CHAPTER 13
CALCULATION OF ORBITAL ELEMENTS

13.1 *Introduction*

We have seen in Chapter 10 how to calculate an ephemeris from the orbital elements. This chapter deals with the rather more difficult problem of determining the orbital elements from the observations.

We saw in Chapter 2 how to fit an ellipse (or other conic section) to five points in a plane. In the case of a planetary orbit, we need also to know the orientation of the plane, which will require two further bits of information. Thus we should be able to determine the shape, size and orientation of the ellipse from seven pieces of information.

This, however, is not quite the same problem facing us in the determination of a planetary orbit. Most importantly, we do not know *all* of the coordinates of the planet at the time of *any* of the observations. We know two of the coordinates – namely the right ascension and declination – but we have no idea at all of the distance. All that an observation gives us is the direction to the planet in the sky at a given instant of time. Finding the geocentric distance at the time of a given observation is indeed one of the more difficult tasks; once we have managed to do that, we have broken the back of the problem.

However, although we do not know the geocentric (or heliocentric) distances, we do have some additional information to help us. For one thing, we know where one of the foci of the conic section is. The Sun occupies one of them – though we don't immediately know which one. Also, we know the instant of time of each observation, and we know that the radius vector sweeps out equal areas in equal times. This important keplerian law is of great value in computing an orbit.

To determine an orbit, we have to determine a set of six *orbital elements.* These are, as previously described, a, e, i, Ω, ω and T for a sensibly elliptic orbit; for an orbit of low eccentricity one generally substitutes an angle such as M_0, the mean anomaly at the epoch, for T. Thus we can calculate the orbit from six pieces of information. We saw in Chapter 10 how to do this if we know the three heliocentric spatial coordinates and the three heliocentric velocity components – but this again is not quite the problem facing us, because we certainly do not know any of these data for a newly-discovered planet.

If, however, we have three suitably-spaced observations, in which we have measured three directions (α , δ) at three instants of time, then we have six data, from which it may be possible to calculate the six orbital elements. It should be mentioned, however, that three observations are *necessary* to obtain a credible solution, but they may not always be *sufficient*. Should all three observations, for example, be on the ecliptic, or near to a stationary point, or if the planet is moving almost directly towards us for a while and consequently hardly appears to move in the sky, it may not be possible to obtain a credible solution. Or again, observations always have some error associated with them, and small observational errors may under some circumstances translate into a wide range of possible solutions, or it may not even be possible to fit a single set of elements to the slightly erroneous observations.

In recent years, the computation of the orbits of near-Earth asteroids has been a matter of interest for the public press, who are likely to pounce on any suggestion that the observations might have been "erroneous" and the orbit "wrong" – as if they were unaware that all scientific measurement always have error associated with them. There is a failure to distinguish *errors* from *mistakes*.

When a new minor planet or asteroid is discovered, as soon as the requisite minimum number of observations have been made that enable an approximate orbit to be computed, the elements and an ephemeris are distributed to observers. The purpose of this *preliminary orbit* is not to tell us whether planet Earth is about to be destroyed by a cataclysmic collision with a near-Earth asteroid, but is simply to supply observers with a good enough ephemeris that will enable them to find the asteroid and hence to supply additional observations. Everyone who is actively involved in the process of observing asteroids or computing their orbits either knows or ought to know this, just as he also knows or ought to know that, as additional observations come in, the orbit will be *revised* and *differential corrections* will be made to the elements. Further, the computed orbit is generally an *osculating orbit*, and the elements are *osculating elements* for a particular *epoch of osculation.* In order to allow for planetary perturbations, the epoch of osculation is changed every 200 days, and new osculating elements are calculated. All of this is routine and is to be expected. And yet there has been an unfortunate tendency in recent years for not only the press but also for a number of persons who would speak for the scientific community, but who may not themselves be experienced in orbital computations, to attribute the various necessary revisions to an orbit to "mistakes" or "incompetence" by experienced orbit computers.

When all the observations for a particular apparition have been amassed, and no more are expected for that apparition, a *definitive orbit* for that apparition is calculated from all available observations. Even then, there will be small variations in the elements obtained by different computers. This is because, among other things, each observation has to be critically assessed and weighted. Some observations may be photographic; the majority these days will be higher-precision CCD observations, which will receive a higher weight. Observations will have been made with a variety of telescopes with very different focal lengths, and there will be variations in the experience of the observers involved. Some observations will have been made in a great hurry in the night immediately following a new discovery. Such observations are valuable for computing the preliminary orbit, but may merit less weight in the definitive orbit. There is no unique way for dealing with such problems, and if two computers come up with slightly different answers as a result of weighting the observations differently it does not mean than one of them is "right" and that the other has made a "mistake". All of this should be very obvious, though some words that have been spoken or written in recent years suggest that it bears repeating.

There are a number of small problems involving the original raw observations. One is that the instant of time of an observation is recorded and reported by an observer in Universal Time. This is the correct thing for an observer to do, and is what is expected of him or her. The computer, however, uses as the argument for the orbital calculation the best representation of a uniformly-flowing dynamical time, which at present is TT, or Terrestrial Time (see chapter 7). The difference for the current year is never known exactly, but has to be estimated. Another difficulty is that observations are not made from the centre of Earth, but from some point on the surface of Earth – a point that is moving as Earth rotates. Thus a small parallactic correction has to be made to the observations – but we do not know how large this correction is until we know the distance of the planet. Or again, the computer needs to know the position of the planet when the sunlight reflected from it left the planet, not when the light eventually arrived at Earth twenty or so minutes later – but we do not know how large the light travel-time correction is until we know the distance of the planet.

There is evidently a good deal involved in computing orbits, and this could be a very long chapter indeed, and never written to perfection to cover all contingencies. In order to get started, however, I shall initially restrict the scope of this chapter to the basic problem of computing elliptical elements from three observations. If and when the spirit moves me I may at a later date expand the chapter to include parabolic and hyperbolic orbits, although the latter pose special problems. Computing hyperbolic elements is in principle no more difficult than computing elliptic orbits; in practice, however, any solar system orbits that are sensibly hyperbolic have been subject to relatively large planetary perturbations, and so the problem in practice is not at all a simple one. Carrying out differential corrections to a preliminary orbit is also something that will have to be left to a later date.

In the sections that follow, I am much indebted to Carlos Montenegro of Argentina who went line-by-line with me through the numerical calculations, resulting in a number of corrections to the original text. Any remaining mistakes (I hope there are few, if any) are my own responsibility.

13.2 *Triangles*

I shall start with a geometric theorem involving triangles, which will be useful as we progress towards our aim of computing orbital elements.

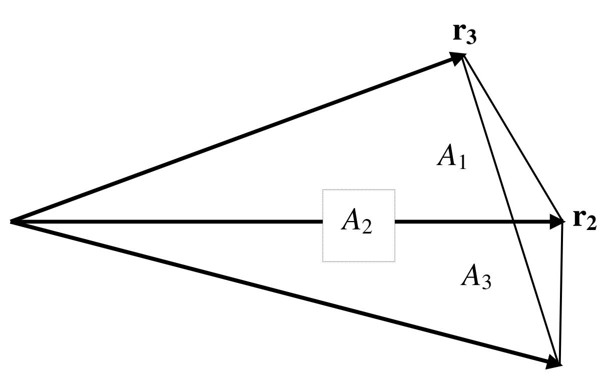

FIGURE XIII.1

Figure XIII.1 shows three coplanar vectors. It is clearly possible to express $\mathbf{r_2}$ as a linear combination of the other two. That is to say, it should be possible to find coefficients such that

$$\mathbf{r_2} = a_1\mathbf{r_1} + a_3\mathbf{r_3} . \qquad 13.2.1$$

The notation I am going to use is as follows:

The area of the triangle formed by joining the tips of $\mathbf{r_2}$ and $\mathbf{r_3}$ is A_1.
The area of the triangle formed by joining the tips of $\mathbf{r_3}$ and $\mathbf{r_1}$ is A_2.
The area of the triangle formed by joining the tips of $\mathbf{r_1}$ and $\mathbf{r_2}$ is A_3.

To find the coefficients in equation 13.2.1, multiply both sides by $\mathbf{r_1} \times$:

$$\mathbf{r_1} \times \mathbf{r_2} = a_3 \, \mathbf{r_1} \times \mathbf{r_3} . \qquad 13.2.2$$

The two vector products are parallel vectors (they are each perpendicular to the plane of the paper), of magnitudes $2A_3$ and $2A_2$ respectively. ($2A_3$ is the area of the *parallelogram* of which the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ form two sides.)

$$\therefore \qquad a_3 = A_3/A_2 . \qquad 13.2.3$$

Similarly by multiplying both sides of equation 13.2.1 by $\mathbf{r_3} \times$ it will be found that

$$a_1 = A_1/A_2 . \qquad 13.2.4$$

Hence we find that

$$A_2\mathbf{r_2} = A_1\mathbf{r_1} + A_3\mathbf{r_3} . \qquad 13.2.5$$

13.3 *Sectors*

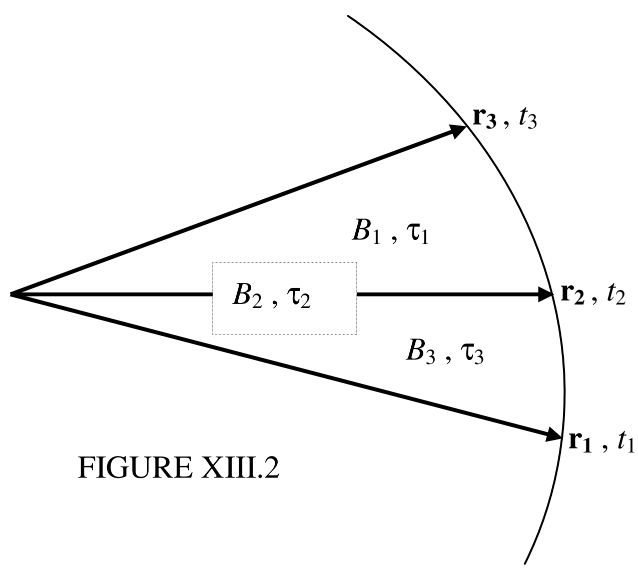

FIGURE XIII.2

Figure XIII.2 shows a portion of an elliptic (or other conic section) orbit, and it shows the radii vectores of the planet's position at instants of time t_1, t_2 and t_3.

The notation I am going to use is as follows:

The area of the sector formed by joining the tips of $\mathbf{r_2}$ and $\mathbf{r_3}$ around the orbit is B_1.
The area of the sector formed by joining the tips of $\mathbf{r_3}$ and $\mathbf{r_1}$ around the orbit is B_2.
The area of the sector formed by joining the tips of $\mathbf{r_1}$ and $\mathbf{r_2}$ around the orbit is B_3.

The time interval $t_3 - t_2$ is τ_1.
The time interval $t_3 - t_1$ is τ_2.
The time interval $t_2 - t_1$ is τ_3.

Provided the arc is fairly small, then to a good approximation (in other words we can approximate the sectors by triangles), we have

$$B_2\mathbf{r_2} \approx B_1\mathbf{r_1} + B_3\mathbf{r_3} . \qquad 13.3.1$$

That is,

$$\mathbf{r_2} \approx b_1\mathbf{r_1} + b_3\mathbf{r_3} , \qquad 13.3.2$$

where

$$b_1 = B_1/B_2 \qquad 13.3.3$$

and

$$b_3 = B_3/B_2 \qquad 13.3.4$$

The coefficients b_1 and b_3 are the *sector ratios*, and the coefficients a_1 and a_3 are the *triangle ratios*.

By Kepler's second law, the sector areas are proportional to the time intervals.

That is

$$b_1 = \tau_1/\tau_2 \qquad 13.3.5$$

and

$$b_3 = \tau_3/\tau_2 . \qquad 13.3.6$$

Thus the coefficients in equation 13.3.2 are *known*. Our aim is to use this approximate equation to find approximate values for the heliocentric distances at the instants of the three observations, and then to refine them in order to satisfy the exact equation 13.2.5. We shall embark upon our attempt to do this in section 13.6, but we should first look at the following three sections.

13.4 *Kepler's Second Law*

In section 13.3 we made use of Kepler's second law, namely that the radius vector sweeps out equal areas in equal times. Explicitly,

$$\dot{B} = \tfrac{1}{2}h = \tfrac{1}{2}\sqrt{GMl}. \qquad 13.4.1$$

We are treating this as a two-body problem and therefore ignoring planetary perturbations. It is nevertheless worth reminding ourselves – from section 9.5 of chapter 9, especially equations 9.5.17, 9.4.3, 9.5.19, 9.5.20 and 9.5.21, of the precise meanings of the symbols in equation 13.4.1. The symbol h is the angular momentum per unit mass of the orbiting body, and l is the semi latus rectum of the orbit. If we are referring to the centre of mass of the two-body system as origin, then h and l are the angular momentum per unit mass of the orbiting body and the semi latus rectum relative to the centre of mass of the system, and M is the *mass function* $M^3/(M+m)^2$ of the system, M and m being the masses of Sun and planet respectively. In chapter 9 we used the symbol $\mathfrak{M}$ for the mass function. If we are referring to the centre of the Sun as origin, then h and l are the angular momentum per unit mass of the planet and the semi latus rectum of the planet's orbit relative to that origin, and M is the sum of the masses of Sun and planet, for which we used the symbol M in chapter 9. In any case, for all but perhaps the most massive asteroids, we are probably safe in regarding the mass of the orbiting body as being negligible compared with the mass of the Sun. In that case there is no distinction between the centre of the Sun and the centre of mass of the two-body system, and the M in equation 13.4.1 is then merely the mass of the Sun. (Note that I have *not* said that the barycentre of the entire solar system coincides with the centre of the Sun. The mass of Jupiter, for example, is nearly one thousandth of the mass of the Sun, and that is by no means negligible.)

The symbol G, of course, stands for the universal gravitational constant. Its numerical value is not known to any very high precision, and consequently the mass of the Sun is not known to any higher precision than G is. Approximate values for them are $G = 6.672 \times 10^{-11}$ N m^2 kg^{-2} and $M = 1.989 \times 10^{30}$ kg. The *product GM*, is known to considerable precision; it is $1.327\ 124\ 38 \times 10^{20}$ m^3 s^{-2}.

Definition: Until June 2012 the *astronomical unit of distance* (au) was defined as the radius of a circular orbit in which a body of negligible mass will, in the absence of planetary perturbations, move around the Sun at an angular speed of exactly 0.017 202 098 95 radians per mean solar day, or $1.990\ 983\ 675 \times 10^{-7}$ rad s^{-1}, or 0.985 607 668 6 degrees per mean solar day. This angular speed is sometimes called the *gaussian constant* and is given the symbol k. With this definition, the value of the astronomical unit is approximately $1.495\ 978\ 70 \times 10^{11}$ m.

However, in June 2012 the International Astronomical Union re-defined the astronomical unit as 149 597 870 700 metres exactly. This means that a body of negligible mass moving around the Sun in a circular orbit will, in the absence of planetary perturbations, move at an angular speed of approximately 0.017 202 098 95 radians per mean solar day, This angular speed is the *gaussian constant* k - but, with the new definition of the au, it is no longer regarded as one of the fundamental astronomical constants. The IAU also recommended that the official abbreviation for the astronomical unit should be au.

If we equate the centripetal acceleration of the hypothetical body moving in a circular orbit of radius 1 AU at angular speed k to the gravitational force on it per unit mass, we see that $ak^2 = GM/a^2$, so that

$$GM = k^2 a^3, \qquad 13.4.2$$

where a is the length of the astronomical unit and k is the gaussian constant.

13.5 *Coordinates*

We need to make use of several coordinate systems, and I reproduce here the descriptions of them from section 10.7 of chapter 10. You may wish to refer back to that chapter as a further reminder.

1. *Heliocentric plane-of-orbit.* $\odot xyz$ with the $\odot x$ axis directed towards perihelion. The polar coordinates in the plane of the orbit are the heliocentric distance r and the true anomaly υ. The z-component of the asteroid is necessarily zero, and $x = r \cos \upsilon$ and $y = r \sin \upsilon$.

2. *Heliocentric ecliptic.* $\odot XYZ$ with the $\odot X$ axis directed towards the First Point of Aries ♈, where Earth, as seen from the Sun, will be situated on or near September 22. The spherical coordinates in this system are the heliocentric distance r, the ecliptic longitude λ, and the ecliptic latitude β, such that $X = r\cos\beta\cos\lambda$, $Y = r\cos\beta\sin\lambda$ and $Z = r\sin\beta$.

3. *Heliocentric equatorial coordinates.* $\odot\xi\eta\zeta$ with the $\odot\xi$ axis directed towards the First Point of Aries and therefore coincident with the $\odot X$ axis . The angle between the $\odot Z$ axis and the $\odot\zeta$ axis is ε, the obliquity of the ecliptic. This is also the angle between the XY-plane (plane of the ecliptic, or of Earth's orbit) and the $\xi\eta$-plane (plane of Earth's equator). See figure X.4.

4. *Geocentric equatorial coordinates.* $\oplus xyz$ with the $\oplus x$ axis directed towards the First Point of Aries. The spherical coordinates in this system are the geocentric distance Δ, the right ascension α and the declination δ, such that $x = \Delta\cos\delta\cos\alpha$, $y = \Delta\cos\delta\sin\alpha$ and $z = \Delta\sin\delta$.

A summary of the relations between them is as follows

$$x = \Delta \cos\alpha \cos\delta = l\Delta = x_o + \xi, \qquad 13.5.1$$

$$y = \Delta \sin\alpha \cos\delta = m\Delta = y_o + \eta, \qquad 13.5.2$$

$$z = \Delta \sin\delta = n\Delta = z_o + \zeta. \qquad 13.5.3$$

Here, (l , m , n) are the direction cosines of the planet's geocentric radius vector. They offer an alternative way to (α , δ) for expressing the direction to the planet as seen from Earth. They are not independent but are related by

$$l^2 + m^2 + n^2 = 1. \qquad 13.5.4$$

The symbols x_o , y_o and z_o are the geocentric equatorial coordinates of the Sun. [I would prefer to use the solar symbol ⊙ (which I can find in the Math B font) as a subscript, but I have not found a way to incorporate this symbol *into the Word equation editor*. If any reader can help me with this, please contact me at jtatum@uvic.ca.]

13.6 *Example*

As we proceed with the theory, we shall try an actual numerical example as we go. We shall suppose that the following three observations are available:

0^h TT	R.A. (J2000.0)	Dec. (J2000.0)
2002 Jul 10	$21^h\ 15^m.40$ $= 318^o.8500$ = 5.564 982 rad	$+16^o\ 13'.8$ $= +16^o.2300$ = +0.283 267 rad
2002 Jul 15	$21^h\ 12^m.44$ $= 318^o.1100$ = 5.552 067 rad	$+16^o\ 03'.5$ $= +16^o.0583$ = + 0.280 271 rad
2002 Jul 25	$21^h\ 05^m.60$ $= 316^o.4000$ = 5.522 222 rad	$+15^o$ 24.8 $= +15^o.4133$ = + 0.269 013 rad

We shall suppose that the times given are 0^h TT, and that the observations were made by an observer at the centre of Earth. In practice, an observer will report his or her observations in Universal Time, and from the surface of Earth. We shall deal with these two refinements at a later time.

The "observations" given above are actually from an ephemeris for the minor planet 2 Pallas published by the Minor Planet Center of the International Astronomical Union. They will not be expected to reproduce exactly the elements also published by the MPC, because the ephemeris positions are rounded off to $0^{\mathrm{m}}.01$ and 0'.1, and of course the MPC elements are computed from all available observations, not just three. But we should be able to compute elements close to the correct ones. Observations are usually given to a precision of about 0.1 arcsec. For the purposes of the illustrative calculation let us start the calculation with the right ascensions and declinations given above to six decimal places as exact.

13.7 *Geocentric and Heliocentric Distances – First Attempt*

Let us write down the three heliocentric equatorial components of equation 13.2.1:

$$\xi_2 = a_1\xi_1 + a_3\xi_3, \qquad 13.7.1$$

$$\eta_2 = a_1\eta_1 + a_3\eta_3, \qquad 13.7.2$$

$$\zeta_2 = a_1\zeta_1 + a_3\zeta_3. \qquad 13.7.3$$

Now write $l\Delta - x_{\mathrm{o}}$ for ξ, etc., from equations 13.5.1,2,3 and rearrange to take the solar coordinates to the right hand side:

$$l_1a_1\Delta_1 - l_2\Delta_2 + l_3a_3\Delta_3 = a_1x_{\mathrm{o}1} - x_{\mathrm{o}2} + a_3x_{\mathrm{o}3}, \qquad 13.7.4$$

$$m_1a_1\Delta_1 - m_2\Delta_2 + m_3a_3\Delta_3 = a_1y_{\mathrm{o}1} - y_{\mathrm{o}2} + a_3y_{\mathrm{o}3}, \qquad 13.7.5$$

$$n_1a_1\Delta_1 - n_2\Delta_2 + n_3a_3\Delta_3 = a_1z_{\mathrm{o}1} - z_{\mathrm{o}2} + a_3z_{\mathrm{o}3}. \qquad 13.7.6$$

As a very first, crude, approximation, we can let $a_1 = b_1$ and $a_3 = b_3$, for we know b_1 and b_3 (in our numerical example, $b_1 = \frac{2}{3}$, $b_3 = \frac{1}{3}$), so we can solve equations 13.7.4,5,6 for the three geocentric distances. However, we shall eventually need to find the correct values of a_1 and a_3.

When we have solved these equations for the geocentric distances, we can then find the heliocentric distances from equations 13.5.1,2 and 3. For example,

$$\xi_1 = l_1\Delta_1 - x_{\mathrm{o}1} \qquad 13.7.7$$

and of course $$r_1^2 = \xi_1^2 + \eta_1^2 + \zeta_1^2. \qquad 13.7.8$$

In our numerical example, we have

$$
\begin{aligned}
l_1 &= +0.722\ 980\ 907\\
l_2 &= +0.715\ 380\ 933\\
l_3 &= +0.698\ 125\ 992\\[1ex]
m_1 &= -0.631\ 808\ 343\\
m_2 &= -0.641\ 649\ 261\\
m_3 &= -0.664\ 816\ 398\\[1ex]
n_1 &= +0.279\ 493\ 876\\
n_2 &= +0.276\ 615\ 882\\
n_3 &= +0.265\ 780\ 465
\end{aligned}
$$

As a check on the arithmetic, the reader can - and should - verify that

$$l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2 = 1$$

This does not verify the *signs* of the direction cosines, for which care should be taken.

From *The Astronomical Almanac* for 2002, we find that

$$
\begin{array}{lll}
x_{o1} = -0.306\ 728\ 3 & y_{o1} = +0.889\ 290\ 0 & z_{o1} = +0.385\ 549\ 5 \quad \text{AU}\\
x_{o2} = -0.386\ 194\ 4 & y_{o2} = +0.862\ 645\ 7 & z_{o2} = +0.373\ 999\ 6\\
x_{o3} = -0.536\ 330\ 8 & y_{o3} = +0.791\ 387\ 2 & z_{o3} = +0.343\ 100\ 4
\end{array}
$$

(For a fraction of a day, which will usually be the case, these coordinates can be obtained by nonlinear interpolation – see chapter 1, section 1.10.)

Equations 13.7.4,5,6 become

$$+0.481\ 987\ 271\Delta_1 - 0.\ 715\ 380\ 933\Delta_2 + 0.\ 232\ 708\ 664\Delta_3 = 0.002\ 931\ 933$$

$$-0.421\ 205\ 562\Delta_1 + 0.641\ 649\ 261\Delta_2 - 0.221\ 605\ 466\Delta_3 = -0.005\ 989\ 967$$

$$+0.186\ 329\ 251\Delta_1 - 0.276\ 615\ 882\Delta_2 + 0.088\ 593\ 488\Delta_3 = -0.002\ 599\ 800$$

I give below the solutions to these equations, which are our first crude approximations to the geocentric distances in AU, together with the corresponding heliocentric distances. I also give, for comparison, the correct values, from the published MPC ephemeris

First crude estimates		MPC	
$\Delta_1 = 2.725\ 71$	$r_1 = 3.485\ 32$	$\Delta_1 = 2.644$	$r_1 = 3.406$
$\Delta_2 = 2.681\ 60$	$r_2 = 3.481\ 33$	$\Delta_2 = 2.603$	$r_2 = 3.404$
$\Delta_3 = 2.610\ 73$	$r_3 = 3.474\ 71$	$\Delta_3 = 2.536$	$r_3 = 3.401$

This must justifiably give cause for some satisfaction, because we now have some idea of the geocentric distances of the planet at the instants of the three observations, though it is a little early to open the champagne bottles. We still have a little way to go, for we must refine our values of a_1 and a_3. Our first guesses, $a_1 = b_1$ and $a_3 = b_3$, are not quite good enough.

The key to finding the geocentric and heliocentric distances is to be able to determine the triangle ratios $a_1 = A_1/A_2$, $a_3 = A_3/A_2$ and the triangle/sector ratios a/b. The sector ratios are found easily from Kepler's second law. We have made our first very crude attempt to find the geocentric and heliocentric distances by assuming that the triangle ratios are equal to the sector ratios. It is now time to improve on that assumption, and to obtain better triangle ratios. After what may seem like a considerable amount of work, we shall obtain approximate formulas, equations 13.8.35a,b, for improved triangle ratios. The reader who does not wish to burden himself with the details of the derivation of these equations may proceed directly to them, near the end of Section 13.7

13.8 *Improved Triangle Ratios*

The equation of motion of the orbiting body is

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}. \qquad 13.8.1$$

If we recall equation 13.4.2, this can be written

$$\ddot{\mathbf{r}} = -k^2\left(\frac{a^3}{r^3}\right)\mathbf{r}. \qquad 13.8.2$$

If we now agree to express r in units of a (i.e. in Astronomical Units of length) and time in units of $1/k$ ($1/k$ = 58.132 440 87 mean solar days), this becomes merely

$$\ddot{\mathbf{r}} = -\frac{1}{r^3}\mathbf{r}. \qquad 13.8.3$$

In these units, GM has the value 1.

Now write the x- and y- components of this equation, where (x , y) are heliocentric coordinates in the plane of the orbit (see sections 13.5 or 10.7).

$$\ddot{x} = -\frac{x}{r^3} \qquad 13.8.4$$

and
$$\ddot{y} = -\frac{y}{r^3}, \qquad 13.8.5$$

where
$$x^2 + y^2 = r^2. \qquad 13.8.6$$

The areal speed is $\frac{1}{2}h = \frac{1}{2}\sqrt{GMl}$, or, in these units, $\frac{1}{2}\sqrt{l}$, where l is the semi latus rectum of the orbit in A.U.

Let the planet be at (x, y) at time t. Then at time $t + \delta t$ it will be at $(x + \delta x, y + \delta y)$, where

$$\delta x = \dot{x}\,\delta t + \tfrac{1}{2!}\ddot{x}(\delta t)^2 + \tfrac{1}{3!}\dddot{x}(\delta t)^3 + \tfrac{1}{4!}\ddddot{x}(\delta t)^4 + \ldots \qquad 13.8.7$$

and similarly for y.

Now, starting from equation 13.8.4 we obtain

$$\dddot{x} = \frac{3x\dot{r}}{r^4} - \frac{\dot{x}}{r^3} \qquad 13.8.8$$

and
$$\ddddot{x} = 3\left(\frac{\dot{x}\dot{r}}{r^4} + \frac{x\ddot{r}}{r^4} - \frac{4x\dot{r}^2}{r^5}\right) - \frac{r^3\ddot{x} - 3r^2\dot{x}\dot{r}}{r^6}. \qquad 13.8.9$$

(The comment in the paragraph preceding equation 3.4.16 may be of help here, in case this is heavy-going.)

Now $\ddot{x}$ and x are related by equation 13.8.4, so that we can write equation 13.8.9 with no time derivatives of x higher than the first, and indeed it is not difficult, because equation 13.8.4 is just $r^3\ddot{x} = -x$. We obtain

$$\ddddot{x} = x\left(\frac{1}{r^6} - \frac{12\dot{r}^2}{r^5} + \frac{3\ddot{r}}{r^4}\right) + \frac{6\dot{x}\dot{r}}{r^4}. \qquad 13.8.10$$

In a similar fashion, because of the relation 13.8.4, *all* higher time derivatives of x can be written with no derivatives of x higher than the first, and a similar argument holds for y.

Thus we can write equation 13.8.7 as

$$x + \delta x = Fx + G\dot{x} \qquad 13.8.11$$

and similarly for y:

$$y + \delta y = Fy + G\dot{y}, \tag{13.8.12}$$

where $$F = 1 - \frac{1}{2r^3}(\delta t)^2 + \frac{\dot{r}}{2r^4}(\delta t)^3 + \frac{1}{24}\left(\frac{1}{r^6} - \frac{12\dot{r}^2}{r^5} + \frac{3\ddot{r}}{r^4}\right)(\delta t)^4 + \ldots \tag{13.8.13}$$

and $$G = \delta t - \frac{1}{6r^3}(\delta t)^3 + \frac{\dot{r}}{4r^4}(\delta t)^4 + \ldots \tag{13.8.14}$$

Now we are going to look at the triangle and sector areas. From figure XIII.1 we can see that

$$\mathbf{A}_1 = \tfrac{1}{2}\mathbf{r}_2\times\mathbf{r}_3, \;\; \mathbf{A}_2 = \tfrac{1}{2}\mathbf{r}_1\times\mathbf{r}_3, \;\; \mathbf{A}_3 = \tfrac{1}{2}\mathbf{r}_1\times\mathbf{r}_2. \tag{13.8.15a,b,c}$$

Also, angular momentum per unit mass is $\mathbf{r} \times \mathbf{v}$ and Kepler's second law tells us that areal speed is half the angular momentum per unit mass and that it is constant and equal to $\frac{1}{2}\sqrt{l}$ (in the units that we are using), so that

$$\dot{\mathbf{B}} = \tfrac{1}{2}\mathbf{r_1}\times\dot{\mathbf{r}}_\mathbf{1} = \tfrac{1}{2}\mathbf{r_2}\times\dot{\mathbf{r}}_\mathbf{2} = \tfrac{1}{2}\mathbf{r_3}\times\dot{\mathbf{r}}_\mathbf{3}. \tag{13.8.16a,b,c}$$

All four of these vectors are parallel and perpendicular to the plane of the orbit, to that their magnitudes are just equal to their z-components. From the usual formulas for the components of a vector product we have, then,

$$A_1 = \tfrac{1}{2}(x_2y_3 - y_2x_3), \quad A_2 = \tfrac{1}{2}(x_1y_3 - y_1x_3), \quad A_3 = \tfrac{1}{2}(x_1y_2 - y_1x_2) \tag{13.8.17a,b,c}$$

and

$$\tfrac{1}{2}\sqrt{l} = \tfrac{1}{2}(x_1\dot{y}_1 - y_1\dot{x}_1) = \tfrac{1}{2}(x_2\dot{y}_2 - y_2\dot{x}_2) = \tfrac{1}{2}(x_3\dot{y}_3 - y_3\dot{x}_3). \tag{13.8.18a,b,c}$$

Now, start from the second observation (x_2, y_2) at instant t_2. We shall try to predict the third observation, using equations 13.8.11-14, in which $x + \delta x$ is x_3 and δt is $t_3 - t_2$, which we are calling (see section 13.3) τ_1. I shall make the subscripts for F and G the same as the subscripts for τ. Thus the F and G that connect observations 2 and 3 will have subscript 1, just as we are using the notation τ_1 for $t_3 - t_2$.

Thus we have

$$x_3 = F_1x_2 + G_1\dot{x}_2 \tag{13.8.19}$$

and $$y_3 = F_1 y_2 + G_1 \dot{y}_2 ,\qquad 13.8.20$$

where $$F_1 = 1 - \frac{1}{2r_2^3}\tau_1^2 + \frac{\dot{r}_2}{2r_2^4}\tau_1^3 + \frac{1}{24}\left(\frac{1}{r_2^6} - \frac{12\dot{r}_2^2}{r_2^5} + \frac{3\ddot{r}_2}{r_2^4}\right)\tau_1^4 + \ldots \qquad 13.8.21$$

and $$G_1 = \tau_1 - \frac{1}{6r_2^3}\tau_1^3 + \frac{\dot{r}_2}{4r_2^3}\tau_1^4 + \ldots \quad . \qquad 13.8.22$$

Similarly, the first observation is given by

$$x_1 = F_3 x_2 + G_3 \dot{x}_2 \qquad 13.8.23$$

and $$y_1 = F_3 y_2 + G_3 \dot{y}_2 ,\qquad 13.8.24$$

where, by substitution of $-\tau_3$ for δt,

$$F_3 = 1 - \frac{1}{2r_2^3}\tau_3^2 - \frac{\dot{r}_2}{2r_2^4}\tau_3^3 + \frac{1}{24}\left(\frac{1}{r_2^6} - \frac{12\dot{r}_2^2}{r_2^5} + \frac{3\ddot{r}_2}{r_2^4}\right)\tau_3^4 + \ldots \qquad 13.8.25$$

and $$G_3 = -\tau_3 + \frac{1}{6r_2^3}\tau_3^3 + \frac{\dot{r}_2}{4r_2^4}\tau_3^4 + \ldots \quad . \qquad 13.8.26$$

From equations 13.8.17,18,19,20,23,24, we soon find that

$$A_1 = \tfrac{1}{2} G_1 \sqrt{l} , \qquad A_2 = \tfrac{1}{2}(F_3 G_1 - F_1 G_3)\sqrt{l} , \qquad A_3 = -\tfrac{1}{2} G_3 \sqrt{l} . \qquad 13.8.27\text{a,b,c}$$

Now we do not yet know $\dot{r}$ or $\ddot{r}$, but we can take the expansions of F and G as far as τ^2. We then obtain, correct to τ^3:

$$A_1 = \tfrac{1}{2}\sqrt{l}\,\tau_1\left(1 - \frac{\tau_1^2}{6r_2^3}\right), \qquad 13.8.28$$

$$A_2 = \tfrac{1}{2}\sqrt{l}\,\tau_2\left(1 - \frac{\tau_2^2}{6r_2^3}\right), \qquad 13.8.29$$

and $$A_3 = \tfrac{1}{2}\sqrt{l}\,\tau_3\left(1 - \frac{\tau_3^2}{6r_2^3}\right). \qquad 13.8.30$$

Thus the triangle ratio $a_1 = A_1/A_2$ is

$$a_1 = \frac{\tau_1}{\tau_2}\left(1 - \frac{\tau_1^2}{6r_2^3}\right)\left(1 - \frac{\tau_2^2}{6r_2^3}\right)^{-1}, \qquad 13.8.31$$

which, to order τ^3, is $$a_1 = \frac{\tau_1}{\tau_2}\left(1 + \frac{(\tau_2^2 - \tau_1^2)}{6r_2^3}\right), \qquad 13.8.32$$

or, with $\tau_2 - \tau_1 = \tau_3$, $$a_1 = \frac{\tau_1}{\tau_2}\left(1 + \frac{\tau_3(\tau_2 + \tau_1)}{6r_2^3}\right). \qquad 13.8.33$$

Similarly, $$a_3 = \frac{\tau_3}{\tau_2}\left(1 + \frac{\tau_1(\tau_2 + \tau_3)}{6r_2^3}\right). \qquad 13.8.34$$

Further, with $\tau_1/\tau_2 = b_1$ and $\tau_3/\tau_2 = b_3$,

$$a_1 = b_1 + \frac{\tau_1\tau_3}{6r_2^3}(1 + b_1) \qquad \text{and} \qquad a_3 = b_3 + \frac{\tau_1\tau_3}{6r_2^3}(1 + b_3). \qquad 13.8.35\text{a,b}$$

These will serve as better approximations for the triangle ratios. Be aware, however, that equations 13.8.35a,b are only approximations, and do not give the *exact* values for a_1 and a_3.

13.9 *Iterating*

We can now use equations 13.8.35a,b and get a better estimate of the triangle ratios. The numerical data are

$$b_1 = 2/3, \quad b_3 = 1/3, \quad r_2 = 3.481\,33,$$

$\tau_1 = t_3 - t_2 = 10$ mean solar days and $\tau_3 = t_2 - t_1 = 5$ mean solar days, but recall that we are expressing time intervals in units of $1/k$, which is 58.132 440 87 mean solar days, and therefore

$$\tau_1 = 0.172\,021 \text{ and } \tau_3 = 0.086\,010.$$

Equations 13.8.35 then result in

$$a_1 = 0.666\,764, \quad a_3 = 0.333\,411.$$

Now we can go back to equation 13.7.4 and start again with our new values for the triangle ratios – und so weiter – until we obtain new values for Δ_1, Δ_2, Δ_3 and r_2. I show below in the first two columns the first crude estimates (already given above), in the second two columns the results of the first iteration, and, in the last two columns, the values given in the published IAU ephemeris.

	First crude estimates		First iteration		MPC	
	Δ	r	Δ	r	Δ	r
1	2.72571	3.48532	2.65825	3.41952	2.644	3.406
2	2.68160	3.48133	2.61558	3.41673	2.603	3.404
3	2.61073	3.47471	2.54579	3.41082	2.536	3.401

We see that we have made a substantial improvement, but we are not there yet. We can now calculate new values of a_1 and a_3 from equations 13.8.35a,b to get

$$a_1 = 0.666\,770 \qquad a_3 = 0.333\,416.$$

We *could* (if we so wished) now go back to equations 13.7.4,5,6, and iterate again. However, this will result in only small changes to a_1, a_3, Δ and r, and we have to bear in mind that equations 13.8.35a,b are only approximations (to order τ^3). Therefore, even if successive iterations converge, they will still not give precise correct answers for Δ and r.

To anticipate, eventually we shall arrive at some exact equations (equations 13.12.25 and 13.12.26) that will allow us to solve the problem. But these equations will not be easy to solve. They have to be solved by iteration using a reasonably good first guess. It is our present aim to obtain a reasonably good first guess for a_1, a_3, Δ and r, in order to prepare for the solution of the exact equations 13.12.25 and 13.12.26. Our current values of a_1 and a_3, while not exact, will enable us to solve equations 13.12.25 and 13.12.26 exactly, so we should now, rather than going back again to equations 13.7.4,5,6, proceed straight to Sections 13.11, 13.12 and 13.13.

Nevertheless, in the following section, we provide (in equations 13.10.9 and 13.10.10), after considerable effort, higher-order expansions for a_1 and a_3. These may be useful, but for reasons explained in the previous paragraph, it may be easier to skip Section 13.10 entirely.

13.10 *Higher-order Approximation*

The reason that we made the approximation to order τ^3 was that, in evaluating the expressions for F_1, G_1, F_3 and G_3, we did not know the radial velocity $\dot{r}_2$. Perhaps we can now evaluate it.

Exercise. Show that the radial velocity of a particle in orbit around the Sun, when it is at a distance r from the Sun, is

Ellipse: $$\dot{r} = \mp\sqrt{\frac{GM}{a_0}}\left(\frac{a^2e^2 - (a-r)^2}{ar^2}\right)^{1/2}, \qquad 13.10.1$$

Parabola: $$\dot{r} = \mp\sqrt{\frac{GM(r-q)}{a_0}}, \qquad 13.10.2$$

Hyperbola: $$\dot{r} = \mp\sqrt{\frac{GM}{a_0}}\left(\frac{(a+r)^2 - a^2e^2}{ar^2}\right)^{1/2}. \qquad 13.10.3$$

Show that the radial velocity is greatest at the ends of a latus rectum.

Here a_0 is the astronomical unit, a is the semi major axis of the elliptic orbit or the semi transverse axis of the hyperbolic orbit, q is the perihelion distance of the parabolic orbit, and e is the orbital eccentricity. The − sign is for pre-perihelion, and the + sign is for post-perihelion.

Unfortunately, while this is a nice exercise in orbit theory, we do not know the eccentricity, so these formulas at present are of no use to us.

However, we can calculate the heliocentric distances at the times of the first and third observations by exactly the same method as we used for the second observation. Here are the results for our numerical example, after one iteration. The units, of course, are AU. Also indicated are the instants of the observations, taking $t_2 = 0$ and expressing the other instants in units of $1/k$ (see section 13.8).

$t_1 = -\tau_3 = -0.086\,010\,494\,75 \qquad r_1 = 3.419\,52$

$t_2 = 0 \qquad r_2 = 3.416\,73$

$t_3 = +\tau_1 = +0.172\,020\,989\,5 \qquad r_3 = 3.410\,82$

We can fit a quadratic expression to this, of the form:

$$r = c_0 + c_1t + c_2t^2 \qquad 13.10.4$$

With our choice of time origin $t_2 = 0$, c_0 is obviously just equal to r_2, so we have just two constants, c_1 and c_2 to solve for. We can then calculate the radial velocity at the time of the second observation from

$$\dot{r}_2 = c_1 + 2c_2t_2. \qquad 13.10.5$$

We can calculate A_1, A_2 and A_3 in the same manner as before, up to τ^4 rather than just τ^3. The algebra is slightly long and tedious, but straightforward. Likewise, the results look long and unwieldy, but there is no difficulty in programming them for a computer, and the actual calculation is, with a modern computer, virtually instantaneous. The results of the algebra that I give below are taken from the book *Determination of Orbits* by A.D. Dubyago (which has been the basis of much of this chapter). I haven't checked the algebra myself, but the conscientious reader will probably want to do so himself or herself.

$$A_1 = \tfrac{1}{2}\sqrt{l}\,\tau_1\left(1 - \frac{\tau_1^2}{6r_2^3} + \frac{\tau_1^3}{4r_2^4}\dot{r}_2\right), \qquad 13.10.6$$

$$A_2 = \tfrac{1}{2}\sqrt{l}\,\tau_2\left(1 - \frac{\tau_2^2}{6r_2^3} + \frac{\tau_2^2(\tau_1 - \tau_3)}{4r_2^4}\dot{r}_2\right), \qquad 13.10.7$$

$$A_3 = \tfrac{1}{2}\sqrt{l}\,\tau_3\left(1 - \frac{\tau_3^2}{6r_2^3} - \frac{\tau_3^3}{4r_2^4}\dot{r}_2\right). \qquad 13.10.8$$

And from these,

$$a_1 = \frac{\tau_1}{\tau_2}\left(1 + \frac{\tau_3(\tau_2 + \tau_1)}{6r_2^3} + \frac{\tau_3(\tau_3(\tau_1 + \tau_3) - \tau_1^2)}{4r_2^4}\dot{r}_2\right). \qquad 13.10.9$$

and

$$a_3 = \frac{\tau_3}{\tau_2}\left(1 + \frac{\tau_1(\tau_2 + \tau_3)}{6r_2^3} - \frac{\tau_1(\tau_1(\tau_1 + \tau_3) - \tau_3^2)}{4r_2^4}\dot{r}_2\right). \qquad 13.10.10$$

This might result in slightly better values for a_1 and a_3. I have not calculated this for our numerical example here, for reasons given in Section 13.9. We can move on to the next section, using our current vales of a_1 and a_3, namely

$$a_1 = 0.666\,770 \quad \text{and } a_3 = 0.333\,416.$$

13.11 *Light-time Correction*

Before going further, however, our current estimates of the geocentric distances are now sufficiently good that we should make the light-time corrections. The observed positions of the planet were not the positions that they occupied at the instants when they were observed. It actually occupied these observed positions at times $t_1 - \Delta_1 / c$, $t_2 - \Delta_2 / c$ and $t_3 - \Delta_3 / c$. Here, c is the speed of light, which, as everyone knows, is 10065.320 astronomical units per $1/k$. The calculation up to this point can now be repeated with these new times. This may seem tedious, but of course with a computer, all one needs is a single statement telling the computer to go to the beginning of the program and to do it again. I am not going to do it with our particular numerical example, since the "observations" that we are using are in fact predicted positions from a Minor Planet Center ephemeris.

13.12 *Sector-Triangle Ratio*

We recall that it is easy to determine the ratio of adjacent sectors swept out by the radius vector. By Kepler's second law, it is just the ratio of the two time intervals. What we really need, however, are the triangle ratios, which are related to the heliocentric distance by equation 13.2.1. Oh, wouldn't it just be so nice if someone were to tell us the ratio of a sector area to the corresponding triangle area! We shall try in this section to do just that.

Notation: Triangle ratios:

$$a_1 = A_1 / A_2, \qquad a_3 = A_3 / A_2. \qquad 13.12.1\text{a,b}$$

Sector ratios:

$$b_1 = B_1 / B_2, \qquad b_3 = B_3 / B_2. \qquad 13.12.2\text{a,b}$$

Sector-triangle ratios:

$$R_1 = \frac{B_1}{A_1} \quad , \quad R_2 = \frac{B_2}{A_2} \quad , \quad R_3 = \frac{B_3}{A_3} \quad , \qquad 13.12.3\text{a,b,c}$$

from which it follows that

$$a_1 = \frac{R_2}{R_1} b_1 \quad , \quad a_3 = \frac{R_2}{R_3} b_3 \quad . \qquad 13.12.4\text{a,b}$$

We also recall that subscript 1 for areas refers to observations 2 and 3; subscript 2 to observations 3 and 1; and subscript 3 to observations 1 and 2. Let us see, then, whether we can determine R_3 from the first and second observations.

Readers who wish to avoid the heavy algebra may proceed direct to equations 13.12.25 and 13.12.26, which will enable the calculation of the sector-triangle ratios.

Let (r_1 , υ_1) and (r_2 , υ_2) be the polar coordinates (i.e. heliocentric distance and true anomaly) in the plane of the orbit of the planet at the instant of the first two observations. In concert with our convention for subscripts involving two observations, let

$$2f_3 = \upsilon_2 - \upsilon_1 . \qquad 13.12.5$$

We have $R_3 = B_3/A_3$. From equation 13.4.1, which is Kepler's second law, we have, in the units that we are using, in which $GM = 1$, $\dot{B} = \frac{1}{2}\sqrt{l}$ and therefore $B_3 = \frac{1}{2}\sqrt{l}\,\tau_3$. Also, from the z-component of equation 13.8.15c, we have $A_3 = \frac{1}{2}r_1 r_2 \sin(\upsilon_2 - \upsilon_1)$.

Therefore
$$R_3 = \frac{\sqrt{l}\,\tau_3}{r_1 r_2 \sin(\upsilon_2 - \upsilon_1)} = \frac{\sqrt{l}\,\tau_3}{r_1 r_2 \sin 2f_3}. \qquad 13.12.6a$$

In a similar manner, we have

$$R_1 = \frac{\sqrt{l}\,\tau_1}{r_2 r_3 \sin(\upsilon_3 - \upsilon_2)} = \frac{\sqrt{l}\,\tau_1}{r_2 r_3 \sin 2f_1} \qquad 13.12.6b$$

$$R_2 = \frac{\sqrt{l}\,\tau_2}{r_3 r_1 \sin(\upsilon_3 - \upsilon_1)} = \frac{\sqrt{l}\,\tau_2}{r_3 r_1 \sin 2f_2}. \qquad 13.12.6c$$

I would like to eliminate l from here.

I now want to recall some geometrical properties of an ellipse and a property of an elliptic orbit. By glancing at figure II.11, or by multiplying equations 2.3.15 and 2.3.16, we immediately see that $r\cos\upsilon = a(\cos E - e)$, and hence by making use of a trigonometric identity we find

$$r\cos^2 \tfrac{1}{2}\upsilon = a(1-e)\cos^2 \tfrac{1}{2}E, \qquad 13.12.7$$

and in a similar manner it is easy to show that

$$r\sin^2 \tfrac{1}{2}\upsilon = a(1+e)\sin^2 \tfrac{1}{2}E. \qquad 13.12.8$$

Here E is the eccentric anomaly.

Also, the mean anomaly at time t is defined as $\frac{2\pi}{P}(t-T)$ and is also equal (via Kepler's equation) to $E - e\sin E$. The period of the orbit is related to the semi major axis of its orbit by Kepler's third law: $P^2 = \frac{4\pi^2}{GM}a^3$. (This material is covered on Chapter 10.) Hence we have (in the units that we are using, in which $GM = 1$):

$$E - e\sin E = \frac{t - T}{a^{3/2}}\,, \qquad 13.12.9$$

where T is the instant of perihelion passage.

Now introduce $$2f_3 = \upsilon_2 - \upsilon_1\,, \qquad 13.12.10$$

$$2F_3 = \upsilon_2 + \upsilon_1\,, \qquad 13.12.11$$

$$2g_3 = E_2 - E_1\,, \qquad 13.12.12$$

$$2G_3 = E_2 + E_1\,. \qquad 13.12.13$$

From equation 13.12.7 I can write

$$\sqrt{r_1 r_2}\cos\tfrac{1}{2}\upsilon_1\cos\tfrac{1}{2}\upsilon_2 = a(1-e)\cos\tfrac{1}{2}E_1\cos\tfrac{1}{2}E_2 \qquad 13.12.14$$

and from equation 13.12.8 I can write

$$\sqrt{r_1 r_2}\sin\tfrac{1}{2}\upsilon_1\sin\tfrac{1}{2}\upsilon_2 = a(1+e)\sin\tfrac{1}{2}E_1\sin\tfrac{1}{2}E_2\,. \qquad 13.12.15$$

I now make use of the sum of the sum-and-difference formulas from page 38 of chapter 3, namely $\cos A\cos B = \frac{1}{2}(\cos S + \cos D)$ and $\sin A\sin B = \frac{1}{2}(\cos D - \cos S)$, to obtain

$$\sqrt{r_1 r_2}\,(\cos F_3 + \cos f_3) = a(1-e)(\cos G_3 + \cos g_3) \qquad 13.12.16$$

and $$\sqrt{r_1 r_2}\,(\cos f_3 - \cos F_3) = a(1+e)(\cos g_3 - \cos G_3)\,. \qquad 13.12.17$$

On adding these, we obtain

$$\sqrt{r_1 r_2}\,\cos f_3 = a(\cos g_3 - e\cos G_3)\,. \qquad 13.12.18$$

I leave it to the reader to derive in a similar manner (also making use of the formula for the semi latus rectum $l = a(1 - e^2)$)

$$\sqrt{r_1 r_2}\,\sin f_3 = \sqrt{a}\sqrt{l}\sin g \qquad 13.12.19$$

and $$r_1 + r_2 = 2a(1 - e\cos g_3\cos G_3)\,. \qquad 13.12.20$$

We can eliminate $e\cos G$ from equations 13.12.18 and 13.12.20:

$$r_1 + r_2 - 2\sqrt{r_1 r_2}\cos f_3 \cos g_3 = 2a\sin^2 g_3 \qquad 13.12.21$$

Also, if we write equation 13.12.9 for the first and second observations and take the difference, and then use the formula on page 35 of chapter 3 for the difference between two sines, we obtain

$$2(g_3 - e\sin g_3 \cos G_3) = \frac{\tau_3}{a^{3/2}} \cdot \qquad 13.12.22$$

Eliminate $e \cos G_3$ from equations 13.12.18 and 13.12.22:

$$2g_3 - \sin 2g_3 + \frac{2\sqrt{r_1 r_2}}{a}\sin g_3 \cos f_3 = \frac{\tau_3}{a^{3/2}} \cdot \qquad 13.12.23$$

Also, eliminate l from equations 13.12.6 and 13.12.19:

$$R_3 = \frac{\tau_3}{2\sqrt{a}\sqrt{r_1 r_2}\cos f_3 \sin g_3} \cdot \qquad 13.12.24$$

We have now eliminated F_3, G_3 and e, and we are left with equations 13.12.21, 23 and 24, the first two of which I now repeat for easy reference:

$$r_1 + r_2 - 2\sqrt{r_1 r_2}\cos f_3 \cos g_3 = 2a\sin^2 g_3 \qquad 13.12.21$$

$$2g_3 - \sin 2g_3 + \frac{2\sqrt{r_1 r_2}}{a}\sin g_3 \cos f_3 = \frac{\tau_3}{a^{3/2}} \cdot \qquad 13.12.23$$

In these equations we already know an approximate value for f_3 (we'll see how when we resume our numerical example); the unknowns in these equations are R_3 , a and g_3, and it is R_3 that we are trying to find. Therefore we need to eliminate a and g_3. We can easily obtain a from equation 13.12.24, and, on substitution in equations 13.12.21 and 23 we obtain, after some algebra:

$$R_3^2 = \frac{M_3^2}{N_3 - \cos g_3} \qquad 13.12.25$$

and

$$R_3^3 - R_3^2 = \frac{M_3^2(g_3 - \sin g_3 \cos g_3)}{\sin^3 g_3}, \qquad 13.12.26$$

where $$M_3 = \frac{\tau_3}{2\left(\sqrt{r_1 r_2}\cos f_3\right)^{3/2}} \qquad 13.12.27$$

and $$N_3 = \frac{r_1 + r_2}{2\sqrt{r_1 r_2}\cos f_3}. \qquad 13.12.28$$

Similar equations for R_1 and R_2 can be obtained by cyclic permutation of the subscripts. Equations 13.12.25 and 26 are two simultaneous equations in R_3 and g_3. Their solution is given as an example in section 1.9 of chapter 1, so we can now assume that we can calculate the sector-triangle ratios.

We can then calculate better triangle ratios from equations 13.12.4 and return to equations 13.7.4, 5 and 6 to get better geocentric distances. From equations 13.7.8 and 9 calculate the heliocentric distances. Make the light-time corrections. (I am not doing this in our numerical example because our original positions were not actual observations, but rather were ephemeris positions.) Then go straight to this section (13.12) again, until you get to here again. Repeat until the geocentric distances do not change.

13.13 *Resuming the Numerical Example*

Let us start with our previous iteration

$$\begin{array}{lll} \Delta_1 = 2.65825 & & r_1 = 3.41952 \\ \Delta_2 = 2.61558 & & r_2 = 3.41673 \\ \Delta_3 = 2.54579 & & r_3 = 3.41082 \end{array}$$

- or rather with the more precise values that will at this stage presumably be stored in our computer.

These are the values that we had reached when we last left the numerical example.

I promised to say how we know f_3. We defined $2f_3$ as $\upsilon_2 - \upsilon_1$, and this is the angle between the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$. Thus

$$\cos 2f_3 = \frac{\xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2}{r_1 r_2}. \qquad 13.13.1$$

The heliocentric coordinates can be obtained from equations 13.5.1, 2 and 3. For example,

$$\xi_1 = l_1\Delta_1 - x_{o1}, \qquad 13.13.2$$

and of course $$r_1 = \sqrt{\xi_1^2 + \eta_1^2 + \zeta_1^2}. \qquad 13.13.3$$

We know how to find the components (ξ, η, ζ) of the heliocentric radius vector (see equations 13.7.8 and 9), and so we can now find f_3. I obtain

$$\cos 2f_3 = 0.999\ 929\ 1\ , \qquad \cos f_3 = 0.999\ 982\ 3\ .$$

This means that the true anomaly is advancing at about $0^{\circ}.68$ in five days. It is interesting to see whether we are on the right track. According to the MPC, Pallas has a period of 4.62 years, which means that, on average, it will move through $1^{\circ}.067$ in five days. But Pallas has a rather eccentric orbit (according to the MPC, $e = 0.23$). The semi major axis of the orbit must be $P^{2/3} = 2.77$ AU (which agrees with the MPC), and therefore its aphelion distance $a(1 + e)$ is about 3.41 AU. Thus Pallas must be close to aphelion in July 2002. By conservation of angular momentum, its angular motion at aphelion must be less than its mean motion by a factor of $(1 + e)^2$, so the increase in the true anomaly in five days should be about $1^{\circ}.067/1.23^2$ or $0^{\circ}.71$. Thus we do seem to be on the right track.

We can now calculate M_3 and N_3 from equations 13.12.27 and 28:

$$M_3^2 = 0.000\ 046\ 313\ 0$$

$$N_3 = 1.000\ 018$$

and so we have the following equations 13.12.25 and 26 for the sector-triangle ratios:

$$R_3^2 = \frac{0.000\ 046\ 313\ 0}{1.000\ 018 - \cos g_3}$$

and $$R_3^3 - R_3^2 = \frac{0.000\ 046\ 313\ 0(g_3 - \sin g_3 \cos g_3)}{\sin^3 g_3}.$$

Since we discussed how to solve these equations in section 1.9 of chapter 1, I merely give the solutions here. The one useful hint worth giving is that you can make the first guess for the iteration for g_3 equal to f_3, which we know ($\cos f_3 = 0.999\ 982\ 3$), and $R_3 = 1$.

$$\cos g_3 = 0.999\ 972\ , \qquad R_3 = 1.000\ 031$$

We can proceed similarly with R_1 and R_2.

Here is a summary:

Subscript	$\cos f$	M^2	N	$\cos g$	R

1	0.999 928 7	$1.859\ 91 \times 10^{-4}$	1.000 072	0.999 886	1.000 124
2	0.999 839 9	$4.180\ 80 \times 10^{-4}$	1.000 161	0.999 743	1.000 279
3	0.999 982 3	$4.631\ 30 \times 10^{-5}$	1.000 018	0.999 972	1.000 031

Our new triangle ratios will be

$$a_1 = \frac{R_2}{R_1} b_1 = \frac{1.000\ 279}{1.000\ 124} \times \frac{2}{3} = 0.666\ 770$$

and

$$a_3 = \frac{R_2}{R_3} b_3 = \frac{1.000\ 279}{1.000\ 031} \times \frac{1}{3} = 0.333\ 416.$$

We can now go back to equations 13.7.4,5 and 6, and calculate the geocentric and heliocentric distances anew. Skip sections 13.8, 13.9 and 13.10, and calculate new sector- triangle ratios and hence new triangle ratios, and repeat until convergence is obtained. After three iterations, I obtained convergence to six significant figures and after seven iterations I obtained convergence to 11 significant figures. The results to six significant figures are as follows:

$$\begin{array}{lll} \Delta_1 = 2.65403 & & r_1 = 3.41539 \\ \Delta_2 = 2.61144 & & r_2 = 3.41268 \\ \Delta_3 = 2.54172 & & r_3 = 3.40681 \end{array}$$

This is not to be expected to agree exactly with the published MPC values, which are based on all available Pallas observations, whereas we arbitrarily chose three approximate ephemeris positions, but, based on these three positions, we have now broken the back of the problem and have found the geocentric and heliocentric distances.

13.14 *Summary So Far*

1. Gather together the three observations (t , α , δ).
2. Convert t from UT to TT. (See Chapter 7.)
3. Calculate or look up and interpolate the solar coordinates.
4. Calculate the geocentric direction cosines of the planet. (Equations 13.5.1-3)
5. Calculate the first approximation to the geocentric distances, using $a_1 = b_1$, $a_3 = b_3$. (Equations 13.7.4-6)
6. Calculate the heliocentric distances. (Equations 13.7.7-8)
7. Improve a_1 and a_3. (Equations 13.8.32-34) Do steps 6 and 7 again.
8. Optional. Calculate $\dot{r}_2$ (equation 13.10.4) and improve a_1 and a_3 again (equations 13.10.9-10) and again repeat steps 6 and 7.
9. Make the light travel time corrections for the planet, and go back to step 3! Repeat 6 and 7 but of course with your best current a_1 and a_3.

10. Calculate f_1 , f_2 , f_3 and the three values of M^2 and N. (Equations 13.13.1, 13.12.27-28) and solve equations 13.12.25-26 for the sector-triangle ratios. The method of solution of these equations is given in chapter 1, section 1.9.
11. Calculate new triangle ratios (equations 13.12.4a,b) – and start all over again!

By this stage we know the geocentric and heliocentric distances, and it is fairly straightforward from this point, at least in the sense that there are no further iterations, and we can just proceed from step to step without having to repeat it all over again. The main problem in computing the angular elements is likely to be in making sure that the angles you obtain (when you calculate inverse trigonometric functions such as arcsin, arccos, arctan) are in the correct quadrant. If your calculator or computer has an ATAN2 facility, make good use of it!

13.15 *Calculating the Elements*

We can now immediately calculate the semi latus rectum from equation 13.12.6a (recalling that $2f_3 = \upsilon_2 - \upsilon_1$, so that everything except l in the equation is already known.) In fact we have three opportunities for calculating the semi latus rectum by using each of equations 13.12.6a,b,c, and this serves as a check on the arithmetic. For our numerical example, I obtain

$$l = 2.61779$$

identically (at least to eleven significant figures) for each of the three permutations.

Now, on referring to equation 2.3.37, we recall that the polar equation to an ellipse is

$$r = \frac{l}{1 + e\cos\upsilon}. \qquad 13.15.1$$

We therefore have, for the first and third observations,

$$e\cos\upsilon_1 = l/r_1 - 1 \qquad 13.15.2$$

and, admitting that $\upsilon_3 = \upsilon_1 + 2f_2$,

$$e\cos(\upsilon_1 + 2f_2) = l/r_3 - 1. \qquad 13.15.3$$

We observe that, in equations 13.15.2 and 13.15.3, the only quantities we do not already know are υ_1 and e – so we are just about to find our first orbital element, the eccentricity!

A hint for solving equations 13.15.2 and 3: Expand $\cos(\upsilon_1 + 2f_2)$. Take $e\sin\upsilon_1$ to the left hand side, and equation 13.15.3 will become

$$e\sin\upsilon_1 = \frac{(l/r_1 - 1).\cos 2f_2 - (l/r_3 - 1)}{\sin 2f_2}. \qquad 13.15.4$$

After this, it is easy to solve equations 13.15.2 and 13.15.4 for e and for υ_1. The other true anomalies are given by $\upsilon_2 = \upsilon_1 + 2f_3$ and $\upsilon_3 = \upsilon_1 + 2f_2$. A check on the arithmetic may (and should) be performed by carrying out the same calculation for the first and second observations and for the second and third observations. For all three, I obtained

$$\underline{e = 0.23875}$$

We have our first orbital element!

(The MPC value for the eccentricity for this epoch is 0.22994 – but this is based on all available observations, and we cannot expect to get the MPC value from just three hypothetical "observations".)

The true anomalies at the times of the three observations are

$$\upsilon_1 = 191^\circ.99814 \qquad \upsilon_2 = 192^\circ.68221 \qquad \upsilon_3 = 194^\circ.05377$$

After that, the semi major axis is easy from equation 2.3.10, $l = a(1 - e^2)$, for the semi latus rectum of an ellipse. We find

$$\underline{a = 2.77602\text{ AU}}$$

The period in sidereal years is given by $P^2 = a^3$, and is therefore 4.62524 sidereal years. This is not one of the six independent elements, since it is always related to the semi major axis by Kepler's third law, so it doesn't merit the extra dignity of being underlined. However, it is certainly worth converting it to mean solar days by multiplying by 365.25636. We find that $P = 1689.39944$ days.

The next element to yield will be the time of perihelion passage. We find the eccentric anomalies for each of the three observations from any of equations 2.3.16, 17a, 17b or 17c. For example:

$$\cos E = \frac{e + \cos\upsilon}{1 + e\cos\upsilon}. \qquad 13.15.5$$

Then the time of perihelion passage will come from equations 9.6.4 and 9.6.5:

$$T = t - \frac{P}{2\pi}(E - e\sin E) + nP. \qquad 13.15.6$$

With $n = 1$ I make this $\underline{T = t_1 + 756^{\mathrm{d}}.1319}$

The next step is to calculate the *P*s and *Q*s. These are defined in equation 10.9.40. They are the direction cosines relating the heliocentric plane-of-orbit basis set to the heliocentric equatorial basis set.

Exercise. Apply equation 10.9.50 to the first and third observations to show that

$$P_x = \frac{\xi_1 r_3 \sin \upsilon_3 - \xi_3 r_1 \sin \upsilon_1}{r_1 r_3 \sin 2f_2} \qquad 13.15.7$$

and

$$Q_x = \frac{\xi_3 r_1 \cos \upsilon_1 - \xi_1 r_3 \cos \upsilon_3}{r_1 r_3 \sin 2f_2}. \qquad 13.15.8$$

From equations 10.9.51 and 52, find similar equations for P_y , Q_y , P_z , Q_z .

The numerical work can and should be checked by calculating these direction cosines also from the first and second, and from the second and third, observations. Check also that $P_x^2 + P_y^2 + P_z^2 = Q_x^2 + Q_y^2 + Q_z^2 = 1$. I get

$$P_x = -0.48044 \qquad P_y = +0.86568 \qquad P_z = -0.14059$$

$$Q_x = -0.87392 \qquad Q_y = -0.45907 \qquad Q_z = +0.15978$$

(Remember that my computer is carrying all significant figures to double precision, though I print out here only a limited number of significant figures. You will not get exactly my numbers unless you, too, carry all significant figures and do not prematurely round off.)

The direction cosines are related to the Eulerian angles, of course, by equations 10.9.41-46 (how could you possibly forget?!). All (!) you have to do, then, is to solve these six equations for the Eulerian angles. (You need six equations to remove quadrant ambiguity from the angles. Remember the ATAN2 function on your computer – it's an enormous help with quadrants.)

Exercise. Show that (or verify at any rate) that:

$$\sin\omega\sin i = P_z\cos\varepsilon - P_y\sin\varepsilon \qquad 13.15.9$$

and

$$\cos\omega\sin i = Q_z\cos\varepsilon - Q_y\sin\varepsilon. \qquad 13.15.10$$

You can now solve this for the argument of perihelion ω. Don't yet try to solve it for the inclination. (Why not?!) Using $\varepsilon = 23^\circ.438\,960$ for the obliquity of the ecliptic of date (calculated from page B18 of the 2002 *Astronomical Almanac*), I get

$$\underline{\omega = 304^\circ.81849}$$

Exercise. Show that (or verify at any rate) that:

$$\sin\Omega = (P_y\cos\omega - Q_y\sin\omega)\sec\varepsilon \qquad 13.15.11$$

and

$$\cos\Omega = P_x\cos\omega - Q_x\sin\omega. \qquad 13.15.12$$

From these, I find:

$$\underline{\Omega = 172^\circ.64776}$$

One more to go!

Exercise. Show that (or verify at any rate) that:

$$\cos i = -(P_x\sin\omega + Q_x\cos\omega)\csc\Omega\,. \qquad 13.15.13$$

You can now solve this with equation 13.15.9 or 13.15.10 (or both, as a check on the arithmetic) for the inclination. I get

$$\underline{i = 35^\circ.20872}$$

Here they are, all together:

$a = 2.77602$ AU	$i = 35^\circ.20872$
$e = 0.23875$	$\Omega = 172^\circ.64776$
$T = t_1 + 756^{\rm d}.1319$	$\omega = 304^\circ.81849$

Have we made any mistakes? Well, presumably after you read chapter 10 you wrote a program to generate an ephemeris. So now, use these elements to see whether they will reproduce the original observations! Incidentally, to construct an ephemeris, there is no need actually to use the elements – you can use the Ps and Qs instead.

13.16 *Topocentric-Geocentric Correction*

In section 13.1 I indicated two small (but not negligible) corrections that needed to be made, namely the ΔT correction (which can be made at the very start of the calculation) and the light-time correction, which can be made as soon as the geocentric distances have been determined – after which it is necessary to recalculate the geocentric distances from the beginning! I did not actually make these corrections in our numerical example, but I indicated how to do them.

There is another small correction that needs to be made. The diameter of Earth subtends an angle of 17".6 at 1 AU, so the observed position of an asteroid depends appreciably on where it is observed from on Earth's surface. Observations are, of course, reported as *topocentric* – i.e. from the place (τοπος) where the observer was situated. They must be corrected by the computer to *geocentric* positions – but of course that can't be done until the distances are known. As soon as the distances are known, the light-time and the topocentric-geocentric corrections can be made. Then, of course, one has to return to the beginning and recompute the distances – possibly more than once until convergence is reached. This section shows how to make the topocentric-geocentric correction.

We have used the notation (x , y , z) for geocentric coordinates, and I shall use (x' , y' , z') for topocentric coordinates. In figure XIII.3 I show Earth from a point in the equatorial plane, and from above the north pole. The radius of Earth is R, and the radius of a small circle of latitude ϕ (where the observer is situated) is $R\cos\phi$. The x- and x'-axes are directed towards the first point of Aries, ♈.

It should be evident from the figure that the corrections are given by

$$x' = x - R\cos\phi\cos\mathrm{LST}, \qquad 13.16.1$$

$$y' = y - R\cos\phi\sin\mathrm{LST} \qquad 13.16.2$$

and

$$z' = z - R\sin\phi\,. \qquad 13.16.3$$

Any observer who submits observations to the Minor Planet Center is assigned an *Observatory Code*, a three-digit number. This code not only identifies the observer, but, associated with the Observatory Code, the Minor Planet Center keeps a record of the quantities R cos ϕ and R sin ϕ in AU. These quantities, in the notation employed by the MPC, are referred to as $-\Delta_{xy}$ and $-\Delta_z$ respectively. They are unique to each observing site.

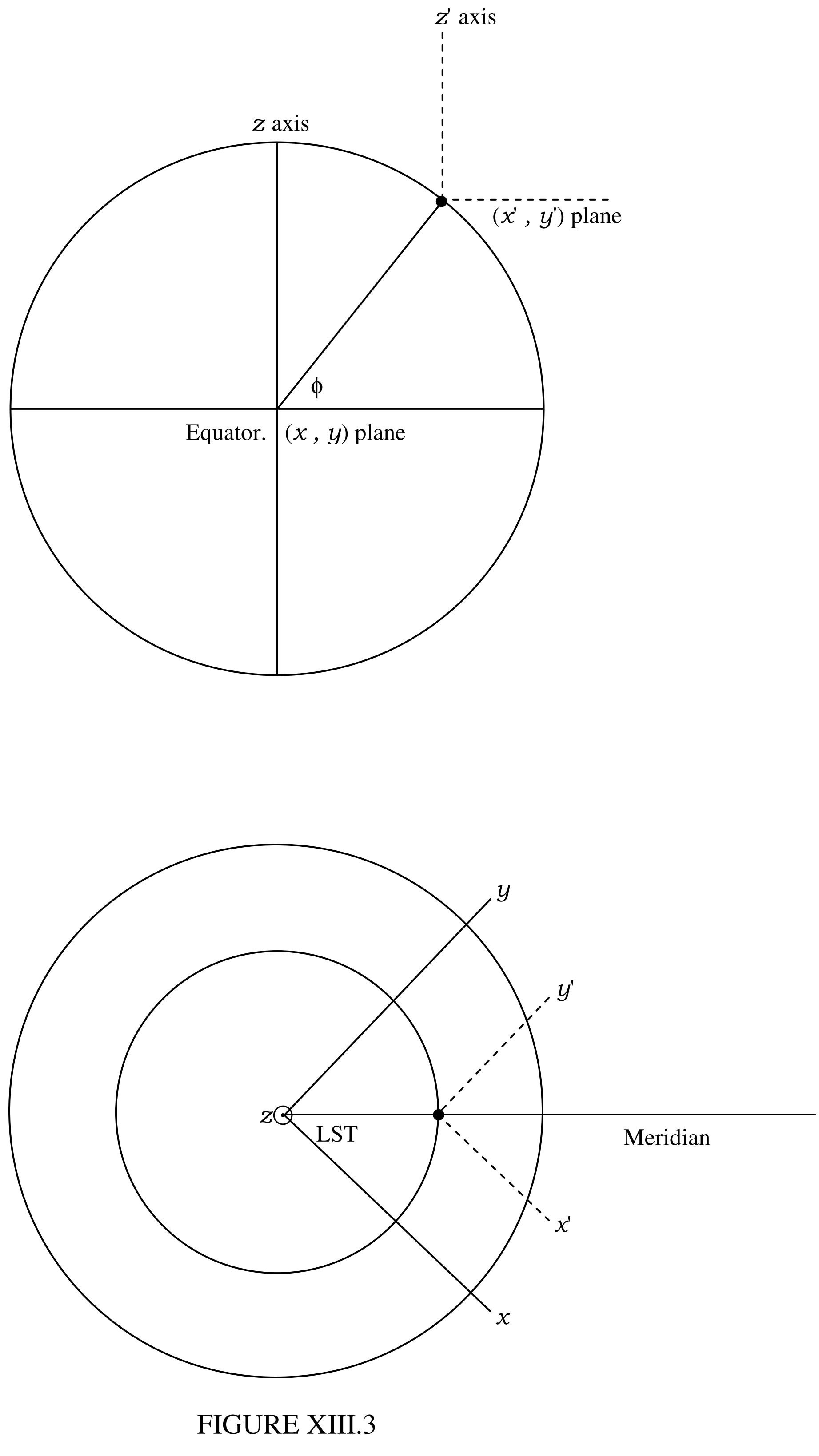

FIGURE XIII.3

13.17 *Concluding Remarks*

Anyone who has done the considerable work of following this chapter in detail is now capable of determining the elements of an elliptic orbit from three observations, if the orbit is an ellipse and if indeed elliptical elements can be obtained from the observations (which is not always the case). No one arriving at this stage would possibly think of himself or herself as an expert in orbit calculations. There is much, much more to be learned, and much of it will come with experience, and be self-taught or picked up from others. There are questions about how to handle more than the requisite three observations, how to correct the elements differentially as new observations become available, how to apply planetary perturbations, how to handle parabolic or hyperbolic orbits. Some of this material *may* (or may not!) be discussed in future chapters. However, often the most difficult thing is getting started, and calculating one's very first orbit from the minimum data. It is hoped that this chapter has helped the reader to attain this.

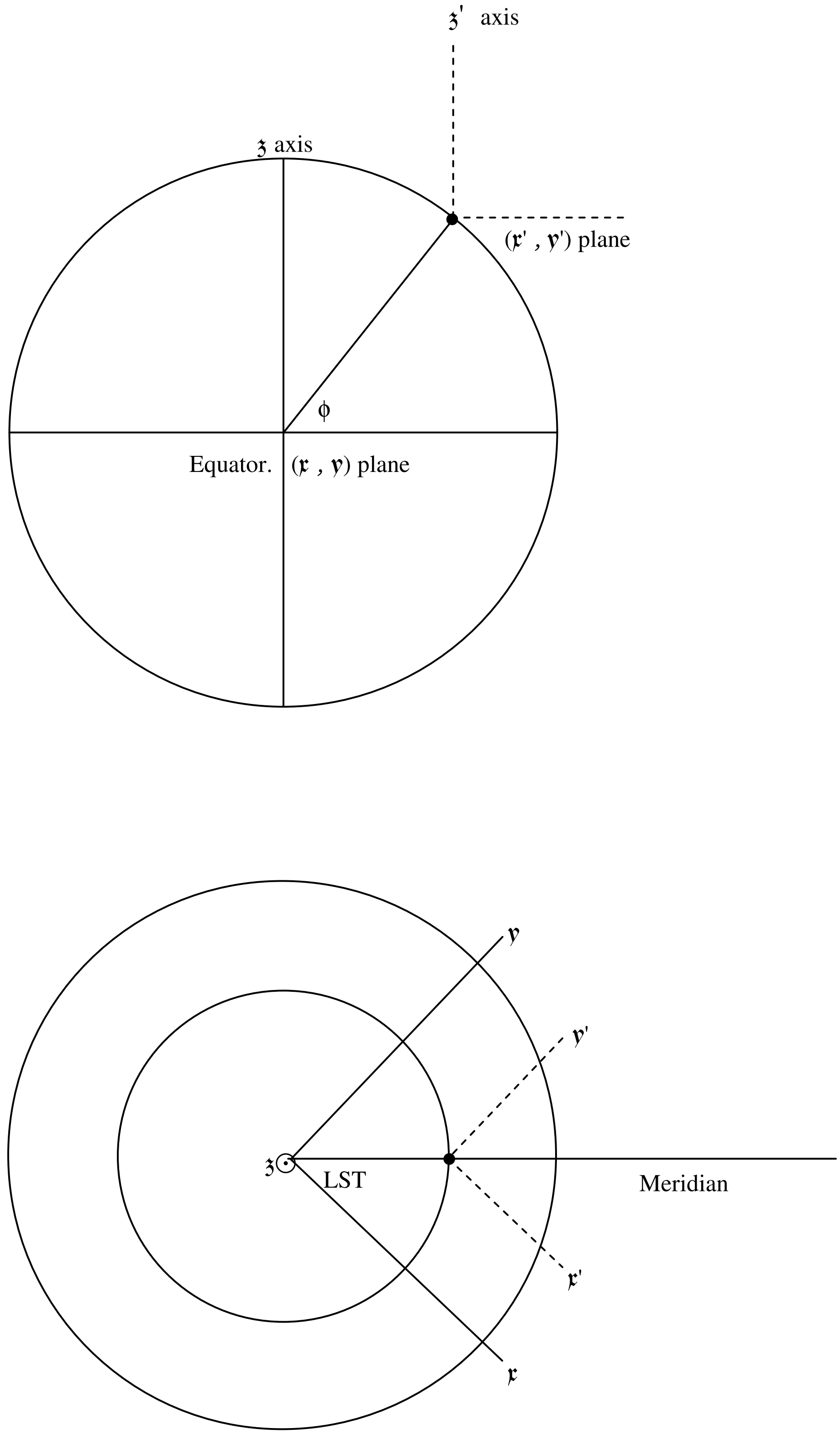

FIGURE XIII.3