper meridian transits of the First Point of Aries, is called a *sidereal day.* (It might be thought that, since the word "sidereal" implies "relative to the stars", this is not a particularly good term. I would have sympathy with this view, and would prefer to call the interval an "equinoctial day". However, the term *sidereal day* is so firmly entrenched that I shall use that term in these notes.) A sidereal day is divided into 24 *sidereal hours*, which are shorter than mean solar hours by a factor of 0.99726957. We shall discuss the motion of ♈ in more detail in a later section. At this stage no great harm is done by considering ♈ in the first approximation to be fixed relative to the stars.

Now some more words. Small circles parallel to the celestial equator (such as the small circle T′XT in figure VI.2) are *parallels of declination*. Great circles that pass through the north and south celestial poles (for example the great circle PXBQ of figure VI.2) and which are fixed on and rotate with the celestial sphere are called by a variety of names. Some call them *declination circles*, because you measure declination up and down these circles. Others call them *hour circles*, because the hour angle or right ascension is constant along them. For those who find it confusing that a given circle can be called either a declination circle or an hour circle, you can get around this difficulty by calling them *colures*. The colure that passes through the First Point of Aries and the diametrically opposite point on the celestial sphere, and which therefore has right ascensions 0^h and 12^h, is the *equinoctial colure.* The colure that is 90^o from this (or, rather, 6 hours from this) and which has right ascensions 6^h and 18^h, is the *solstitial colure*.

The time that has elapsed, in sidereal hours, since the First Point of Aries transited (upper transit) the meridian, that is to say the hour angle of the first point of Aries, or the angle from A to ♈ in figure VI.2, is called the *Local Sidereal Time*. It is evident from figure VI.2 that the Local Sidereal Time is also equal to ♈B + AB. But ♈B is the right ascension of the star X and AB is its hour angle. Therefore *the local sidereal time* (the hour angle of the First Point of Aries) *is equal to the right ascension of any star plus its hour angle.*

The sidereal time at the longitude of Greenwich (0^o longitude) is tabulated daily in the *Astronomical Almanac* and the local sidereal time at your location is equal to the local sidereal time at Greenwich minus your geographical longitude. Most observatories have two clocks running in the dome at all times. One gives Universal Time, while the other, which runs a little faster, gives the local sidereal time. But you always have a sidereal clock available, for a glance at figure VI.2 will tell you that the local sidereal time is equal to the right ascension of stars at upper meridian transit.

6.4. *Conversion between Equatorial and Altazimuth Coordinates.*

Whereabouts in the sky will a given star be at a certain time? This as a typical problem involving conversion between equatorial and altazimuth coordinates. We have to solve a spherical triangle. That is no problem – we already know how to do that. The problem is: which triangle?

The first problem, however, arises from the phrase "at a certain time". In particular, if we want to know where a star is, for example, at 2002 November 24, at 10:00 p.m. Pacific Standard Time as seen from Victoria, whose longitude is $123^o\ 25'.0$ W, we need to know the *local sidereal time* at that instant.

The calculation might go something like this.

From the *Astronomical Almanac* we find that the local sidereal times at Greenwich at 0^h UT on November 25 and 26, 2002, are

November 25: $04^h\ 16^m\ 59^s$
November 26: 04 20 56

We want the local sidereal time at November $24^d\ 22^h\ 00^m$ PST
= November $25^d\ 06^h\ 00^m$ UT

By interpolation we find that the local sidereal time at Greenwich at that instant is $10^h\ 17^m\ 58^s$.

The longitude of Victoria is $08^h\ 13^m\ 40^s$, and therefore the local sidereal time at Victoria is $02^h\ 04^m\ 18^s$.

We have overcome the first obstacle, and we now know the local sidereal time (LST).

We'll ask ourselves now what are the altitude and azimuth of a star whose right ascension and declination are α and δ. We also need the latitude of the observer (= altitude of the north celestial pole), which I'll call ϕ. The hour angle H of the star is LST $-\alpha$.

The triangle that we have to solve is the triangle PZX. Here P, Z and X are, respectively, the north celestial pole, the zenith and the star. That is, we solve the triangle formed by *the star and the poles of the two coordinate systems* of interest. I draw the celestial sphere in figure VI.3 as seen from the west. I have marked in the hour angle H, the co-declination $90^\circ - \delta$, the altitude ϕ of the pole, the zenith distance z and the azimuth A measured from the north point westwards.

In triangle PZX, we know ϕ, δ and H, so we immediately find the zenith distance z by application of the cosine formula (equation 3.5.2) and the azimuth A from the cotangent formula (equation 3.5.5).

Problem. Show that the hour angle H of a star of declination δ when it sets for an observer at latitude ϕ is given by $\cos H = -\tan\delta\tan\phi$. This will enable you now to find the Local Sidereal Time of starset, since LST = hour angle plus right ascension, and then you can convert to your zone solar time.

Show also that the azimuth A of starset, westward from the north point, is given by $\tan A = -\sin\phi\tan H$.

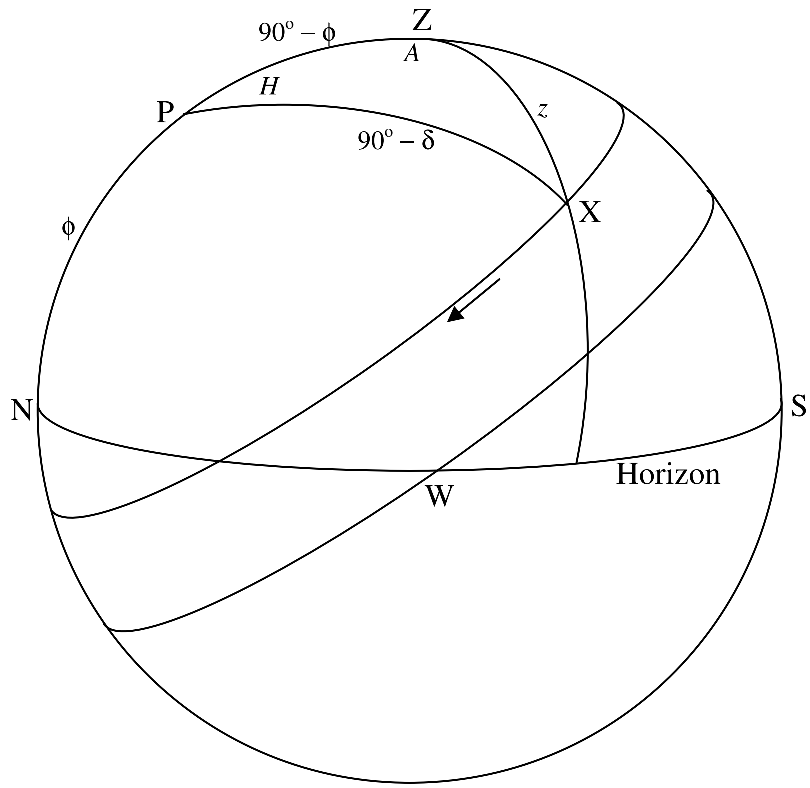


FIGURE VI.3

6.5 *Ecliptic Coordinates.*

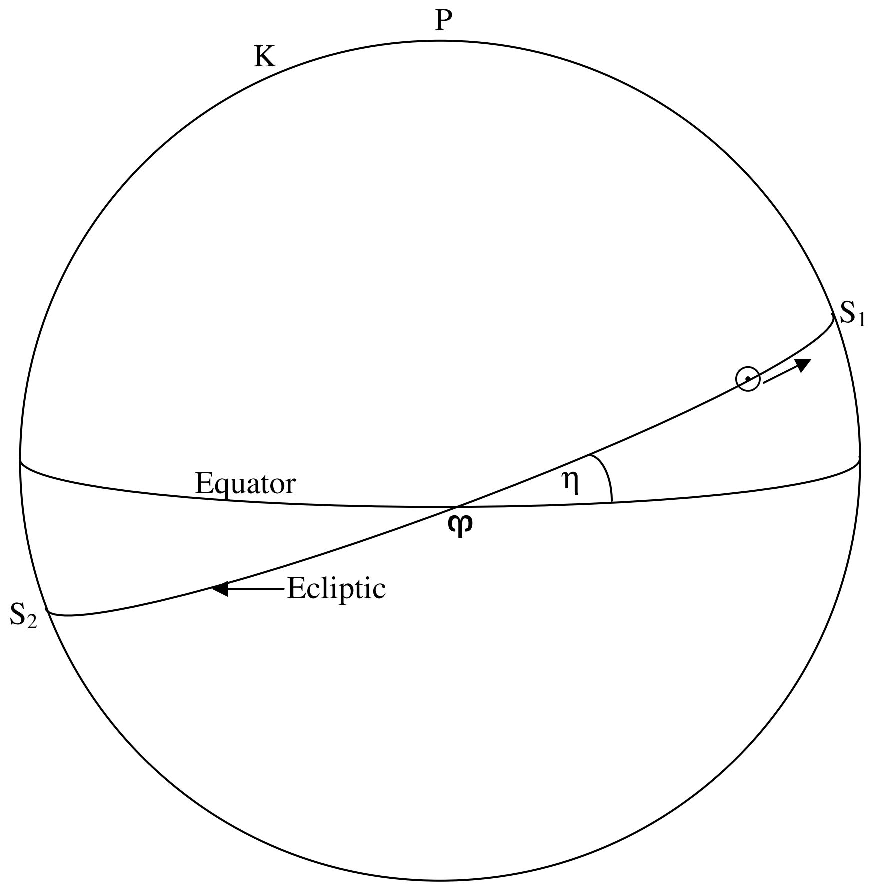


FIGURE VI.4

In figures VI.2 and 3 we were concerned mainly with the daily rotation of the celestial sphere. In figure VI.4 we shall be concerned mainly with the annual motion of the Sun relative to the stars on the celestial sphere. In contrast to figures VI.2 and 3, I have drawn the celestial equator, not the observer's horizon, horizontally, and the north celestial pole, not the observer's zenith, is at the top of the diagram. It is found that, for an observer on Earth, the Sun moves eastward relative to the stars during the course of the year, its right ascension continuously increasing; this apparent motion of the Sun relative to the stars is, of course, a consequence of the Earth revolving around the Sun.

Relative to the stars, it is found that, during the course of a year, the Sun moves eastward along a great circle that is inclined to the equator at an angle of about $23^{\circ}.4$. This great circle is called the *ecliptic*, and it is the projection of the plane of Earth's orbit on the celestial sphere. The angle between the ecliptic and the equator is called the *Obliquity of the Ecliptic.* The ecliptic crosses the equator at two points. The Sun reaches one of these points on about March 22 each year on its way north at which time the Sun's declination changes from negative to positive. This point, the ascending node of the Sun's path on the equator, is the *First Point of Aries*, which we introduced in section 6.3. As mentioned there, and for reasons that will be explained in section 6.7, it is actually in the constellation Pisces rather than Aries. Nevertheless it is still known as the First Point of Aries and is denoted by the astrological symbol ♈ for Aries. It is the point from which right ascensions are measured. The instant of time when the Sun crosses the equator from south to north at the First Point of Aries is the *March Equinox*. Days and nights are of equal length all over the world on that date ("equinox" = "equal night"), and that date marks the first day of Spring in the northern hemisphere. For that reason it is also called the "vernal equinox" (Latin *verna* = "spring") – but that is hardly fair to southern hemisphere astronomers, for it marks the beginning of the southern autumn.

About three months later, on or near June 21, the Sun reaches the point S_1 at the *June Solstice* (called by those who live in the Northern hemisphere, the summer solstice). The declination of the Sun is then at its highest point, +23.4 degrees. At that instant the rate of change of the Sun's declination is zero, which explains the origin of the word "solstice", which implies that the Sun is momentarily standing still. The Sun is then in the constellation Gemini. After a further three months, the Sun has descended back to the equator on its way south, at the *September equinox* (the "autumnal equinox" for northerners) on or near September 23, when the Sun is in the constellation Virgo. And after a further three months the Sun reaches its most southerly declination at the *December solstice* ("winter solstice" to northerners) on or near December 21, when the Sun is in the constellation Sagittarius.

Also drawn in figure VI.4 is the *North Pole of the Ecliptic*, K, which is in Draco. The *South Pole of the Ecliptic* is in Dorado.

The ecliptic and its pole K form the basis of the ecliptic coordinate system, illustrated in figure VI.5. The *ecliptic longitude* λ and the *ecliptic latitude* β of a star X are shown in figure VI.5, which should be self explanatory. In order to convert between equatorial and ecliptic coordinates, the triangle to solve is triangle PKX. The arc KX is $90^\circ - \beta$ and the angle PKX is $90^\circ - \lambda$. What are the arc PK, the arc PX and the angle KPX?

[Answers: PK = η PX = $90^\circ - \delta$ KPX = $90^\circ + \alpha$]

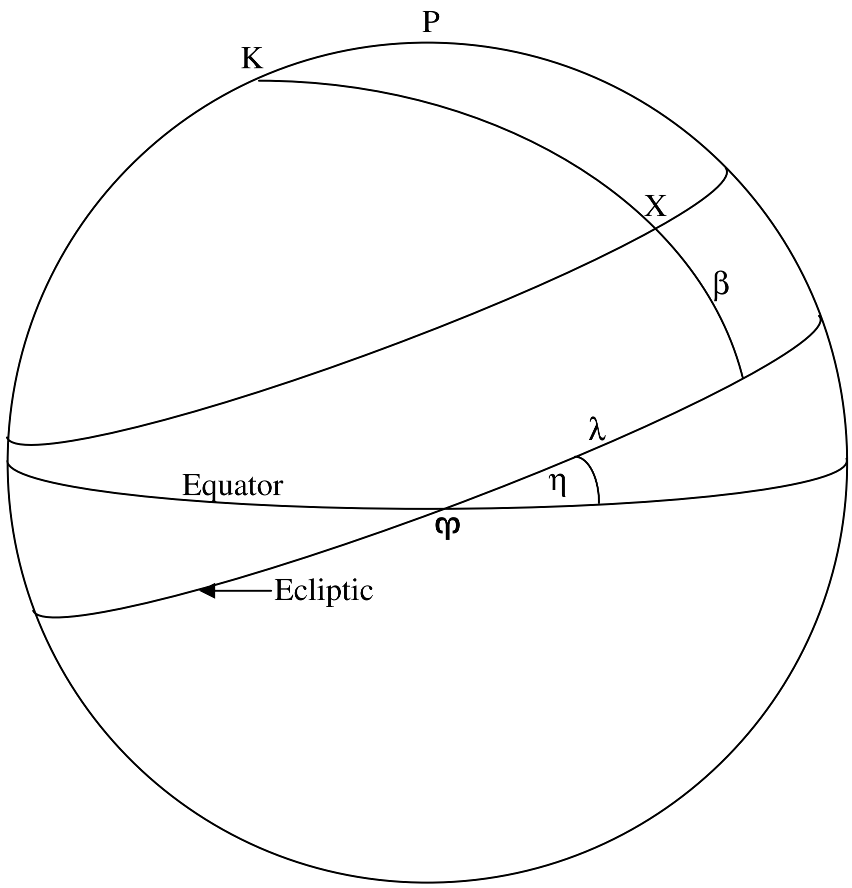


FIGURE VI.5

6.6 *The Mean Sun*

The bright yellow (or white) ball of fire that *appears* in the sky and which you could see with your eyes if ever you were foolish enough to look directly at it is the *Apparent Sun.* It is moving eastward along the ecliptic, and its right ascension is increasing all the time. Consequently consecutive upper transits across the meridian take about four minutes longer than consecutive transits of a star or of the First Point of Aries. The hour angle of the Apparent Sun might have been called the local apparent solar time, except that we like to start our days at midnight rather than at midday. Therefore the *Local Apparent Solar Time* is the *hour angle of the Apparent Sun plus twelve hours*. It is "local", because the hour angle of the apparent Sun depends continuously on the longitude of the observer. It is the time indicated by a sundial. In order to convert it to a standard zone time, we must know, among other things, our longitude.

The Apparent Sun has some drawbacks as an accurate timekeeper, particularly because *its right ascension does not increase at a uniform rate throughout the year.* The motion of the Apparent Sun, is, of course, just a reflection of Earth's annual orbital motion around the Sun. The Earth moves rather faster at perihelion (on or near January 4) than at aphelion (on or near July 4); consequently the Apparent Sun moves faster along the ecliptic in January than in July. Even if this were not so, however, and the Sun were to move at a uniform rate along the ecliptic, its right ascension would not increase at a uniform rate. This is because right ascension is measured along the celestial equator rather than along the ecliptic. If the Sun were moving uniformly along the ecliptic, its right ascension would be increasing faster at the solstices (where its motion is momentarily parallel to the equator) than at the equinoxes, (where is motion is inclined at $23^{o}.4$ to the ecliptic). So there are these two reasons why the right ascension of the apparent Sun does not increase uniformly throughout the year.

To get over these two difficulties we have to invent two imaginary suns. One of them accompanies the apparent (i.e. the real!) Sun in its journey around the ecliptic. The two start together at perihelion. This Dynamic Sun moves at a constant rate, so that the Apparent Sun (which moves faster in January when Earth is at perihelion) moves ahead of the imaginary sun. By the time Earth reaches aphelion in July, however, the Apparent Sun is slowing down, and the Dynamic Sun manages to catch up with the Apparent Sun. After that, the Dynamic Sun surges ahead, leaving the Apparent Sun behind. But the Apparent Sun starts to gain speed again, and catches up again with the Dynamic Sun at perihelion in January. The Apparent Sun and the Dynamic Sun coincide twice per year, at perihelion and at aphelion.

Now we imagine a second imaginary sun – a rather important one, known as the Mean Sun. The Mean Sun moves at a constant rate *along the equator*, its right ascension moving uniformly all through the year. It coincides with the Dynamic Sun at ♈. At this time, the right ascension of the Dynamic Sun is increasing rather slowly, because it is moving along the ecliptic, at an angle to the equator. Its right ascension increases most rapidly at the solstices, and by the time of the first solstice it has caught up with the Mean Sun. After that, it moves ahead of the Mean Sun for a while, but it soon slows down as its motion begins to make an ever steeper angle to the equator, and Dynamic Sun and the Mean Sun coincide again at the second equinox. Indeed these two suns coincide four times a year – at each of the equinoxes and solstices.

Local Mean Solar Time is the *hour angle of the Mean Sun plus twelve hours*, and the difference Local Apparent Solar Time minus Local Mean Solar Time is called the *Equation of Time.* The equation of time is the sum of two periodic functions. One is the *equation of the centre*, which is the difference in right ascensions of the Apparent Sun and the Dynamic Sun, and it has a period of one year. The second is the *reduction to the equator*, which has a period of half a year. The value of the equation of time varies through the year, and it can amount to a little more than 16 minutes in early November. Local Mean Solar Time, while uniform (or as uniform as the rotation of the Earth) still depends on the longitude of the observer. For that reason, all the inhabitants of a zone on Earth roughly between longitudes $7^{o}.5$ East and West agree to use a standard the Local

Mean Solar Time at Greenwich, also called Greenwich Mean Time, GMT, or Universal Time, UT. Similar zones about 15 degrees wide have been established around the world, within each of which the time differs by an integral; number of hours from Greenwich Mean Time.

We shall discuss in Chapter 7 small distinctions between various versions of Universal Time as well as Ephemeris Time and Terrestrial Dynamical Time.

6.7 *Precession*

The First Point of Aries is the point where the ecliptic crosses the equator at the point occupied by the Sun at the March equinox. It is the point from which right ascensions are measured. We have hitherto treated it as if it were fixed relative to the stars, although we have hinted from time to time that this is not exactly so. Indeed we have said that it is essential, when stating the right ascension and declination of a star, to state the date of the equinox to which it refers.

In figure VI.6, I have drawn the ecliptic horizontally, and the celestial equator inclined at an angle of $23^\circ.4$. You can see the north pole of the ecliptic, K, and the north celestial pole P. The great circle P♈ (not drawn) is the equinoctial colure, and the right ascension of ♈ is 0^h. The right ascension and declination of K are 18^h, $+66^\circ.6$, which is a point between the stars δ and ζ Draconis.

Neither the north celestial pole P nor the "First Point of Aries" ♈ are fixed, however. The north celestial pole P describes a small circle of radius $23^\circ.4$ around K, and the equinox ♈ regresses westwards along the ecliptic in a period of 25,800 years. This motion, called the *precession of the equinoxes* (or just "precession" for short) is not quite uniform, but is nearly so and will be treated as such in this section. The complete cycle of 25,800 years corresponds to a westward regression of ♈ along the ecliptic of $50''.2$ per year or $0''.137$ per day. The component of that motion along the celestial equator is $0''.137 \cos 23^\circ.4 = 0''.126 = 0^s.008$ per day. That is why the length of the mean sidereal day (which is defined as the interval between two consecutive upper meridian transits of the first point of Aries) is $0^s.008$ shorter than the sidereal rotation period of Earth.

The precession of P around K means that the entire system of equatorial coordinates (right ascension and declination) moves continuously, and the right ascensions and declinations of all the stars are continuously changing. No matter where P is in its journey around K, however, the equatorial coordinates of ♈ and of K are always 0^h , 0° and 18^h , $+66^\circ.5$. However, equatorial coordinates of the stars must always be referred to the equinox and equator of a stated epoch.

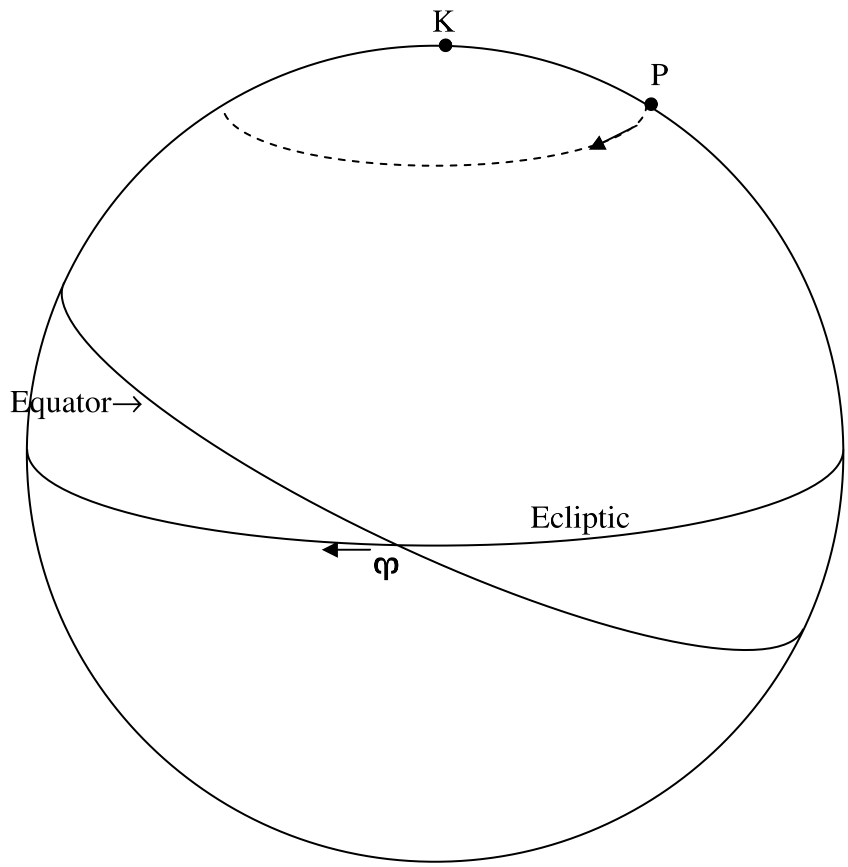


FIGURE VI.6

During much of the twentieth century, the epoch referred to by many catalogues and atlases was B1950.0. That is the beginning of the *Besselian Year* of 1950, at the instant (shortly before midnight on the night of 1949 Dec 31 / 1950 Jan 1) when the right ascension of the Mean Sun was $18^h\ 40^m$. Most catalogues since 1984 have referred right ascensions and declinations to the mean equinox and equator of J2000.0. That is the beginning of the *Julian Year* 2000, at the instant when Greenwich Mean Time (UT) indicated midnight. For example, in the older catalogues, the right ascension and declination of Arcturus would be given as

$$\alpha_{1950.0} = 14^h\ 13^m.4 \qquad \delta_{1950.0} = +19^o\ 26',$$

whereas in more recent catalogues they are given as

$$\alpha_{2000.0} = 14^h\ 15^m.8 \qquad \delta_{2000.0} = +19^o\ 11'.$$

Thus it can be seen that for precise work the difference is not at all negligible, and to state the equatorial coordinates of an object without also stating the epoch of the equinox and equator to which the coordinates are referred is not generally useful. Of course, when setting the circles of a telescope for the night's observations, what one needs are the right ascension and declination referred to the equinox and equator *of date* – i.e. for the date in question. It is therefore essential for a practical observer to know how to correct for precession.

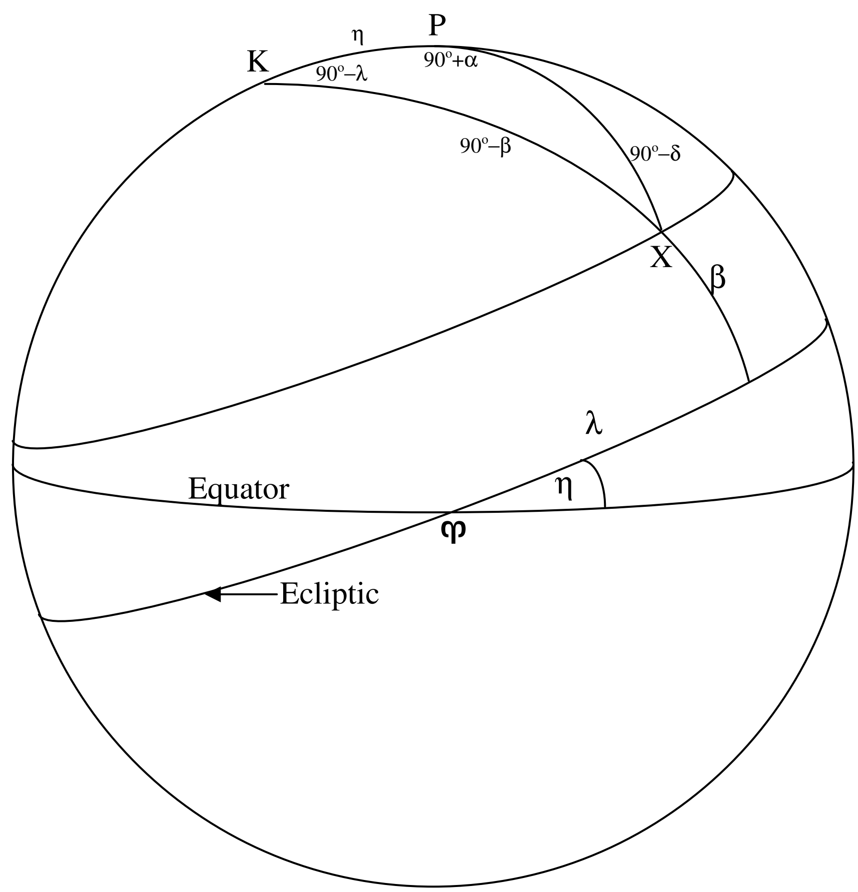


FIGURE VI.7

Apply the cosine formula (equation 3.5.2) to triangle PKX to obtain

$$\sin\delta = \cos\eta\,\sin\beta \;+\; \sin\eta\,\cos\beta\sin\lambda. \qquad 6.7.1$$

Since ♈ is regressing down the ecliptic, the ecliptic longitude λ of the star X is increasing. If it is increasing at a rate $\dot{\lambda}$ (= 50″.2 per year), the rate of change of its declination can be obtained by differentiation of equation 6.7.1 with respect to time, bearing in mind that β and η are constant:

$$\cos\delta\,\dot{\delta} \;=\; \sin\eta\,\cos\beta\,\cos\lambda\,\dot{\lambda}. \qquad 6.7.2$$

But $(\cos\beta\,\cos\lambda)/\cos\delta$ is obtained from the sine formula (equation 3.5.1):

$$\frac{\cos\beta}{\cos\alpha} = \frac{\cos\delta}{\cos\lambda}. \qquad 6.7.3$$

Hence we obtain for the rate of change of declination of a star due to precession:

$$\dot{\delta} = \dot{\lambda}\sin\eta\cos\alpha\,. \qquad 6.7.4$$

To obtain the rate of change of right ascension, we can write equation 6.7.3 as

$$\cos\alpha = \cos\beta\sec\delta\cos\lambda \qquad 6.7.5$$

and then differentiate with respect to time:

$$-\sin\alpha\,\dot{\alpha} = \cos\beta\sec\delta\left(\tan\delta\,\dot{\delta}\cos\lambda - \sin\lambda\,\dot{\lambda}\right), \qquad 6.7.6$$

which I am going to write as

$$-\sin\alpha\,\dot{\alpha} = \cos\beta\sec\delta\cos\lambda\left(\tan\delta\,\dot{\delta} - \tan\lambda\,\dot{\lambda}\right). \qquad 6.7.7$$

We can get $\cos\beta\sec\delta\cos\lambda$ from equation 6.7.5, and of course we have $\dot{\delta}$ from equation 6.7.4, but we still need to find an expression for $\tan\lambda$ in terms of equatorial coordinates. We can do this from the cotangent formula (equation 3.5.4), in which the inner angle is $90^\circ + \alpha$ and the inner side is η:

$$-\cos\eta\sin\alpha = \sin\eta\tan\delta - \cos\alpha\tan\lambda. \qquad 6.7.8$$

On substitution of equations 6.7.4, 6.7.5 and 6.7.8 into equation 6.7.7 we obtain, after a very small amount of algebra, for the rate of change of right ascension of a star due to precession:

$$\dot{\alpha} = \dot{\lambda}\left(\cos\eta + \sin\alpha\tan\delta\sin\eta\right). \qquad 6.7.9$$

With $\dot{\lambda} = 50''.2$ per year and $\eta = 23^\circ.4$, equations 6.7.4 and 6.7.9 become

$$\dot{\delta} = 19''.9\cos\alpha \qquad \text{per year} \qquad 6.7.10$$

and

$$\dot{\alpha} = 46''.1 + 19''.9\sin\alpha\tan\delta \qquad \text{per year} \qquad 6.7.11$$

or

$$\dot{\alpha} = 3^s.07 + 1^s.33\sin\alpha\tan\delta \qquad \text{per year.} \qquad 6.7.12$$

These formulae should be adequate for all but very precise calculations.

Problem: Use equations 6.7.10 and 6.7.12 to verify the data about Arcturus – and please let me know if it isn't right!

At the time of Hipparcos (who discovered the phenomenon of precession as long ago as the second century B.C.), the spring equinox was in the constellation Aries – indeed at its eastern boundary. Hence it was called the First Point of Aries. Over the centuries, precession has carried the equinox westward right across the constellation Aries, and because of this, together with the way in which the constellation boundaries were formally fixed in 1928, the equinox is now near the western boundary of Pisces and is only a few degrees from Aquarius. It is still called, however, by its traditional name of the First Point of Aries.
Incidentally, the ecliptic actually passes through the constellation Ophiuchus, which is not one of the traditional twelve "Signs of the Zodiac", and it is sometimes said that this is a result of precession over the centuries. This is not the case. Precession does not alter the plane of the ecliptic, and the ecliptic continues to pass through the same constellations regardless of where the equinox is along it. The inclusion of Ophiuchus is merely a result of the way in which the constellation boundaries were formally fixed in 1928.

The physical cause of the precession.

The daily motion of the stars around the north celestial pole is, of course, a reflection of Earth's rotation on its axis; and the annual motion of the Sun along the ecliptic, which is inclined at $23^{\circ}.4$ to the celestial equator, is a reflection of the annual orbital motion of Earth around the Sun, the plane of Earth's rotational equator being inclined at $23^{\circ}.4$ to the plane of its orbit – i.e. to the ecliptic. Although this obliquity of $23^{\circ}.4$ is approximately constant, the direction of Earth's rotational axis is not fixed, but it precesses around the normal to the ecliptic plane with a period of 25,800 years.

From the point of view of classical mechanics, Earth is an *oblate symmetric top*. That is to say, it has an axis of symmetry and two of its principal moments of inertia are equal and are less than the moment of inertia about the axis of symmetry. The phenomena of precession of such a body are well understood and are studied in courses of classical mechanics. It is necessary, however, to be clear in one's mind about the distinction between *torque-free precession* and *torque-induced precession*.

The phenomenon of *torque-free precession* is the precession that occurs when a symmetric top is spinning about an axis that does not coincide with its symmetry axis and it is spinning freely with no external torques acting upon it. In such circumstances, the angular momentum vector is fixed in magnitude and direction. The symmetry axis precesses about the fixed angular momentum vector while the instantaneous axis of rotation precesses about the symmetry axis. The rotation of Earth does indeed exhibit this type of behaviour, but this is *not* the precession that we are talking about in connection with the precession of the equinoxes. The instantaneous axis of rotation of Earth is only a very few metres away from its symmetry axis and the period of the torque-free precession is about 432 days. This gives rise to a phenomenon known as *variation of latitude*, and it results in the latitudes of locations of Earth's surface varying quasi-periodically with an amplitude of less than a fifth of an arcsecond. The precession of the equinoxes that we have been discussing in this section is something entirely different.

The figure of Earth is approximately an oblate spheroid. If we call the equatorial radius a and the polar radius c, the *geometric ellipticity* $(a-c)/a$ is about 1/297.0. If we call the corresponding principal moments of inertia A and C, the *dynamical ellipticity* $(C-A)/C$ is about 1/305.1. Earth's equator is inclined to the ecliptic, and, because of the equatorial bulge, the spinning Earth is subject to torques from both the Sun and the Moon (whose orbit is inclined to the ecliptic by about 5 degrees). The magnitude of the torque is proportional to the diameter of Earth times the *gravitational field gradient* $2GM/r^3$, and the direction of the torque vector is perpendicular to the angular momentum vector.

Exercise: Look up the masses of Sun and Moon, and their mean distances from Earth. Show that M/r^3 for the Moon is about twice that for the Sun. Thus the torque on Earth exerted by the Moon is about twice the torque exerted by the Sun.

Now if a symmetric top is spinning about its axis of symmetry with angular momentum **L** and if it is subject to an external torque $\boldsymbol{\tau}$, its angular momentum will change (not in magnitude, but in direction), and **L** will precess with an angular velocity $\boldsymbol{\Omega}$ given by

$$\boldsymbol{\tau} = \boldsymbol{\Omega} \times \mathbf{L}. \qquad 6.7.10$$

Equation 6.7.10 does not give the direction of $\boldsymbol{\Omega}$ uniquely – that depends on the initial conditions. Figure VI.8 illustrates the situation. The equatorial bulge is much exaggerated. The figure is drawn in a reference frame that is revolving around the Sun with the Earth, so there is no net gravitational force on Earth (the gravitational attraction of the Sun is counteracted by the centrifugal force). In this frame, there is a little force F acting towards the Sun on the sunward-facing bulge, and an equal force acting away from the Sun on the opposite side. This amounts to a torque of magnitude $\tau = Fd \sin \eta$, where η is the obliquity of the ecliptic and d is the diameter of Earth. Thus if we equate the magnitudes of both sides of equation 6.7.10, we obtain for the angular speed of the precession

$$\Omega = Fd\,/\,L, \qquad 6.7.11$$

which is independent of η. This, then is the cause of the precession of the equinoxes, except that, for the purpose of figure VI.8, I referred only to the Sun. You have yourself calculated that the influence of the Moon is about twice that of the Sun, and the combined effect of the Moon and the Sun is called the *luni-solar precession.* There is a small additional precession resulting from the influence of the other planets in the solar system.

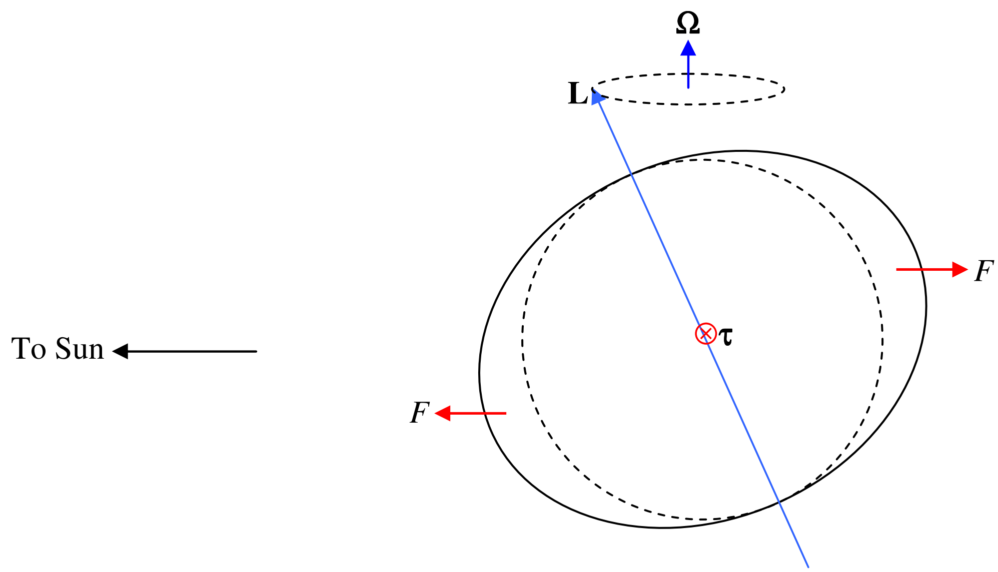


FIGURE VI.8

6.8 *Nutation*

Those who have studied the gyrations of a spinning top will recall that, in addition to precessing, a top may *nutate*, or nod up and down (Latin *nutare*, to nod), the amplitude and type of nutation depending on the initial conditions. Earth's axis does indeed nutate, but not from the same cause. Those who have studied tops will understand that damping more or less rapidly damps out the amplitude of the nutation, and, since Earth is a non-rigid, flexible body, this type of nutation has long ago damped out.

Earth's axis of rotation nutates because it is subject to varying torques from Sun and Moon – the former varying because of the eccentricity of Earth's orbit, and the latter because of both the eccentricity and inclination of the Moon's orbit. This means that the equinox ♈ does not move at uniform speed along the ecliptic, and the obliquity of the ecliptic varies quasi-periodically. These two effects are known as the *nutation in longitude* and the *nutation in the obliquity*. While several effects involving both the Sun and the Moon are involved, the most important term in the general expressions for both nutation in longitude and nutation in obliquity involve the longitude of the nodes of the Moon's orbit, which are known to regress with a period of 18.6 years. Thus both nutations, in the first approximation, have a period of 18.6 years. The nutation in longitude has an amplitude of 17″.2, and the nutation in the obliquity has an amplitude of 9″.2. In addition, planetary perturbations cause a secular (i.e. not periodic) decrease in the obliquity of about 0″.47 per year.

A further point that should be mentioned is that the plane of the ecliptic is not quite invariable. What is invariable in the absence of external torques on the solar system is the direction of the angular momentum vector of the solar system; the plane perpendicular to this is called the *invariable plane* of the solar system.

This section and the previous section have described briefly in a rather qualitative way the motion of the equinox along the ecliptic with a period of 25,800 years (i.e. precession) – a motion that is not quite uniform on account of the nutations in longitude and the obliquity. This brief account may suffice for most purposes of the observational astronomer and for the aim of this chapter, which is a general overview of the celestial sphere. A more thorough and detailed treatment of precession and nutation will have to wait for a special chapter devoted to the subject.

6.9 *The Length of the Year.*

The time taken for Earth to revolve around the Sun with respect to the stars, which is the same thing as the time taken for the Apparent Sun to move around the ecliptic with respect to the stars, is a *Sidereal Year*, which is $365^{d}.25636$, where the "d" denotes a mean solar day. The length of the seasons, however, is determined by the motion of the Apparent Sun relative to ♈. Because ♈ is moving westward along the ecliptic, the time that the Apparent Sun takes to move around the ecliptic relative to ♈, which is called the *Tropical Year*, is a little less than the sidereal year. We have seen, however that the motion of ♈ along the ecliptic is not quite uniform, and we have to average out the effects of nutation. Thus the *Mean Tropical Year* is the average time for the ecliptic longitude of the Apparent Sun to increase by 360^{o}, which is $365^{d}.24219$.

The calendar that we use in everyday life is the *Gregorian Calendar*, in which there are 365 days in most years, but 366 days in years that are divisible by 4 unless they are also divisible by 100 other than those that are also divisible by 400. Thus *leap years* (those that have 366 days) include 1996, 2000, 2004, but not 2005 or 1900. (2000 was a leap year because, although it is divisible by 100, it is also divisible by 400.) The average length of the *Gregorian Year* is 365.2425, which is close enough to the Mean Tropical Year for present-day purposes, but which is of concern to calendar reformers and will be of some concern to our remote descendants.

The *Anomalistic Year* is the interval between consecutive passages of the Earth through perihelion. The perihelion of Earth's orbit is slowly advancing n the same direction as the Earth's motion, so the anomalistic year is a little longer than the sidereal year, and is equal to $365^{d}.25964$.

Figure VI.9 illustrates a way of thinking about the relation between the sidereal and tropical years. We are looking down on the ecliptic from the direction of the north ecliptic pole. We see the Sun moving counterclockwise at angular speed ω_{sid} and ♈ moving clockwise at angular speed $\omega_{♈}$. The angular speed of the Sun relative to ♈

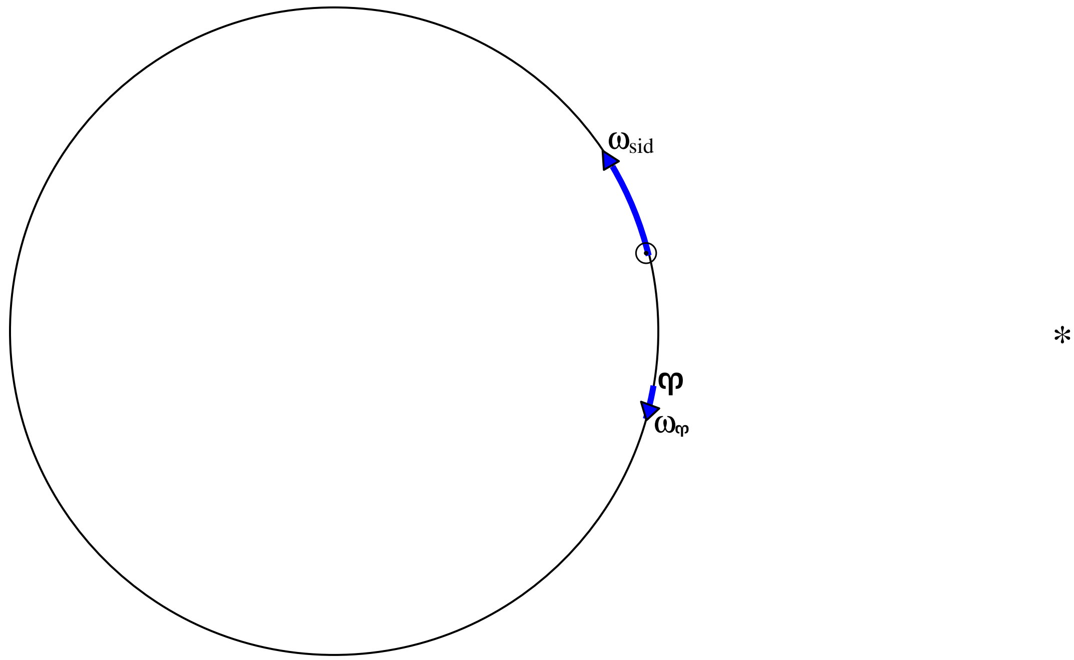


FIGURE VI.9

is $\omega_{trop} = \omega_{sid} + \omega_{♈}$. But period P and angular speed ω are related by $\omega = 2\pi/P$.

Therefore:

$$\frac{1}{P_{trop}} = \frac{1}{P_{sid}} + \frac{1}{P_{♈}}. \qquad 6.9.1$$

Thus $P_{sid} = 365^d.25636$ and $P_{♈} = 25800$ years $= 9.424 \times 10^6$ days. Hence $P_{trop} = 365^d.2422$. Using the same argument, see if you can calculate how long it takes for the perihelion of Earth's orbit to advance by 360° – bearing in mind that the perihelion is advancing, not regressing.

One more point worth noting is that, during a sidereal year, the Sun has upper transited across the meridian 365.25636 times, whereas a fixed star has transited 366.25636 times. Expressed another way, while Earth turns on its axis 365.25636 times relative to the Sun, relative to the stars it has made one extra turn during its revolution around the Sun. Thus

$$\frac{\text{Length of sidereal day}}{\text{Length of solar day}} = \frac{365.25636}{366.25636}.$$

Thus the length of the sidereal day is $23^h\ 56^m\ 04^s$.

6.10 *Problems*

In Section 3.5 of Chapter 5, I suggested that it might be a good ides to write a computer program, which would last you for life, that would solve any problem involving plane or spherical triangles. If you did that, the following problems will be easy. If you didn't, you are now about to suffer.

6.10.1

The equatorial coordinates (J2000.0) of Antares and Deneb are, respectively

Antares	$\alpha = 16^{\mathrm{h}}\, 29^{\mathrm{m}}.5$	$\delta = -26^{\circ}\ 26'$
Deneb	$20\ \ 37.6$	$+45\ \ 17$

Calculate the positions of the poles of the great circle joining these two stars.

I put one star in the northern hemisphere, and the other in the south, and I put the stars in the third and fourth quadrants of right ascension, just to be awkward.

6.10.2

The parallax of Antares is $0''.00540$, and the parallax of Deneb is $0''.00101$. How far apart are the stars (a) in parsecs? (b) in km? (c) in light-years? The speed of light is $2.997\ 92 \times 10^{8}\ \mathrm{m\ s^{-1}}$, the radius of Earth's orbit is $1.495\ 98 \times 10^{8}$ km, and a tropical year is 365.24219 mean solar days.

6.10.3

A meteor starts at	$\alpha = 23^{\mathrm{h}}\, 24^{\mathrm{m}}.0$	$\delta = +04^{\circ}\ 00'$
and finishes at	$\alpha = 01^{\mathrm{h}}\, 36^{\mathrm{m}}.0$	$\delta = +10^{\circ}\ 00'$

A second meteor, from the same shower (i.e. from the same meteoroid stream) starts at

	$\alpha = 00^{\mathrm{h}}\, 06^{\mathrm{m}}.0$	$\delta = +03^{\circ}\ 00'$
and finishes at	$\alpha = 02^{\mathrm{h}}\, 12^{\mathrm{m}}.0$	$\delta = +05^{\circ}\ 30'$.

Calculate the position of the radiant (i.e. the position on the sky where the two paths, projected backwards, intersect).

Again you'll notice that I chose the coordinates to be as awkward as I could.

6.11 *Solutions*

6.11.1

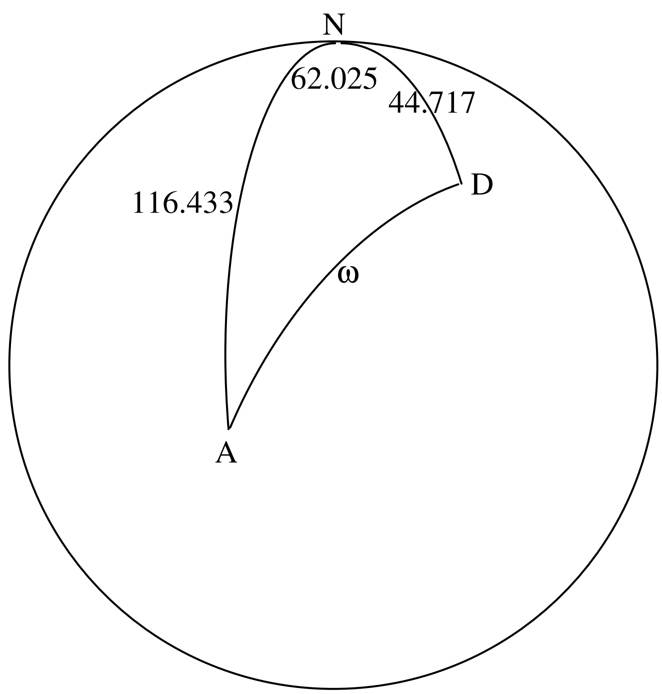
I think the first thing that I would do, would be to convert the coordinates to degrees and decimals (or maybe even radians and decimals, though I do it below in <u>degrees</u> and decimals):

Antares:	$\alpha = 247.375$	$\delta = -26.43\dot{3}$
Deneb	$\alpha = 309.400$	$\delta = +45.28\dot{3}$.

We already did a similar problem in Chapter 3, Section 3.5, Example 2, so I shan't do it again. I make the answer:

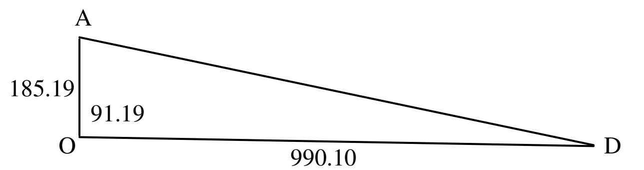
One pole:	$\alpha = 11^{h}\,47^{m}.3$	$\delta = +\ 56^{o}\ 11'$
The other pole:	$\alpha = 23^{h}\,47^{m}.3$	$\delta = +123^{o}\ 49'$

6.11.2



I have drawn the North Celestial Pole N, and the colures from N to Antares (A) and to Deneb (D), together with their north polar distances in degrees. I have also marked the difference between their right ascensions, in degrees. We can immediately calculate, from the cosine rule for spherical triangles, equation 3.5.2, the angular distance ω between the two stars in the sky. I make it $\omega = 91^\circ.190\ 79$.

Now that we know the angle between the stars, we can use a plane triangle to calculate the distance between them:



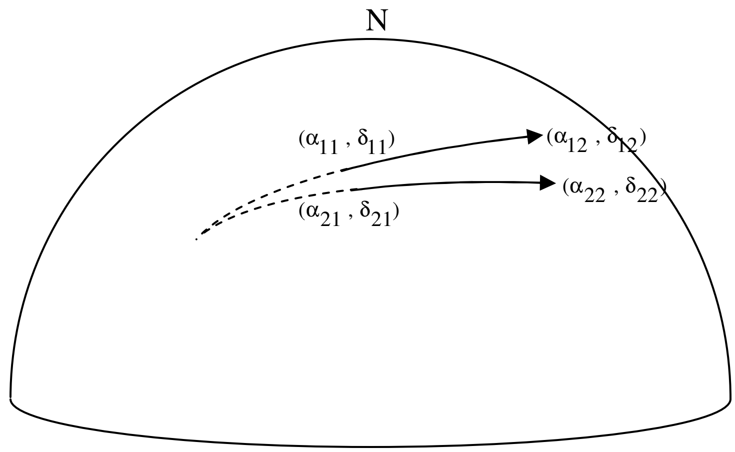
I have marked Antares (A), Deneb (D) and us (O), and the distances from us to the two stars, in parsecs. (That's the reciprocal of their parallaxes in arcsec.) I have also marked the angles, in degrees, between Antares and Deneb. We can now use the cosine rule for planes triangles, equation 3.2.2, to find the distance AD. I make it 1011 parsecs.

A parsec is the distance at which an astronomical unit (approximately the radius of Earth's orbit) would subtend an angle of one arcsecond. This also means, if you come to think of it, that the number of astronomical units in a parsec is equal to the number of arcseconds in a radian, which is $360 \times 3600 \div (2\pi) = 2.062648 \times 10^5$. The distance between the stars is therefore $1011 \times 2.062648 \times 10^5$ astronomical units. Multiply this by $1.495\ 98 \times 10^8$, to get the distance in km. I make the distance 3.120×10^{16} km.

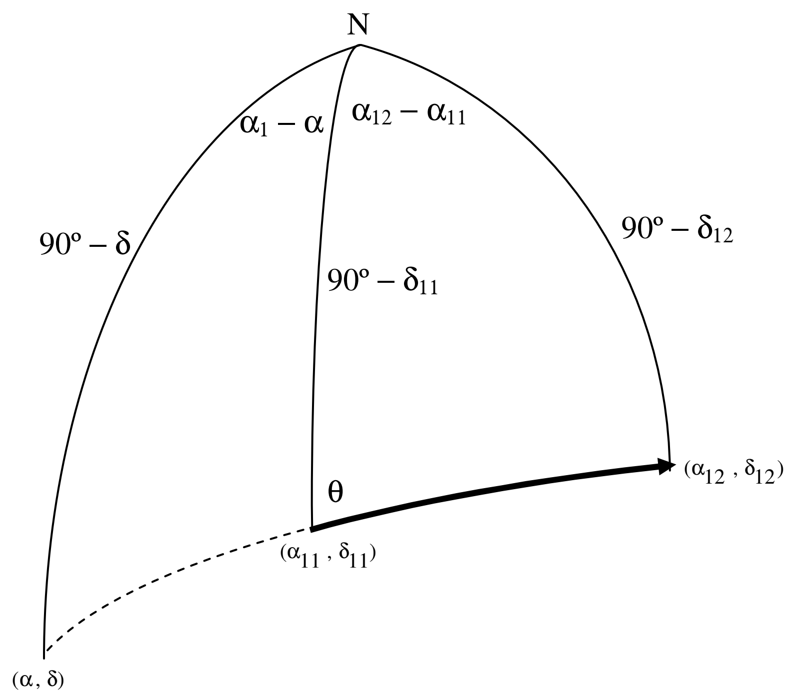
This would take light 1.040596×10^8 seconds to travel, or 3298 years, so the distance between the stars is 3298 light-years.

6.11.3

Let's see if we can develop a formula for a general case. We'll have the first meteor start at $(\alpha_{11}, \delta_{11})$ and finish at $(\alpha_{12}, \delta_{12})$. The second meteor starts at $(\alpha_{21}, \delta_{21})$ and finishes at $(\alpha_{22}, \delta_{22})$. We have to find the coordinates (α, δ) of the point from which the two meteors diverge.



This is not a particularly easy problem – but is one that is obviously useful for meteor observers. I'll just outline some suggestions here, and leave the reader to work out the details. I'll draw below one of the meteors, and the radiant, and the North Celestial Pole:



Use the cotangent rule (equation 3.5.5) on the righthand triangle to get an expression for cot θ:

$$\sin\delta_{11}\cos(\alpha_{12} - \alpha_{11}) = \cos\delta_{11}\tan\delta_{12} - \sin(\alpha_{12} - \alpha_{11})\cot\theta.$$

Use the cotangent rule (equation 3.5.5) on the lefthand triangle to get another expression for cot θ:

$$\sin\delta_{11}\cos(\alpha_{11}-\alpha) = \cos\delta_{11}\tan\delta_{12} + \sin(\alpha_{11}-\alpha)\cot\theta.$$

Equate these two expression for cot θ (i.e. eliminate θ between the two equations). This will give you a single equation containing the two unknowns, α and δ, everything else in the equation being a known quantity. (This will be obvious if you are actually doing a numerical example.)

Now do the same thing for the second meteor, and you will get a second equation in α and δ. In principle you are now home free, though there may be a bit of heavy algebra and trigonometry to go through before you finally get there.

I make the answer as follows:

$$\tan\alpha = \frac{\cos\alpha_{22}\tan\delta_{22} - \cos\alpha_{12}\tan\delta_{12} + a_1\sin\alpha_{12} - a_2\sin\alpha_{22}}{\sin\alpha_{12}\tan\delta_{12} - \sin\alpha_{22}\tan\delta_{22} + a_1\cos\alpha_{12} - a_2\cos\alpha_{22}},$$

where

$$a_1 = \frac{\tan\delta_{11}}{\sin(\alpha_{11}-\alpha_{12})} - \frac{\tan\delta_{12}}{\tan(\alpha_{11}-\alpha_{12})}$$

and

$$a_2 = \frac{\tan\delta_{21}}{\sin(\alpha_{21}-\alpha_{22})} - \frac{\tan\delta_{22}}{\tan(\alpha_{21}-\alpha_{22})}.$$

Then

$$\tan\delta = \cos(\alpha-\alpha_{12})\tan\delta_{12} + \sin(\alpha-\alpha_{12})[\csc(\alpha_{11}-\alpha_{12})\tan\delta_{11} - \cot(\alpha_{11}-\alpha_{12})\tan\delta_{12}]$$

or

$$\tan\delta = \cos(\alpha-\alpha_{22})\tan\delta_{22} + \sin(\alpha-\alpha_{22})[\csc(\alpha_{21}-\alpha_{22})\tan\delta_{21} - \cot(\alpha_{21}-\alpha_{22})\tan\delta_{22}].$$

Either of these two equations for tan δ should give the same result. In the computer program I use for this calculation, I get it to calculate tan δ from *both* equations, just as a check for mistakes.

This may look complicated, but all terms are just calculable numbers for any particular case. If the equinoctial colure gets in the way (as it did – deliberately – in the numerical example I gave), I suggest just add 24 hours to all right ascensions.

For the numerical example I gave, I make the coordinates of the radiant to be:

$$\alpha = 22^{h}\,01^{m}.3 \qquad \delta = -\,00^{\circ}\,37'.$$