CHAPTER 15
SPECIAL PERTURBATIONS

[This chapter is under development and it may be a rather long time before it is complete. It is the intention that it may deal with special perturbations, differential corrections, and the computation of a definitive orbit. However, it will probably proceed rather slowly and whenever the spirit moves me.]

15.1 *Introduction*

Chapter 14 dealt with the subject of *general perturbations.* That is, if the perturbation R can be expressed as an explicit algebraic function, the rates of change of the orbital elements with time can be calculated by explicit algebraic expressions known as *Lagrange's Planetary Equations.* By way of example we derived Lagrange's equations for the case of a satellite in orbit around an oblate planet, in which the departure of the gravitational potential from that of a spherically symmetric planet could be expressed in simple algebraic form.

Lagrange's equations are important and interesting from a theoretical point of view. However, in the practical matter of calculating the perturbations of the orbit of an asteroid or a comet resulting from the gravitational field of the other planets in the solar system, that is not how it is done. The perturbing forces are functions of time which must be computed numerically rather than from a simple formula. Such perturbations are generally referred to as *special perturbations*. While long-established computer programs, such as RADAU15, may be available to carry out the necessary rather long computations without the user having to understand the details, it is the intention in this chapter to indicate in principle how such a program may be developed from scratch.

Jupiter is by far the greatest perturber, but for high-precision work it may be necessary to include perturbations from the other major planets, Mercury to Neptune. Pluto may also be considered. However, it is now known that Pluto is a good deal less massive than it was once estimated to be, so it is a nice question as to whether or not to include Pluto. Besides, Pluto is probably not the most massive of the transneptunian objects - Eris is believed to be a little larger and hence possibly more massive. The main belt object Ceres may be more important than either of these. The total mass of the remaining asteroids is usually considered negligible in this context.

It will be evident that any computer program intended to compute special perturbations will have to include, as subroutines, programs for calculating, day-by-day, the positions and distances of each of the perturbing planets to be included in the computation. Computer programs are available to provide these. In what follows, it will be assumed that the reader has access to such a program (I do!) or is otherwise able to compute the planetary positions, and we move on from there to see how we calculate the planetary perturbations.

15.2 *Orbital elements and the position and velocity vector*

The six elements used to describe the orbit of an asteroid are the familiar

$$a, e, i, \Omega, \omega, T$$

Because of the precession and nutation of Earth, the angular elements must, of course, be referred to a particular equinox and equator, usually chosen to be that of the standard epoch J2000.0, which means 12h 00m TT on 2000 January 01. (The "J" stands for "Julian Year".)

The element T is the instant of perihelion passage. If the orbit is nearly circular, the instant of perihelion passage is ill-defined, and if the orbit is exactly circular, it is not defined at all. In such cases, instead of T, we may give either the *mean anomaly* M_0 or the *mean longitude* L_0 at a specified epoch (see Chapter 10). This epoch need not be (and usually is not) the same as the standard epoch referred to in the previous paragraph.

Suppose that, at some instant of time (to be known, for reasons to be explained later, as the *epoch of osculation*), the heliocentric ecliptic coordinates of an asteroid or comet in an elliptic orbit are (X, Y, Z) and the components of the velocity vector are $(\dot{X}, \dot{Y}, \dot{Z})$. We have shown in Chapter 10, Section 10.10) how to calculate, from these, the six elements $a, e, i, \Omega, \omega, T$ of the orbit at that instant. Conversely, given the orbital elements, we could reverse the calculation and calculate the components of the position and velocity vectors. Thus an orbit may equally well be described by the six numbers

$$X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}.$$

That is to say the components, at some specified instant of time, of the position and velocity vector in heliocentric ecliptic coordinates.

We could equally well give the components, at some instant of time, of the position and velocity vectors in heliocentric equatorial coordinates:

$$\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta}$$

We saw in Section 10.9 that yet another set of six numbers,

$$P_x, Q_x, P_y, Q_y, P_z, Q_z$$

will also suffice to describe an orbit.

It is assumed here that the reader is familiar with all four of these alternative sets of elements, and can convert between them. Indeed, before reading on, it may be a useful exercise to prepare a computer program that will convert instantly between them. This may not be a trivial task, but I strongly recommend doing so before reading further. The facility to convert instantly between one set and another is an enormous help. To convert between ecliptic and equatorial coordinates, you will need, of course, the obliquity of the ecliptic at that instant - it varies, of course, with time.)

The reader will have noticed the frequent occurrence of the phrase "at that instant" in the previous paragraphs. If the asteroid were not subject to perturbations from the other planets, it would retain its orbital elements forever. However, because of the planetary perturbations, the elements $a, e, i, \Omega, \omega, T$ computed from $X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}$ or from $\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta}$ at a particular instant of time are valid only for that instant. The elements will change with time. Therefore in quoting the elements of an asteroidal orbit, it is entirely necessary to state clearly and without ambiguity the instant of time to which these elements are referred. The unperturbed orbit, and the real perturbed orbit, will coincide in position and velocity at that instant. The real and unperturbed orbits will "kiss" or *osculate* at that instant, which is therefore known as the epoch of osculation.

The elements $a, e, i, \Omega, \omega, T$ calculated for a particular epoch of osculation may suffice for the computation of an ephemeris for weeks to come. But after months the observed position of the object will start to deviate from its calculated ephemeris position. It is then necessary to calculate a new set of elements for a later epoch of osculation. Depending on circumstances, orbital elements may be recalculated every year, or every 200 days or every 40 days or every 10 days, or at some other convenient interval.

It will be the purpose in what follows to do the following. Given that at some instant (i.e. at some epoch of osculation) the elements are $a, e, i, \Omega, \omega, T$ (or the position and velocity vectors are $\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}, \dot{\zeta}$), how do we calculate the elements at some subsequent epoch, taking into account planetary perturbations?

As pointed out at the end of Section 15.1, we shall need to know the positions and distances of the major planets as a function of time. We suppose that we have subroutines in our program that we can call upon to calculate these data at any date. As mentioned above, the equations of motion can be written in equatorial or ecliptic coordinates, though it is more likely that, for the positions of the major planets, we shall have available their positions in *equatorial* coordinates.

15.3 *The equations of motion*

First let us consider the motion of an asteroid under the gravitational influence of the Sun alone, ignoring perturbations from the other planets. We take the mass of the Sun to be M and the mass of the asteroid to be m. The force on the asteroid – and, of course, by Newton's third law, the force on the Sun – is $\frac{GMm}{r^2}$, where r is the distance between the two bodies. The two bodies are, of course, in motion around their common centre of mass, which, in the case of an asteroid, is very close to the centre of the Sun.

The acceleration of the asteroid towards the centre of mass is $\frac{GM}{r^2}$, and the acceleration of the Sun towards the centre of mass is $\frac{Gm}{r^2}$. If we refer the motion to the Sun as origin, we see that the acceleration of the asteroid towards the Sun is $\frac{G(M+m)}{r^2}$. In vector form we may write this as

$$\ddot{\mathbf{r}} = -\frac{G(M+m)}{r^3}\mathbf{r}, \qquad 15.3.1$$

where $\mathbf{r}$ is a vector directed from the Sun towards the asteroid, with heliocentric rectangular components (x, y, z). These heliocentric coordinates could be either ecliptic coordinates, for which we have hitherto used the symbols (X, Y, Z) ; or they could be equatorial coordinates, for which we have hitherto used the symbols (ξ, η, ζ). The symbols (x, y, z) will be understood here to refer to either, at our convenience. It is more likely that we shall have available the *equatorial* rather than the ecliptic coordinates. The direction cosines of $\mathbf{r}$ are $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$, and consequently the rectangular components of equation 15.3.1 are

$$\ddot{x} = -\frac{G(M+m)}{r^3}x \qquad 15.3.2$$

$$\ddot{y} = -\frac{G(M+m)}{r^3}y \qquad 15.3.3$$

$$\ddot{z} = -\frac{G(M+m)}{r^3}z \qquad 15.3.4$$

These are the equations of motion of the asteroid with respect to the Sun as origin. The quantities x, y, z, r $(=\sqrt{x^2 + y^2 + z^2})$ are, of course, functions of time. The solution of these equations describe the elliptical (or other conic section) orbits of the asteroid and all the other properties that we have discussed in previous chapters.

If we are using ecliptic coordinates (X, Y, Z), the X-axis is directed towards the First Point of Aries, the Y-axis is directed along the direction of increasing ecliptic longitude, and the Z-axis is directed towards the north pole of the ecliptic.

If we are using equatorial coordinates (ξ, η, ζ), the ξ-axis is directed towards the First Point of Aries, the η-axis is directed along the direction of 6 hours right ascension, and the ζ-axis is directed towards the north celestial pole. The Earth will be on the X- or ξ-axis in September (not March).

Now let us introduce a third body, a perturbing planet, such as, perhaps, Jupiter. We'll suppose that its mass is m_1, that its distance from the Sun is r_1 and its distance from the asteroid is ρ_1 (see figure XV.I, in which S is the Sun, A is the asteroid, and P is the perturbing planet). This is now a three-body problem and a general solution in terms of algebraic functions is not possible, and it has to be solved by numerical computation.

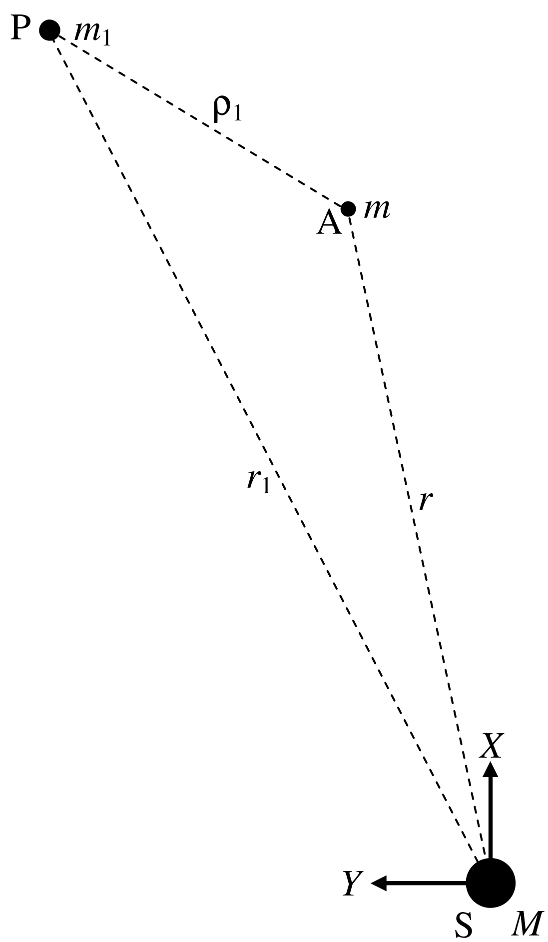

FIGURE XV.I

In addition to the accelerations of the asteroid towards the Sun and the Sun towards the asteroid described on page 3, used in developing equations 15.3.1-4, we now have also to consider the accelerations of the asteroid and the Sun towards the perturbing planet, as indicated in figure XV.II.

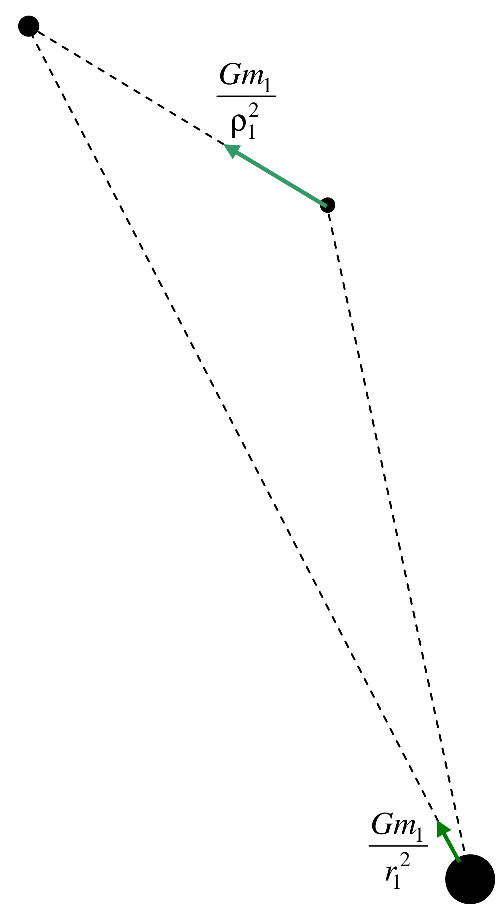

FIGURE XV.II

The x-components of these are $\frac{Gm_1}{\rho_1^2} \times \frac{x_1 - x}{\rho_1}$ and $\frac{Gm_1}{r_1^2} \times \frac{x_1}{r_1}$, and so the additional acceleration of A, relative to the Sun, in the X-direction is $Gm_1\left(\frac{x_1 - x}{\rho_1^3} - \frac{x_1}{r_1^3}\right)$, and this has now to be added to the right hand side of equation 15.3.2:

$$\ddot{x} = -\frac{G(M+m)}{r^3}x + Gm_1\left(\frac{x_1 - x}{\rho_1^3} - \frac{x_1}{r_1^3}\right) \qquad 15.3.5$$

Neither G nor M are known to great precision, but the product GM is known to very great precision. Indeed in computational practice we make use of the *Gaussian constant* $k = \sqrt{\frac{GM}{a_0}}$, where a_0 is the astronomical unit of length. This constant has dimension T^{-1} and is equal to the angular velocity of a particle of negligible mass in circular orbit of radius 1 au around the Sun, which is 0.017 202 098 95 radians per mean solar day. Therefore in computational practice, equation 15.3.5 is generally written as

$$\ddot{x} = -\frac{k^2(1+m)}{r^3}x + k^2 m_1\left(\frac{x_1 - x}{\rho_1^3} - \frac{x_1}{r_1^3}\right), \qquad 15.3.6$$

in which the units of mass, length and time are, respectively, solar mass, astronomical unit, and mean solar day. Recall that m is the mass of the asteroid whose orbit we are computing, and m_1 is the mass of the perturbing planet, and that the origin of coordinates is the centre of the Sun. Similar equations apply to the y- and z-components:

$$\ddot{y} = -\frac{k^2(1+m)}{r^3}y + k^2 m_1\left(\frac{y_1 - y}{\rho_1^3} - \frac{y_1}{r_1^3}\right), \qquad 15.3.7$$

$$\ddot{z} = -\frac{k^2(1+m)}{r^3}z + k^2 m_1\left(\frac{z_1 - z}{\rho_1^3} - \frac{z_1}{r_1^3}\right). \qquad 15.3.8$$

If we add the perturbations from all the major planets from Mercury (M) to Neptune (N), these equations become, of course,

$$\ddot{x} = -\frac{k^2(1+m)}{r^3}x + k^2 \sum_{i=\mathrm{M}}^{\mathrm{N}} m_i\left(\frac{x_i - x}{\rho_i^3} - \frac{x_i}{r_i^3}\right) \qquad 15.3.9$$

and similar equations in y and z.

In the case of an asteroid or a comet, it may be permissible to neglect m in this equation (i.e. set $m = 0$), but not, of course, m_1. We shall do that here, so the equation of motion in x becomes

$$\ddot{x} = -k^2\frac{x}{r^3} + k^2\sum_{i=\mathrm{M}}^{\mathrm{N}} m_i\left(\frac{x_i - x}{\rho_i^3} - \frac{x_i}{r_i^3}\right), \qquad 15.3.10$$

with similar equations in y and z.

The x, x_i. ρ_i, r_i, etc., are numerical data, which have to be supplied by independent computations (subroutines) for all the planets. As stated at the end of the previous Section, we suppose that we have subroutines in our program that we can call upon to calculate these data at any date. We also pointed out that the equations of motion are valid for either ecliptic or equatorial coordinates, although the coordinates of the planets are more likely to be available is *equatorial* rather than ecliptic coordinates. They are all functions of time, so that, in effect, we have to develop numerical methods for integrating equations of the form, where $f(t)$ is not an algebraic expression, but rather a table of numerical values.

$$\ddot{x} = f(t). \qquad 15.3.11$$

That is to say

$$\frac{d\dot{x}}{dt} = f(t). \qquad 15.3.12$$

We suppose that we know $\dot{x}$ at the epoch of osculation. Then we can find $\dot{x}$ at any subsequent date by any standard technique of numerical integration, such as Simpson's or Weddle's Rules, or Gaussian quadrature, or by a Runge-Kutta process. Thus we now have a table of $\dot{x}$ as a function of time:

$$\dot{x} = g(t) \qquad 15.3.13.$$

That is to say

$$\frac{dx}{dt} = g(t). \qquad 15.3.14$$

We integrate a second time, until we arrive at both x and $\dot{x}$ at some subsequent epoch of osculation (perhaps 200, or 40, days into the future). Repeat with the y and z components, so we eventually have a new set of $(x, y, z, \dot{x}, \dot{y}, \dot{z})$ for a later epoch, and hence also of $a, e, i, \Omega, \omega, T$.