CHAPTER 10
LINE PROFILES

10.1 *Introduction.*

Spectrum lines are not infinitesimally narrow; they have a finite width. A graph of radiance or intensity per unit wavelength (or frequency) versus wavelength (or frequency) is the *line profile*. There are several causes of line broadening, some internal to the atom, others external, and each produces its characteristic profile. Some types of profile, for example, have a broad core and small wings; others have a narrow core and extensive, broad wings. Analysis of the exact shape of a line profile may give us information about the physical conditions, such as temperature and pressure, in a stellar atmosphere.

10.2 *Natural Broadening (Radiation Damping)*

The classical oscillator model of the atom was described in section 9.2.1. In this model, the motion of the optical electron, when subject to the varying electromagnetic field of a light wave, obeys the differential equation for forced, damped, oscillatory motion:

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{e\hat{E}}{m}\cos\omega t. \qquad 10.2.1$$

Because the oscillating (hence accelerating) electron itself radiates, the system loses energy, which is equivalent to saying that the motion is damped, and γ is the damping constant.

Electromagnetic theory tells us that the rate of radiation of energy from an accelerating electron is

$$\frac{2}{3}\cdot\frac{e^2\ddot{x}^2}{4\pi\varepsilon_0 c^3}. \qquad 10.2.2$$

(The reader, as always, should check the dimensions of this and all subsequent expressions.)

For an electron that is oscillating, the average rate of loss of energy per cycle is

$$\frac{2}{3}\cdot\frac{e^2\overline{\ddot{x}^2}}{4\pi\varepsilon_0 c^3}. \qquad 10.2.3$$

Here the bar denotes the average value over a cycle.

If the amplitude and angular frequency of the oscillation are a and ω_0, the maximum acceleration is $a\omega_0^2$ and the mean square acceleration is $\frac{1}{2}a^2\omega_0^4$. The energy (kinetic plus potential) of the oscillating electron is

$$W = \tfrac{1}{2}ma^2\omega_0^2. \qquad 10.2.4$$

Thus we can write for the average rate of loss per cycle of energy from the system by electromagnetic radiation:

$$\frac{2}{3}\cdot\frac{e^2\omega_0^2}{4\pi\varepsilon_0 mc^3}\cdot W \qquad 10.2.5$$

The energy therefore falls off according to

$$\dot{W} = -\frac{1}{3}\cdot\frac{e^2\omega_0^2}{4\pi\varepsilon_0 mc^3}\cdot W. \qquad 10.2.6$$

The radiated wavelength is given by $\lambda = 2\pi c/\omega_0$, so that equation 10.2.6 becomes

$$\dot{W} = -\frac{2\pi e^2}{3\varepsilon_0 mc\lambda^2}\cdot W. \qquad 10.2.7$$

It will be recalled from the theory of lightly damped oscillations that the solution to equation 10.2.1 shows that the amplitude falls off with time as $\exp(-\frac{1}{2}\gamma t)$, and that the *energy* falls off as $\exp(-\gamma t)$. Thus we identify the coefficient of W on the right hand side of equation 10.2.7 as the *classical radiation damping constant* γ:

$$\gamma = \frac{2\pi e^2}{3\varepsilon_0 mc\lambda^2}\cdot \qquad 10.2.8$$

Numerically, if γ is in s^{-1} and λ is in m,

$$\gamma = \frac{2.223\times10^{-5}}{\lambda^2}\cdot \qquad 10.2.9$$

We are now going to calculate the rate at which energy is transported per unit area by an electromagnetic wave, and also to calculate the rate at which an optically thin slab of a gas of classical oscillators absorbs energy, and hence we are going to calculate the classical absorption coefficient. We start by recalling, from elementary electromagnetism, that the energy held per unit volume in an electric field is $\frac{1}{2}\mathbf{D}\cdot\mathbf{E}$ and the energy held per unit volume in a magnetic field is $\frac{1}{2}\mathbf{B}\cdot\mathbf{H}$. In an isotropic medium, these become $\frac{1}{2}\epsilon E^2$ and $\frac{1}{2}\mu H^2$, and, *in vacuo*, they become $\frac{1}{2}\epsilon_0 E^2$ and $\frac{1}{2}\mu_0 H^2$.

For an oscillating electric field of the form $E = \hat{E}\cos\omega t$, the average energy per unit volume per cycle is $\frac{1}{2}\epsilon_0\overline{E^2} = \frac{1}{4}\epsilon_0\hat{E}^2$. Similarly for an oscillating magnetic field, the average energy per unit volume per cycle is $\frac{1}{4}\mu_0\hat{H}^2$. An electromagnetic wave consists of an electric and a magnetic wave moving at speed c, so the rate at which energy is transmitted across unit area is $\left(\frac{1}{4}\epsilon_0\hat{E}^2 + \frac{1}{4}\mu_0\hat{H}^2\right)c$, and the two parts are equal, so that the rate at which energy is transmitted per unit area by a plane electromagnetic wave is $\frac{1}{2}\epsilon_0\hat{E}^2 c$.

Now we are modelling the classical oscillator as an electron bound to an atom, and being subject to a periodic force $\frac{e\hat{E}}{m}\cos\omega t$ from an electromagnetic wave. The rate of absorption of energy by such an oscillator (see, for example, Chapter 12 of Classical Mechanics is

$$\frac{\gamma e^2\hat{E}^2\omega^2}{2m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}.$$

We imagine a plane electromagnetic wave arriving at (irradiating) a slab of gas containing $\mathcal{N}$ classical oscillators per unit area, or n per unit volume. The rate of arrival of energy per unit area, we have seen, is $\frac{1}{2}\epsilon_0\hat{E}^2 c$. The rate of absorption of energy per unit area is

$$\frac{\mathcal{N}\gamma e^2\hat{E}^2\omega^2}{2m[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}.$$

The absorptance (see Chapter 2, section 2.2) is therefore

$$a = \frac{\mathcal{N}\gamma e^2\omega^2}{m\epsilon_0 c[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}. \qquad 10.2.10$$

and the linear absorption coefficient is

$$\alpha = \frac{n\gamma e^2\omega^2}{m\epsilon_0 c[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]}. \qquad 10.2.11$$

[A reminder here might be in order. *Absorptance* a is defined in section 2.2, and in the notation of figure IX.1, the absorptance at wavelength λ would be $(I_\lambda(\mathrm{c}) - I_\lambda(\lambda))/I_\lambda(\mathrm{c})$. *Absorption coefficient* α is defined by equation 5.2.1: $-dI/I = \alpha\,dx$. For a thick slice of gas, of thickness t, this integrates, in the notation of figure IX.1, to $I_\lambda(\lambda) = I_\lambda(\mathrm{c})\exp(-\alpha t)$. But for an optically thin gas, which is what we are considering, unless stated otherwise, in this chapter, this becomes $(I_\lambda(\mathrm{c}) - I_\lambda(\lambda))/I_\lambda(\mathrm{c}) = \alpha t$. Thus, *for an optically thin gas*, absorptance is just absorption coefficient times thickness of the gas. And the relation between particle density n and column density $\mathcal{N}$ is $\mathcal{N} = nt$.]

We can write $\omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega)$. Let us also write ω as $2\pi\nu$. Also, in the near vicinity of the line, let us make the approximation $\omega_0 + \omega = 2\omega$. We then obtain for the absorption coefficient, in the vicinity of the line,

$$\alpha = \frac{\gamma n e^2}{16\pi^2 mc\varepsilon_0\left[(\nu-\nu_0)^2 + \left(\frac{\gamma}{4\pi}\right)^2\right]}. \qquad 10.2.12$$

Exercise: Make sure that I have made no mistakes in deriving equations 10.2.10,11 and 12, and check the dimensions of each expression as you go. Let me know if you find anything wrong.

Now the equivalent width in frequency units of an absorption line in an optically thin layer of gas of geometric thickness t is (see equation 9.1.6)

$$W^{(\nu)} = t\int_{-\infty}^{\infty} \alpha\, d(\nu - \nu_0). \qquad 10.2.13$$

Exercise: (a) For those readers who (understandably) object that expression 10.2.12 is valid only in the immediate vicinity of the line, and therefore that we cannot integrate from $-\infty$ to $+\infty$, integrate expression 10.2.11 from 0 to ∞.

(b) For the rest of us, integrate equation10.2.11 from $\nu - \nu_0 = -\infty$ to $+\infty$. A substitution $4\pi(\nu - \nu_0) = \gamma\tan\theta$ will probably be a good start.

We obtain

$$W^{(\nu)} = \frac{\mathcal{N}e^2}{4mc\varepsilon_0} = 2.654\times10^{-6}\,\mathcal{N}, \qquad 10.2.14$$

where $W^{(\nu)}$ is in Hz and $\mathcal{N}$ is in m^{-2}. Thus the classical oscillator model predicts that the equivalent width in frequency units is independent of the frequency (and hence wavelength) of the line, and also independent of the damping constant. If we express the equivalent width in wavelength units (see equation 9.1.3), we obtain:

$$W = \frac{\mathcal{N}e^2\lambda^2}{4mc^2\varepsilon_0}. \qquad 10.2.15$$

This is the same as equation 9.2.2.

When we discussed this equation in Chapter 9, we pointed out that the equivalent widths of real lines differ from this prediction by a factor f_{12}, the absorption oscillator strength, and we also pointed out that $\mathcal{N}$ has to be replaced by $\mathcal{N}_1$, the column density of atoms in the initial (lower) level. Thus, from this point, I shall replace $\mathcal{N}$ with $\mathcal{N}_1 f_{12}$. However, in this chapter we are not so much concerned with the equivalent width, but with the line profile and the actual width. The width of an emission line in this context is commonly expressed as the full width at half maximum (FWHM) and the width of an absorption line as the full width at half minimum (FWHm). (These are on no account to be confused with the *equivalent width*, which is discussed in section 9.1.) Note that some writers use the term "half-width". It is generally not possible to know what a writer means by this.

In terms of the notation of figure IX.1 (in which "c" denotes "continuum"), but using a frequency rather than a wavelength scale, the absorptance at frequency ν is

$$a(\nu) = \frac{I_\nu(\mathrm{c}) - I_\nu(\nu)}{I_\nu(\mathrm{c})}. \qquad 10.2.16$$

The profile of an absorption line is thus given by

$$I_\nu(\nu) = I_\nu(\mathrm{c})(1 - a(\nu)). \qquad 10.2.17$$

For radiation damping we have

$$a(\nu) = \frac{\gamma\mathcal{N}_1 f_{12}e^2}{16\pi^2 mc\varepsilon_0\left[(\nu - \nu_0)^2 + \left(\frac{\gamma}{4\pi}\right)^2\right]}. \qquad 10.2.18$$

The maximum value of the absorptance (at the line centre) is

$$a(\nu_0) = \frac{\mathcal{N}_1 f_{12}e^2}{mc\varepsilon_0\gamma}. \qquad 10.2.19$$

This quantity is also $\frac{I_\nu(\mathrm{c})-I_\nu(\nu_0)}{I_\nu(\mathrm{c})}$ and it is also known as the *central depth d* of the line. (Be sure to refer to figure IX.1 to understand its meaning.) I shall use the symbol d or $a(\nu_0)$ interchangeably, according to context.

It is easy to see that the value of $\nu-\nu_0$ at which the absorptance is half its maximum value is $\gamma/(4\pi)$. That is to say, the full width at half maximum (FWHM) of the absorptance, which I denote as w, is, in frequency units:

$$w = \frac{\gamma}{2\pi}. \qquad 10.2.20$$

(In wavelength units, it is λ^2/c times this.) This is also the FWHm of the absorption profile.

Equation 10.2.18 can be written

$$\frac{a(\nu)}{a(\nu_0)} = \frac{1}{4\left(\frac{\nu-\nu_0}{w}\right)^2+1}. \qquad 10.2.21$$

The absorption line profile (see equation 10.2.1) can be written

$$\frac{I_\nu(\nu)}{I_\nu(\mathrm{c})} = 1-\frac{d}{4\left(\frac{\nu-\nu_0}{w}\right)^2+1}. \qquad 10.2.22$$

Notice that at the line centre, $I_\nu(\nu_0)/I_\nu(\mathrm{c})$ = 1 minus the central depth; and a long way from the line centre, $I_\nu(\nu) = I_\nu(\mathrm{c})$, as expected. This type of profile is called a *Lorentz profile.*

From equations 10.2.14 (but with $\mathcal{N}_1 f_{12}$ substituted for $\mathcal{N}$), 10.2.19 and 10.2.20, we find that

$$\text{Equivalent width} = \frac{\pi}{2}\times \text{central depth} \times \text{FWHm}$$

$$1.571 \times \text{central depth} \times \text{FWHm}. \qquad 10.2.23$$

This is true whether equivalent width and FWHm are measured in frequency or in wavelength units. (It is a pity that, for theoretical work, frequency is more convenient than wavelength, since frequency is proportional to energy, but experimentalists often (not invariably!) work with gratings, which disperse light linearly with respect to wavelength!)

Indeed the equivalent width of any type of profile can be written in the form

$$\text{Equivalent width} = \text{constant} \times \text{ central depth} \times \text{FWHm}, \qquad 10.2.24$$

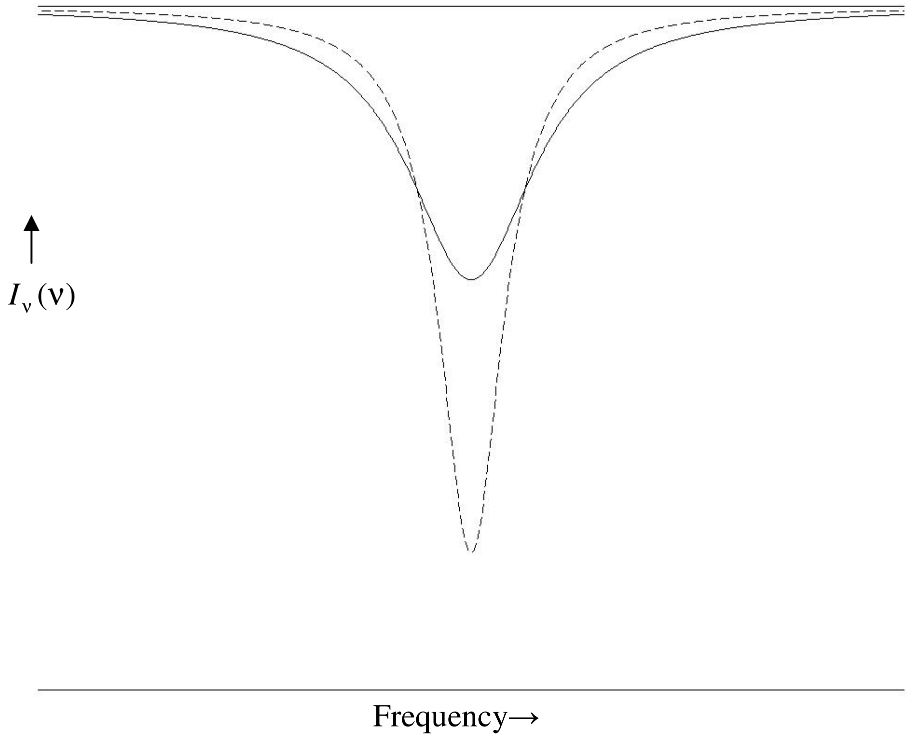
the value of the constant depending upon the type of profile.

In photographic days, the measurement of equivalent widths was a very laborious procedure, and, if one had good reason to believe that the line profiles in a spectrum were all lorentzian, the equivalent with would be found by measuring just the FWHm and the central depth. Even today, when equivalent widths can often be determined by computer from digitally-recorded spectra almost instantaneously, there may be occasions where low-resolution spectra do not allow this, and all that can be honestly measured are the central depths and equivalent widths. The type of profile, and hence the value to be used for the constant in equation 10.2.14, requires a leap of faith.

It is worth noting (consult equations 10.2.4,19 and 20) that the equivalent width is determined by the column density of the absorbing atoms (or, rather, on $\mathcal{N}_1 f_{12}$), the FWHm is determined by the damping constant, but the central depth depends on both. You can determine the damping constant by measuring the FWHm.

The form of the Lorentz profile is shown in figure X.1 for two lines, one with a central depth of 0.8 and the other with a central depth of 0.4. Both lines have the same equivalent width, the product *wd* being the same for each. Note that this type of profile has *a narrow core, skirted by extensive wings.*

FIGURE X.I



Of course a visual inspection of a profile showing a narrow core and extensive wings, while suggestive, doesn't prove that the profile is strictly lorentzian. However, equation 10.2.22 can be rearranged to read

$$\frac{I_\nu(\mathrm{c})}{I_\nu(\mathrm{c})-I_\nu(\nu)} = \frac{4}{w^2 d}(\nu-\nu_0)^2 + \frac{1}{d^2}. \qquad 10.2.25$$

This shows that if you make a series of measurements of $I_\nu(\nu)$ and plot a graph of the left hand side versus $(\nu-\nu_0)^2$, you should obtain a straight line if the profile is lorentzian, and you will obtain the central depth and equivalent width (hence also the damping constant and the column density) from the intercept and slope as a bonus. And if you don't get a straight line, you don't have a Lorentz profile.

It will be recalled that the purely classical oscillator theory predicted that the equivalent widths of all lines (in frequency units) of a given element is the same, namely that given by equation 10.2.14. The obvious observation that this is not so led us to introduce the emission oscillator strength, and also to replace $\mathcal{N}$ by $\mathcal{N}_1$. Likewise, equation 10.2.20 predicts that the FWHm (in wavelength units) is the same for all lines. (Equation 10.2.20 gives the FWHm in frequency units. To understand my caveat "in wavelength units", refer also to equations 10.2.8 and 10.2.9. You will see that the predicted FWHm in wavelength units is $\frac{e^2}{3\varepsilon_0 mc^2} = 1.18 \times 10^{-14}$ m, which is exceedingly small, and the core, at least, is beyond the resolution of most spectrographs.) Obviously the damping constants for real lines are much larger than this. For real lines, the classical damping constant γ has to be replaced with the *quantum mechanical damping constant* Γ.

At present I am describing in only a very qualitative way the quantum mechanical treatment of the damping constant. Quantum mechanically, an electromagnetic wave is treated as a perturbation to the hamiltonian operator. We have seen in section 9.4 that each level has a finite lifetime – see especially equation 9.4.7. The mean lifetime for a level m is $1/\Gamma_m$. Each level is not infinitesimally narrow. That is to say, one cannot say with infinitesimal precision what the energy of a given level (or state) is. The uncertainty of the energy and the mean lifetime are related through Heisenberg's uncertainty principle. The longer the lifetime, the broader the level. The energy probability of a level m is given by a Lorentz function with parameter Γ_m, given by equation 9.4.7 and equal to the reciprocal of the mean lifetime. Likewise a level n has an energy probability distribution given by a Lorentz function with parameter Γ_n. When an atom makes a transition between m and n, naturally, there is an energy uncertainty in the emitted or absorbed photon, and so there is a distribution of photons (i.e. a line profile) that is a Lorentz function with parameter $\Gamma = \Gamma_m + \Gamma_n$. This parameter Γ must replace the classical damping constant γ. The FWHm of a line, in frequency units, is now $\Gamma/(2\pi)$, which varies from line to line.

Unfortunately it is observed, at least in the spectrum of main sequence stars, if not in that of giants and supergiants, that the FWHms of most lines *are about the same*! How frustrating! Classical theory predicts that all lines have the same FWHm. We know classical theory is wrong, so we go to the trouble of doing quantum mechanical theory, which predicts different FWHms from line to line. And then we go and observe main sequence stars and we find that the lines all have the same FWHm (admittedly much broader than predicted by classical theory.)

The explanation is that, in main sequence atmospheres, lines are additionally broadened by *pressure broadening*, which also gives a Lorentz profile, which is generally broader than, and overmasks, radiation damping. (The pressures in the extended atmospheres of giants and supergiants are generally much less than in main sequence stars, and consequently lines are narrower.) We return to pressure broadening in a later section.

10.3 *Thermal Broadening.*

Let us start with an assumption that the radiation damping broadening is negligible, so that, for all practical purposes the spread of the frequencies emitted by a collection of atoms in a gas is infinitesimally narrow. The observer, however, will not see an infinitesimally thin line. This is because of the motion of the atoms in a hot gas. Some atoms are moving hither, and the wavelength will be blue-shifted; others are moving yon, and the wavelength will be red-shifted. The result will be a broadening of the lines, known as *thermal broadening*. The hotter the gas, the faster the atoms will be moving, and the broader the lines will be. We shall be able to measure the kinetic temperature of the gas from the width of the lines.

First, a brief reminder of the relevant results from the kinetic theory of gases, and to establish our notation.

Notation: c = speed of light

$\mathbf{V}$ = velocity of a particular atom = $u\hat{\mathbf{x}} + \upsilon\hat{\mathbf{y}} + w\hat{\mathbf{z}}$

V = speed of that atom = $\left(u^2 + \upsilon^2 + w^2\right)^{\frac{1}{2}}$

V_m = modal speed of all the atoms $= \sqrt{\dfrac{2kT}{m}} = 1.414\sqrt{\dfrac{kT}{m}}$

$\overline{V}$ = mean speed of all the atoms $= \sqrt{\dfrac{8kT}{\pi m}} = 1.596\sqrt{\dfrac{kT}{m}} = 1.128V_\mathrm{m}$

V_RMS = root mean square speed of all the atoms

$$= \sqrt{\frac{3kT}{m}} = 1.732\sqrt{\frac{kT}{m}} = 1.225V_\mathrm{m}$$

The Maxwell distribution gives the distribution of speeds. Consider a gas of N atoms, and let $N_V dV$ be the number of them that have speeds between V and $V + dV$. Then

$$\frac{N_V dV}{N} = \frac{4}{\sqrt{\pi}V_\mathrm{m}^3} V^2 \exp\left(-\frac{u^2}{V_\mathrm{m}^2}\right) dV. \qquad 10.3.1$$

More relevant to our present topic is the distribution of a velocity component. We'll choose the x-component, and suppose that the x-direction is the line of sight of the observer as he or she peers through a stellar atmosphere. Let $N_u du$ be the number of atoms with velocity components between u and du. Then the gaussian distribution is

$$\frac{N_u du}{N} = \frac{1}{\sqrt{\pi}V_m} \exp\left(-\frac{u^2}{V_\mathrm{m}^2}\right) du, \qquad 10.3.2$$

which, of course, is symmetric about $u = 0$.

Now an atom with a line-of-sight velocity component u gives rise to a Doppler shift $\nu - \nu_0$, where (provided that $u^2 << c^2$) $\frac{\nu - \nu_0}{\nu_0} = \frac{u}{c}\cdot$ If we are looking at an emission line, the left hand side of equation 10.3.2 gives us the line profile $I_\nu(\nu)/I_\nu(\nu_0)$ (provided the line is optically thin, as is always assumed in this chapter unless specified otherwise). Thus the line profile of an emission line is

$$\frac{I_\nu(\nu)}{I_\nu(\nu_0)} = \exp\left[-\frac{c^2}{V_\mathrm{m}^2}\frac{(\nu - \nu_0)^2}{\nu_0^2}\right]. \qquad 10.3.3$$

This is a *gaussian*, or *Doppler*, profile.

It is easy to show that the full width at half maximum (FWHM) is

$$w = \frac{V_\mathrm{m}\nu_0}{c}\sqrt{\ln 16} = 1.6651\frac{V_m \nu_0}{c}\cdot \qquad 10.3.4$$

This is also the full width at half minimum (FWHm) of an absorption line, in frequency units. This is also the FWHM or FWHm in wavelength units, provided that λ_0 be substituted for ν_0.

The profile of an absorption line of central depth d (= $\frac{I_\nu(\mathrm{c}) - I_\nu(\nu_0)}{I_\nu(\mathrm{c})}$) is

$$\frac{I_\nu(\nu)}{I_\nu(\mathrm{c})} = 1 - d \exp\left[-\frac{c^2}{V_\mathrm{m}^2}\frac{(\nu - \nu_0)^2}{\nu_0^2}\right], \qquad 10.3.5$$

which can also be written

$$\frac{I_\nu(\nu)}{I_\nu(\mathrm{c})} = 1 - d\exp\left[-\frac{(\nu-\nu_0)^2\ln 16}{w^2}\right]. \qquad 10.3.6$$

(Verify that when $\nu - \nu_0 = \frac{1}{2}w$, the right hand side is $1-\frac{1}{2}d$. Do the same for equation 10.2.22.)

In figure X.2, I draw two gaussian profiles, each of the same equivalent width as the lorentzian profiles of figure X.1, and of the same two central depths, namely 0.4 and 0.8. We see that a gaussian profile is "all core and no wings". A visual inspection of a profile may lead one to believe that it is probably gaussian, but, to be sure, one could write equation 10.3.6 in the form

$$\ln\left[\frac{I_\nu(\mathrm{c})-I_\nu(\nu)}{I_\nu(\mathrm{c})}\right] = \ln d - \frac{(\nu-\nu_0)^2\ln 16}{w^2} \qquad 10.3.7$$

and plot a graph of the left hand side versus $(\nu - \nu_0)^2$. If the profile is truly gaussian, this will result in a straight line, from which w and d can be found from the slope and intercept.

Integrating the Doppler profile to find the equivalent width is slightly less easy than integrating the Lorentz profile, but it is left as an exercise to show that

$$\text{Equivalent width} = \sqrt{\frac{\pi}{\ln 16}} \times \text{central depth} \times \text{FWHm}$$

$$= 1.064 \times \text{central depth} \times \text{FWHm}. \qquad 10.3.8$$

Compare this with equation 10.2.23 for a Lorentz profile.

FIGURE X.2

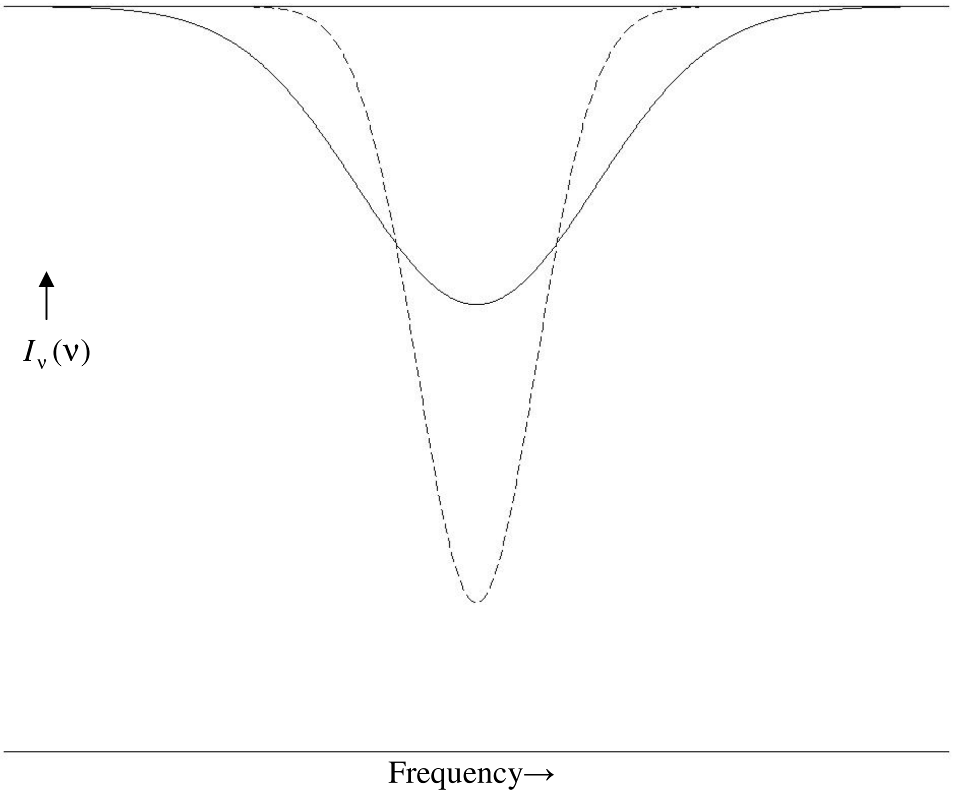
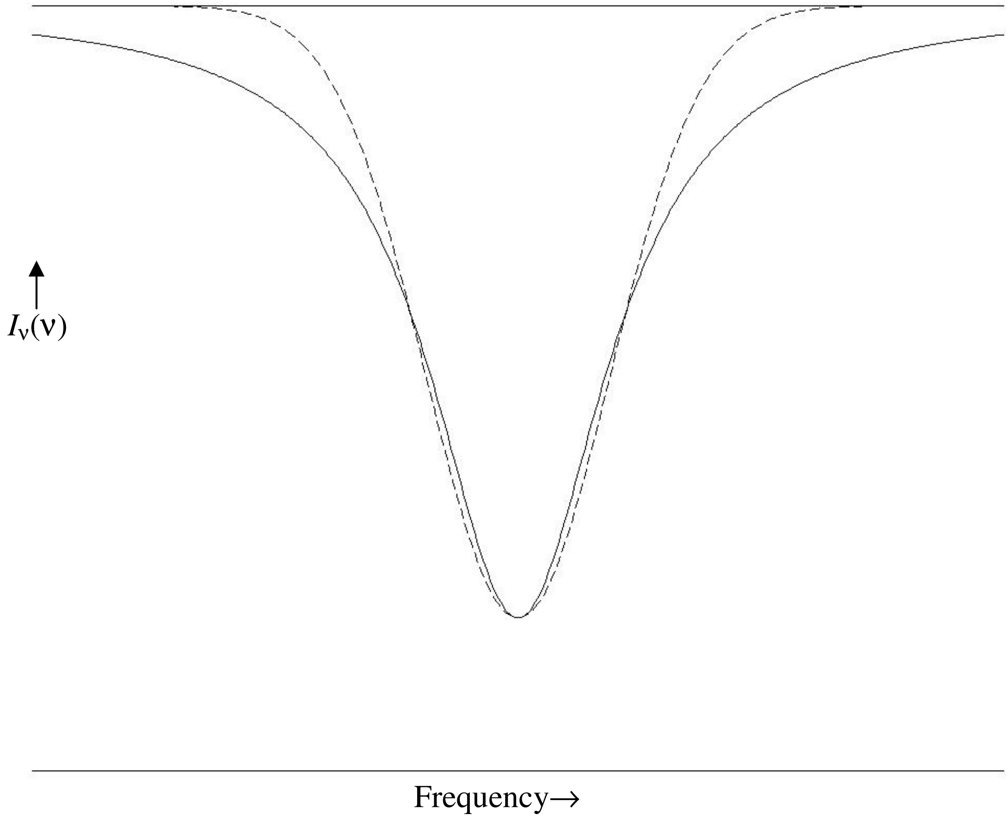


Figure X.3 shows a lorentzian profile (continuous) and a gaussian profile (dashed), each having the same central depth and the same FWHm. The ratio of the lorenzian equivalent width to the gaussian equivalent width is $\frac{\pi}{2} \div \sqrt{\frac{\pi}{\ln 16}} = \sqrt{\pi \ln 2} = 1{\cdot}476.$

FIGURE X.3



10.4 *Microturbulence*

In the treatment of microturbulence in a stellar atmosphere, we can suppose that there are many small cells of gas moving in random directions with a maxwellian distribution of speeds. The distinction between microturbulence and macroturbulence is that in microturbulence the size of the turbulent cells is very small compared with the optical depth, so that, in looking down through a stellar atmosphere we are seeing many cells of gas whose distribution of velocity components is gaussian. In macroturbulence the size of the cells is not very small compared with the optical depth, so that , in peering through the haze of an atmosphere, we can see at most only a very few cells.

If the distribution of velocity components of the microturbulent cells is supposed gaussian, then the line profiles will be just like that for thermal broadening, except that, instead of the modal speed $V_{\rm m} = \sqrt{2kT/m}$ of the atoms we substitute the modal speed ξ_m of the microturbulent cells. Thus the line profile resulting from microturbulence is

$$\frac{I_\nu(\nu)}{I_\nu(\nu_0)} = 1 - \ d\exp\left[-\frac{c^2}{\xi_{\rm m}^2}\frac{(\nu-\nu_0)^2}{\nu_0^2}\right]. \qquad 10.4.1$$

The FWHm in frequency units is $\frac{\xi_{\rm m}\nu_0\sqrt{\ln 16}}{c}$ or, in wavelength units, $\frac{\xi_{\rm m}\lambda_0\sqrt{\ln 16}}{c}$.

If the thermal and microturbulent broadening are comparable in size, we still get a gaussian profile, except that for $V_{\rm m}$ or $\xi_{\rm m}$ we must substitute $\sqrt{V_{\rm m}^2+\xi_{\rm m}^2} = \sqrt{2kT/m+\xi_{\rm m}^2}$. (This actually requires formal proof, and this will be given as an exercise in section 5.)

Since either thermal broadening or microturbulence will result in a gaussian profile, one might think that it would not be possible to tell, from a spectrum exhibiting gaussian line profiles, whether the broadening was caused primarily by high temperature or by microturbulence. But a little more thought will show that in principle it *is* possible to distinguish, and to determine separately the kinetic temperature and the modal microturbulent speed. Think about it, and see if you can devise a way.

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THINKING

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The key is, in purely thermal broadening, the light atoms (such as lithium) move faster than the heavier atoms (such as cadmium), the speeds being inversely proportional to the square roots of their atomic masses. Thus the lines of the light atoms will be broader than the lines of the heavy atoms. In microturbulence all atoms move *en masse* at the same speed and are therefore equally broad. We have seen, beneath equation 10.3.7, that the

FWHm, in frequency units, is $w = \frac{\nu_0}{c}\sqrt{\left(2kT/m + \xi_\mathrm{m}^2\right)\ln 16}$. If we form the quantity $X = \frac{w^2c^2}{\nu_0^2 \ln 16}$ for a lithium line and for a cadmium line, we will obtain

$$X_\mathrm{Li} = \frac{2kT}{m_\mathrm{Li}} + \xi_\mathrm{m}^2 \qquad \text{and} \qquad X_\mathrm{Cd} = \frac{2kT}{m_\mathrm{Cd}} + \xi_\mathrm{m}^2 , \tag{10.4.2}$$

from which T and ξ_m are immediately obtained.

Problem. A Li line at 670.79 nm has a gaussian FWHm = 9 pm (picometres) and a Cd line at 508.58 nm has a gaussian FWHm = 3 pm. Calculate the kinetic temperature and the modal microturbulent speed.

10.5 *Combination of Profiles*

Several broadening factors may be simultaneously present in a line. Two mechanisms may have similar profiles (e.g. thermal broadening and microturbulence) or they may have quite different profiles (e.g. thermal broadening and radiation damping). We need to know the resulting profile when more than one broadening agent is present.) Let us consider an emission line, and let $x = \lambda - \lambda_0$. Let us suppose that the lines are broadened, for example, by thermal broadening, the thermal broadening function being $f(x)$. Suppose, however, that, in addition, the lines are also broadened by radiation damping, the radiation damping profile being $g(x)$. At a distance ξ from the line centre, the contribution to the line profile is the height of the function $f(\xi)$ weighted by the function $g(x - \xi)$. That is to say the resulting profile $h(x)$ is given by

$$h(x) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)d\xi. \tag{10.5.1}$$

The reader should convince him- or herself that this is exactly the same as

$$h(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi. \tag{10.5.2}$$

This profile is called the *convolution* of the two constituent profiles, and is often written symbolically

$$h = f * g . \tag{10.5.3}$$

Let us consider, for example, the convolution of two gaussian functions, for example the convolution of thermal and microturbulent broadening.

Suppose one of the gaussian functions is

$$G_1(x) = \frac{1}{g_1}\cdot\sqrt{\frac{\ln 2}{\pi}}\exp\left(-\frac{x^2\ln 2}{g_1^2}\right) = \frac{0.46972}{g_1}\exp\left(-\frac{0.69315x^2}{g_1^2}\right). \qquad 10.5.4$$

Here $x = \lambda - \lambda_0$. The area under the curve is unity, the HWHM is g₁ and the peak is $\frac{1}{g_1}\sqrt{\frac{\ln 2}{\pi}}$. (Verify these.) Suppose that the second gaussian function is

$$G_2(x) = \frac{1}{g_2}\cdot\sqrt{\frac{\ln 2}{\pi}}\exp\left(-\frac{x^2\ln 2}{g_2^2}\right). \qquad 10.5.5$$

It can now be shown, using equation 10.5.1 or 10.5.2, that the convolution of G_1 and G_2 is

$$G(x) = G_1(x) * G_2(x) = \frac{1}{g}\cdot\sqrt{\frac{\ln 2}{\pi}}\exp\left(-\frac{x^2\ln 2}{g^2}\right), \qquad 10.5.6$$

where $$g^2 = g_1^2 + g_2^2. \qquad 10.5.7$$

We used this result already in section 10.4 when, in adding microturbulent to thermal broadening, we substituted $\sqrt{V_\mathrm{m}^2 + \xi_\mathrm{m}^2}$ for V_m. In case you find the integration to be troublesome, I have done it in an Appendix to this Chapter.

Now let's consider the combination of two lorentzian functions. Radiation damping gives rise to a lorentzian profile, and we shall see later that pressure broadening can also give rise to a lorentzian profile. Let us suppose that the two lorentzian profiles are

$$L_1(x) = \frac{l_1}{\pi}\cdot\frac{1}{x^2 + l_1^2} \qquad 10.5.8$$

and $$L_2(x) = \frac{l_2}{\pi}\cdot\frac{1}{x^2 + l_2^2}. \qquad 10.5.9$$

Here $x = \lambda - \lambda_0$. The area under the curve is unity, the HWHM is l_1 and the peak is $1/(\pi l)$. (Verify these.) It can be shown that

$$L(x) = L(x) * L_2(x) = \frac{l}{\pi}\cdot\frac{1}{x^2 + l^2}, \qquad 10.5.10$$

where
$$l = l_1 + l_2 . \tag{10.5.11}$$

Details of the integration are in the Appendix to this Chapter.

Let us now look at the convolution of a gaussian profile with a lorentzian profile; that is, the convolution of

$$G(x) = \frac{1}{g}\cdot\sqrt{\frac{\ln 2}{\pi}}\exp\left(-\frac{x^2 \ln 2}{g}\right) \tag{10.5.12}$$

with
$$L(x) = \frac{l}{\pi}\cdot\frac{1}{x^2 + l^2}. \tag{10.5.13}$$

We can find the convolution from either equation 10.5.1 or from equation 10.5.2, and we obtain either

$$V(x) = \frac{l}{g}\sqrt{\frac{\ln 2}{\pi^3}}\int_{-\infty}^{\infty} \frac{\exp\left(-[(\xi - x)^2 \ln 2]/g^2\right)}{\xi^2 + l^2} d\xi \tag{10.5.14}$$

or
$$V(x) = \frac{l}{g}\sqrt{\frac{\ln 2}{\pi^3}}\int_{-\infty}^{\infty} \frac{\exp[-(\xi^2 \ln 2)/g^2]}{(\xi - x)^2 + l^2} d\xi . \tag{10.5.15}$$

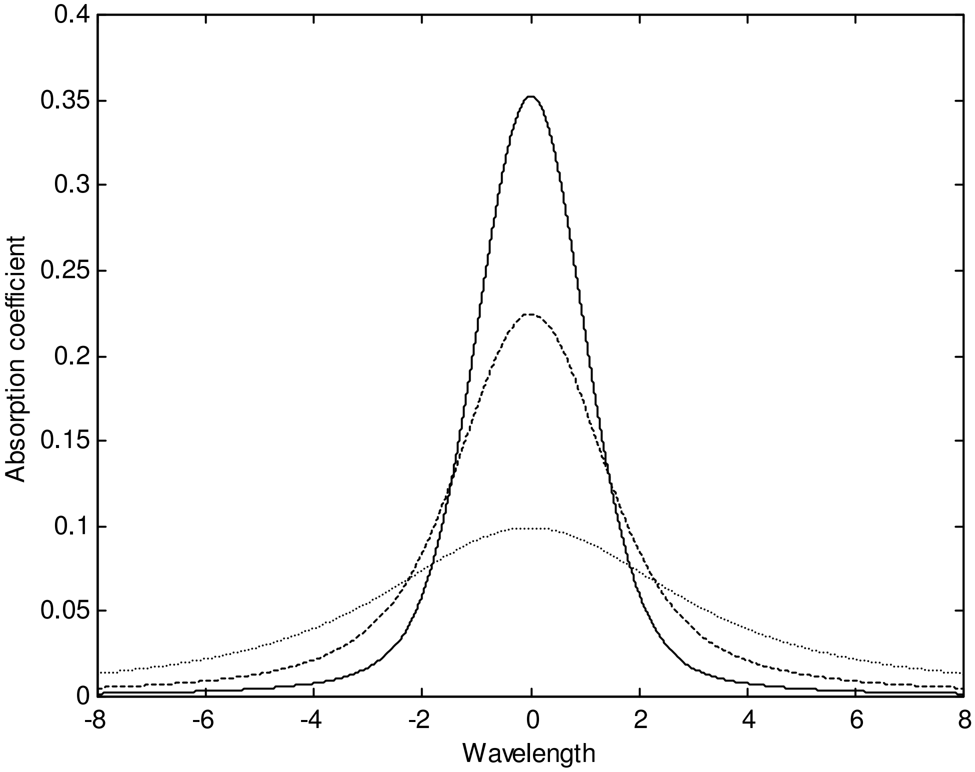
The expression 10.5.14 or 10.5.15, which is a convolution of a gaussian and a lorentzian profile, is called a *Voigt profile.* (A rough attempt at pronunciation would be something like *Focht*.)

A useful parameter to describe the "gaussness" or "lorentzness" of a Voigt profile might be

$$k_{\rm G} = \frac{g}{g + l}, \tag{10.5.16}$$

which is 0 for a pure lorentz profile and 1 for a pure gaussian profile. In figure X.4 I have drawn Voigt profiles for $k_{\rm G}$ = 0.25, 0.5 and 0.75 (continuous, dashed and dotted, respectively). The profiles are normalized so that all have the same area. A nice exercise for those who are more patient and competent with computers than I am would be to draw 1001 Voigt profiles, with $k_{\rm G}$ going from 0 to 1 in steps of 0.001, perhaps normalized all to the same height rather than the same area, and make a movie of a gaussian profile gradually morphing to a lorentzian profile. Let me know if you succeed!

FIGURE X.4



As for the gauss-gauss and lorentz-lorentz profiles, I have appended some details of the integration of the gauss-lorentz profile in the Appendix to this Chapter.

The FWHM or FWHm in wavelength units of a gaussian profile (i.e. $2g$) is

$$w_{\mathrm{G}} = \frac{\left(2kT/m + \xi_{\mathrm{m}}^2\right)^{\frac{1}{2}} \lambda_0 \sqrt{\ln 16}}{c} = \frac{1.665\left(2kT/m + \xi_{\mathrm{m}}^2\right)^{\frac{1}{2}} \lambda_0}{c}. \qquad 10.5.17$$

The FWHM or FWHm in frequency units of a lorentzian profile is

$$w_{\mathrm{L}} = \Gamma/(2\pi) = 0.1592\Gamma, \qquad 10.5.18$$

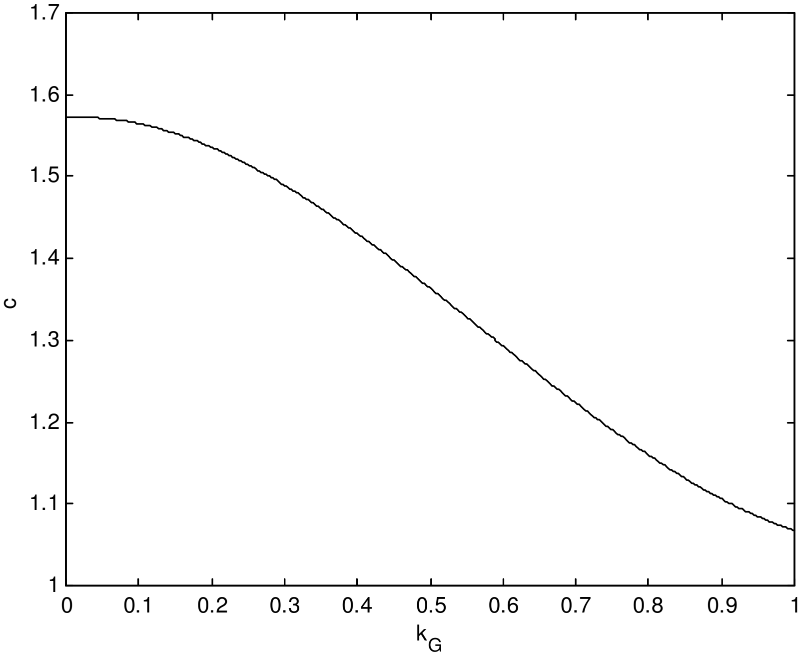
Here Γ is the sum of the radiation damping constant (see section 2) and the contribution from pressure broadening $2/\bar{t}$ (see section 6). For the FWHM or FWHm in wavelength units (i.e. $2l$), we have to multiply by λ_0^2/c.

Integrating a Voigt profile.

The area under Voigt profile is $2\int_0^\infty V(x)dx$, where $V(x)$ is given by equation 10.5.14, which itself had to be evaluated with a numerical integration. Since the profile is symmetric about $x = 0$, we can integrate from 0 to ∞ and multiply by 2. Even so, the double integral might seem like a formidable task. Particularly troublesome would be to integrate a nearly lorentzian profile with extensive wings, because there would then be the problem of how far to go for an upper limit. However, it is not at all a formidable task. The area under the curve given by equation 10.5.14 is unity! This is easily seen from a physical example. The profile given by equation 10.5.14 is the convolution of the lorentzian profile of equation 10.5.13 with the gaussian profile of equation 10.5.12, both of which were normalized to unit area. Let us imagine that an emission line is broadened by radiation damping, so that its profile is lorentzian. Now suppose that it is further broadened by thermal broadening (gaussian profile) to finish as a Voigt profile. (Alternatively, suppose that the line is scanned by a spectrophotometer with a gaussian sensitivity function.) Clearly, as long as the line is always optically thin, the additional broadening does not affect the integrated intensity.

Now we mentioned in sections 2 and 3 of this chapter that the equivalent width of an absorption line can be calculated from $c \times$ central depth × FWHm, and likewise the area of an emission line is $c \times$ height × FWHM, where c is 1.064 (= $\sqrt{\pi / \ln 16}$) for a gaussian profile and 1.571 (= π/2) for a lorentzian profile. We know that the integral of $V(x)$ is unity, and it is a fairly straightforward matter to calculate both the height and the FWHM of $V(x)$. From this, it becomes possible to calculate the constant c as a function of the gaussian fraction k_G. The result of doing this is shown in figure X.4A.

FIGURE X.4A



This curve can be fitted with the empirical equation

$$c = a_0 + a_1 k_G + a_2 k_G^2 + a_3 k_G^3 , \qquad 10.5.19$$

where $a_0 = 1.572$, $a_1 = 0.05288$, $a_2 = -1.323$ and $a_3 = 0.7658$. The error incurred in using this formula nowhere exceeds 0.5%; the mean error is 0.25%.

The Voigt Profile in Terms of the Optical Thickness at the Line Center.

Another way to write the Voigt profile that might be useful is

$$\tau(x) = Cl\tau(0)\int_{-\infty}^{\infty} \frac{\exp[-(\xi - x)^2 \ln 2 / g^2]}{\xi^2 + l^2}\, d\xi. \qquad 10.5.20$$

Here $x = \lambda - \lambda_0$ and ξ is a dummy variable, which disappears when the definite integral is performed. The gaussian HWHM is $g = \lambda_0 V_m \sqrt{\ln 2}/c$, and the lorentzian HWHM is $l = \lambda_0^2 \Gamma/(4\pi c)$. The optical thickness at $\lambda - \lambda_0 = x$ is $\tau(x)$, and the optical thickness at the line centre is $\tau(0)$. C is a dimensionless coefficient, whose value depends on the gaussian fraction $k_G = g/(g + l)$. C is clearly given by

$$Cl\int_{-\infty}^{\infty} \frac{\exp[-\xi^2 \ln 2 / g^2]}{\xi^2 + l^2}\, d\xi = 1. \qquad 10.5.21$$

If we now let $l = l' g/\sqrt{\ln 2}$ *and* $\xi = \xi' g/\sqrt{\ln 2}$, and also make use of the symmetry of the integrand about $\xi = \xi' = 0$, this becomes

$$2Cl'\int_0^{\infty} \frac{\exp(-\xi'^2)}{\xi'^2 + l'^2}\, d\xi' = 1. \qquad 10.5.22$$

On substitution of $\xi' = \dfrac{2l't}{1 - t^2}$ (in order to make the limits finite), we obtain

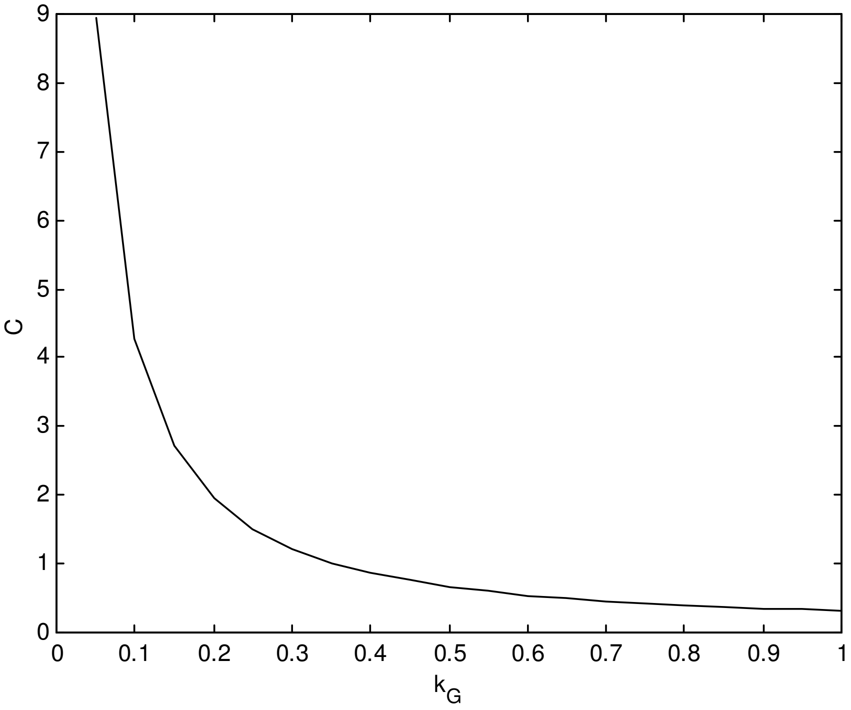
$$4C\int_0^1 \frac{\exp[-\{2l't/(1 - t^2)\}^2]}{1 + t^2}\, dt = 1, \qquad 10.5.23$$

which can readily be numerically integrated for a given value of l'. Recall that $l/g = 1/k_G - 1$ and hence that $l' = (1/k_G - 1)\sqrt{\ln 2}$. The results of the integration are as follows. The column C_{approx} is explained following figure X.4B.

k_G	C	C_{approx}
0.05	8.942 417	9.325 6
0.10	4.264 473	4.288 9
0.15	2.719 106	2.716 4
0.20	1.957 257	1.956 6
0.25	1.508 719	1.511 1
0.30	1.216 486	1.219 6
0.35	1.013 114	1.015 3
0.40	0.864 815	0.865 5
0.45	0.752 806	0.751 9
0.50	0.665 831	0.663 9
0.55	0.596 758	0.594 3
0.60	0.540 859	0.538 6
0.65	0.494 893	0.493 4
0.70	0.456 569	0.456 2
0.75	0.424 227	0.425 1
0.80	0.396 642	0.398 5
0.85	0.372 889	0.375 3
0.90	0.352 263	0.354 2
0.95	0.334 214	0.334 5
1.00	0.318 310	0.315 3

The last entry, the value of C for $k_G = 1$, a pure gaussian profile, is $1/\pi$. These data are graphed in figure X.4B.

FIGURE X.4B



The empirical formula $C_{\text{approx}} = ak_{\text{G}}^{-b} + c_0 + c_1k_{\text{G}} + c_2k_{\text{G}}^2 + c_3k_{\text{G}}^3$, 10.5.24

where $a = +0.309\ 031$ $b = +1.132\ 747$ $c_0 = +0.165\ 10$
$c_1 = -0.829\ 99$ $c_2 = +1.217\ 82$ $c_3 = -0.546\ 65$

fits the curve tolerably well within (but not outside) the range $k_{\text{G}} = 0.15$ to 1.00 .

10.6 *Pressure Broadening*

This is a fairly difficult subject, and I am no expert in it. The reader will forgive me if I accordingly treat it rather briefly and descriptively.

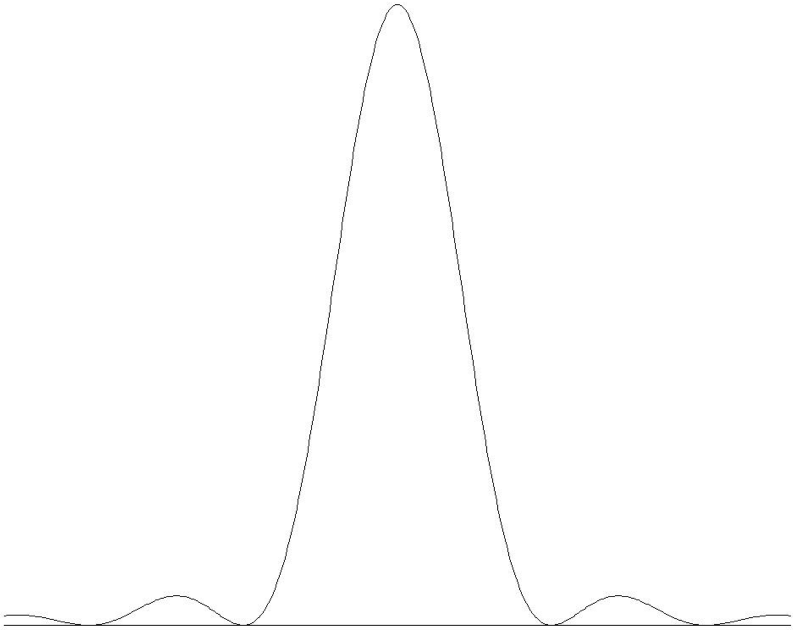
The phenomena of pressure broadening (also known as collisional broadening) are often divided into effects resulting from the short time interval between atomic collisions, and effects resulting at the moment of collision. I shall begin by describing the first of these phenomena.

The only possible absolutely monochromatic unbroadened infinitesimally narrow line with a single, uniquely defined frequency is a sine wave of infinite extent. A sine wave of finite length is not a true sine wave of a single frequency, but it has a spread of component frequencies, which can be determined by Fourier analysis. This, by the way, is the reason behind Heisenberg's uncertainty principle (*Unsicherheitsprinzip*). If the wavefunction that describes a particle is very limited in extent, then the position of the particle is relatively well determined. On the other hand, the limited extent of the wavefunction means that it has a correspondingly broad Fourier spread of constituent wavelengths, and hence the momentum is correspondingly uncertain.

The atmospheres of giant and supergiant stars are relatively thin; pressure broadening is slight and lines tend to be narrow. In the atmospheres of main sequence stars, however, collisions between atoms are frequent. The frequent occurrence of collisions interrupts the wave trains and divides them into short wave-packets, with a corresponding spread of component frequencies. Thus the spectrum lines are broadened.

The Fourier distribution of amplitudes of component frequencies of a sine wave that is truncated by a box function is the same as the Fourier distribution of amplitudes of a light wave that is diffracted by a single slit. That is to say it is a sinc function of the form $(\sin \Delta\nu)/\Delta\nu$ and the intensity distribution is the square of this. The shorter the intercollision time, the wider the spread of constituent frequencies, just as a narrow slit produces a wide diffraction pattern. Thus one might expect the profile of a pressure broadened line to resemble a single slit diffraction pattern, which, it will be recalled, looks like figure X.5.

FIGURE X.5



The profile would indeed look like that if all intercollision times were exactly equal and all wave-train fragments were of exactly the same length. There is, however, a Poisson distribution of intercollision times, and so the above profile has to be convolved with this Poisson distribution. While I don't do the calculation here, the resulting profile is a Lorentz profile except that the damping constant Γ is replaced by $2/\bar{t}$, where $\bar{t}$ is the mean time between collisions. The mean time between collisions is given, from kinetic theory of gases, by

$$\bar{t} = \frac{1}{nd^2}\sqrt{\frac{m}{16\pi kT}}. \qquad 10.6.1$$

Here m, d and n are, respectively, the masses, diameters and number density of the atoms. Hence, if the kinetic temperature is independently known, the number density of the particles can be determined from the FWHm of a pressure-broadened line.

It will be recalled that classical radiation damping theory predicts the same FWHm for all lines, with a classical damping constant γ. Quantum mechanical theory predicts a damping constant Γ and hence FWHm that differs from line to line. Yet in the spectrum of a main sequence star, one quite often finds that all lines of a given element have the same FWHm and hence the same effective damping constant. This is because the width of a Lorentz profile is determined more by pressure broadening than by radiation damping.

There are further broadening effects caused by interactions that take place at the moment of collision. If an atom is approached by an electron or an ion, it will temporarily be in an electric field, and consequently the lines will be broadened by Stark effect, which may be either linear (proportional to the electric field E) or quadratic ($\propto E^2$), or neutral-neutral reactions give rise to interactions between temporarily induced dipole moments (van der Waals forces), and these all have different dependences on interatomic distance. Neutral magnesium is very sensitive to quadratic Stark effect, and hydrogen is sensitive to linear Stark effect. The entire subject is quite difficult, and I leave it here except to point out two small details. Very often the broadening is not symmetric, lines typically having wider wings to the long wavelength side than on the short wavelength side. This is because the effect of the interactions is to lower and broaden the energy levels of a transition, the lower energy level generally being lowered more than the upper. A second point is that the hydrogen Balmer lines are often much broadened by linear Stark effect, and this can be recognized because the Stark pattern for the Balmer series is such that there are no undisplaced Stark components for even members of the series – Hβ, Hδ, Hζ, etc. Thus results in a central dip to these lines in an emission spectrum or a central bump in an absorption line.

10.7 *Rotational Broadening*

The lines in the spectrum of a rotating star are broadened because light from the receding limb is redshifted and light from the approaching limb is blueshifted. (I shall stick to astronomical custom and refer to a "redshift" as a shift towards a longer wavelength, even though for an infrared line a "redshift" in this sense would be a shift away from the red! A "longward" shift doesn't quite solve the problem either, for the following reason. While it is true that relativity makes no distinction between a moving source and a moving observer, in the case of the Doppler effect in the context of sound in air, if the *observer* is moving, there may be a change in the pitch of the perceived sound, but there is no change in wavelength!) It may be remarked that early-type stars (type F and earlier) tend to be much faster rotators than later-type stars, and consequently early-type stars show more rotational broadening. It should also be remarked that pole-on rotators do not, of course, show rotational broadening (even early-type fast rotators).

We shall start by considering a star whose axis of rotation is in the plane of the sky, and which is of uniform radiance across its surface. We shall then move on to oblique rotators, and then to limb-darkened stars. A further complication that could be considered would be non-uniform rotation. Thus, the Sun does not rotate as a solid body, but the angular speed at low latitudes is faster than at higher latitudes – the so-called "equatorial acceleration".

In figure X.6, on the left hand we see the disc of a star as seen on the sky by an observer. PQ is the axis of rotation, supposed to be in the plane of the sky, and AB is the equator. X is a point on the surface of the star at coordinates (x, y), latitude θ. The star is supposed to be rotating with an equatorial speed υ_e. What we are going to show is that all points on the chord LMN have the same radial velocity away from or towards the observer, and consequently all light from points on this chord has the same Doppler shift.

The right hand part of the figure shows the star seen from above the pole P. The small circle is the parallel of latitude CD shown on the left hand part of the figure.

M is a point on the equator and also on the chord LMN. Its speed is v_e and the daial component of its velocity is $v_e \sin \alpha$. The speed of the point M is $v_e \cos \theta$, and its radial velocity is $v_e \cos \theta \sin \mathrm{OPX}$. But $x = \mathrm{PM} \sin \alpha = a \sin \alpha$ and $x = \mathrm{PX} \sin \mathrm{OPX}$ $= a \cos \theta \sin \mathrm{OPX}$. Therefore $\cos \theta \sin \mathrm{OPX} = \sin \alpha$. Therefore the radial velocity of X is $v_e \sin \alpha$, which is the same as that of M, and therefore all points on the chord LMN have radial velocity $v_e \sin \alpha = v_e x/a$.

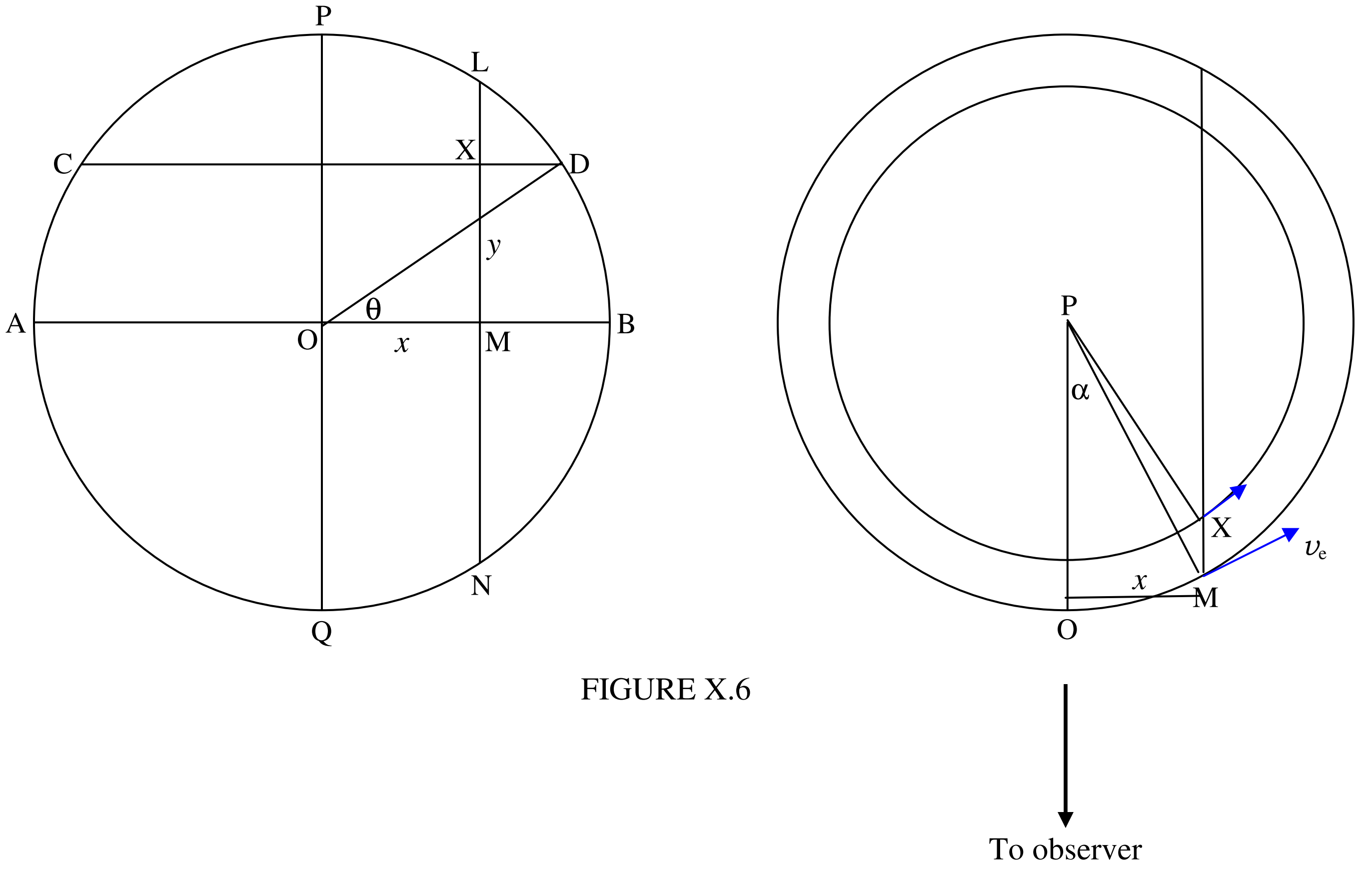


FIGURE X.6

Therefore all points on the chord x = constant are subject to the same Doppler shift

$$\frac{\Delta\lambda}{\lambda} = \frac{v_e x}{ac}. \qquad 10.7.1$$

The ordinate of an emission line profile at Doppler shift $\Delta\lambda$ compared with its ordinate at the line centre is equal to the ratio of the length of the chord x = constant to the diameter $2a$ of the stellar disk:

$$\frac{I_\lambda(\Delta\lambda)}{I_\lambda(0)} = \left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} = \left(1 - \frac{c^2(\Delta\lambda)^2}{v_{\rm e}^2\lambda^2}\right)^{\frac{1}{2}}. \qquad 10.7.2$$

In the above, we have assumed that the axis of rotation is in the plane of the sky, or that the inclination i of the equator to the plane of the sky is $90^{\rm o}$. If the inclination is not $90^{\rm o}$, the only effect is that all radial velocities are reduced by a factor of sin i, so that equation 10.7.2 becomes

$$\frac{I_\lambda(\Delta\lambda)}{I_\lambda(0)} = \left(1 - \frac{c^2(\Delta\lambda)^2}{v_{\rm e}^2\sin^2 i.\lambda^2}\right)^{\frac{1}{2}}, \qquad 10.7.3$$

and this is the line profile. It is an ellipse, and if we write $\frac{I_\lambda(\Delta\lambda)}{I_\lambda(0)} = X$ and $\frac{\Delta\lambda}{\lambda} = Y$

equation 10.7.3 can be written

$$\frac{x^2}{\left(\frac{v_e \sin i}{c}\right)^2} + \frac{y^2}{1^2} = 1. \qquad 10.7.4$$

The basal width of the line (which has no asymptotic wings) is $\frac{2v_{\rm e}\sin i}{c}$ and the FWHM is $\frac{\sqrt{3}v_{\rm e}\sin i}{c}\cdot$ The profile of an absorption line of central depth d is

$$\frac{I_\lambda(\Delta\lambda)}{I_\lambda(0)} = 1 - d\left(1 - \frac{c^2(\Delta\lambda)^2}{v_{\rm e}^2\sin^2 i.\lambda^2}\right)^{\frac{1}{2}}, \qquad 10.7.5$$

It is left as an exercise to show that

$$\text{Equivalent width} = \frac{\pi}{\sqrt{12}} \times \text{ central depth} \times \text{ FWHm} = 0.9069\, dw. \qquad 10.7.6$$

From the width of a rotationally broadened line we can determine $v_{\rm e}\sin i$, but we cannot determine $v_{\rm e}$ and i separately without additional information. Likewise, we cannot determine the angular speed of rotation unless we know the radius independently.

It might be noted that, for a rotating *planet*, visible only by reflected light, the Doppler effect is doubled by reflection, so the basal width of a rotationally broadened line is $\frac{4v_{\rm e}\sin i}{c}$.

Now let us examine the effect of limb darkening. I am going to use the words *intensity* and *radiance* in their strictly correct senses as described in Chapter 1, and the symbols I and L respectively. That is, radiance = intensity per unit projected area. For spectral intensity and spectral radiance – i.e. intensity and radiance per unit wavelength interval, I shall use a subscript λ.

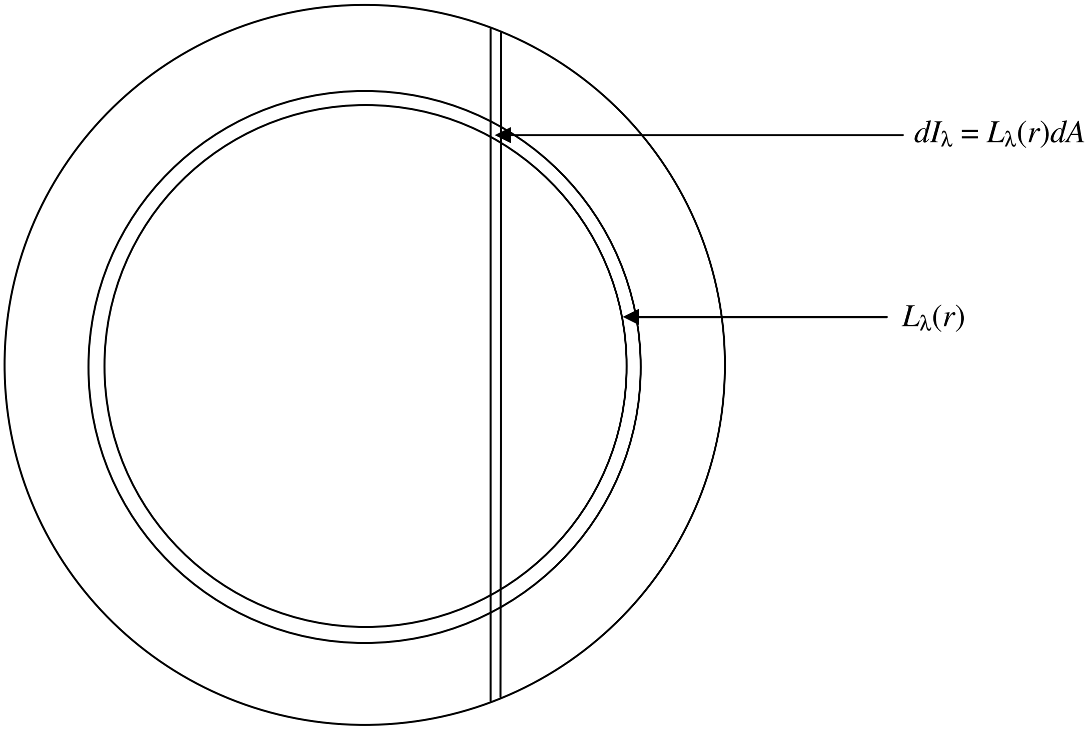


FIGURE X.7

We suppose that the spectral radiance at a distance r from the centre of the disc is $L_\lambda(r)$. The intensity from an elemental area dA on the disc is $dI_\lambda = L_\lambda(r)dA$. The area between the vertical strip and the annulus in figure X.7 is a little parallelogram of length dy and width dx, so that $dA = dxdy$. Here $y^2 = r^2 - x^2$, so that $dy = \frac{rdr}{y} = \frac{rdr}{\sqrt{r^2-x^2}}$.

Therefore $dA = \frac{rdrdx}{\sqrt{r^2-x^2}}$. The total intensity from the strip of width dx, which is $dI_\lambda(\Delta\lambda)$, where $\frac{\Delta\lambda}{\lambda} = \frac{x\upsilon_{\mathrm{e}}\sin i}{ac}$, is

$$dI_\lambda(\Delta\lambda) = 2\int_x^a \frac{L_\lambda(r)rdr}{\sqrt{r^2-x^2}}\cdot dx\,. \tag{10.7.7}$$

The (emission) line profile is

$$\frac{I_\lambda(\Delta\lambda)}{I_\lambda(0)} = \frac{\displaystyle\int_x^a \frac{L_\lambda(r)rdr}{\sqrt{r^2-x^2}}dx}{\displaystyle\int_0^a L_\lambda(r)dr}, \tag{10.7.8}$$

which is the line profile. As an exercise, see if you can find an expression for the line profile if the limb=darkening is given by $L_\theta = L(0)[1-u(1-\cos\theta)]$, and show that if the limb-darkening coefficient $u = 1$, the profile is parabolic.

Equation 10.7.8 enables you to calculate the line profile, given the limb darkening. The more practical, but more difficult, problem, is to invert the equation and, from the observed line profile, find the limb darkening. Examples of this integral, and its inversion by solution of an integral equation, are given by Tatum and Jaworski, J. Quant. Spectr. Rad. Transfer, **38**, 319, (1987).

Further pursuit of this problem would be to calculate the line profile of a uniform star that is rotating faster at the equator than at the poles, and then for a star that is both limb-darkened and equatorially accelerated – and then see if it is possible to invert the problem uniquely and determine both the limb darkening and the equatorial acceleration from the line profile. That would be quite a challenge.

10.8 *Instrumental Broadening*

Even if the radiation damping profile of a line is negligible and if it is subject to negligible thermal, pressure and rotational broadening, it still has to suffer the indignity of instrumental broadening. Almost any type of spectrograph will broaden a line. The broadening produced by a prism is inversely proportional to the size of the prism, and the broadening produced by a grating is inversely proportional to the number of grooves in the grating. After a spectrum is produced (and broadened) by a spectrograph, it may be scanned by a further instrument such as a microphotometer, or even if it is recorded digitally, it is still further broadened by the point spread function. The instrumental broadening can in principle be determined experimentally by measuring the instrumentally-produced profile of an intrinsically very narrow line. Then, when the instrument is used to examine a broad line, the observed profile is the convolution of the true profile and the instrumental profile. We can write this symbolically as

$$\mathrm{O} = \mathrm{T} \ast \mathrm{I}\,. \tag{10.8.1}$$

Here O, T and I are respectively the observed, true and instrumental profiles, and the asterisk denotes the convolution. The mathematical problem is to deconvolve this equation so that, given the instrumental profile and the observed profile it is possible to recover the true profile. This is done by making use of a mathematical theorem known as Borel's theorem, which is that the Fourier transform of the convolution of two functions is equal to the product of the Fourier transforms of each. That is

$$\overline{\mathrm{O}} = \overline{\mathrm{T}} \times \overline{\mathrm{I}}, \qquad 10.8.2$$

where the bar denotes the Fourier transform. Numerical fast Fourier transform computer programs are now readily available, so the procedure is to calculate the Fourier transforms of the observed and instrumental profile, divide the former by the latter to obtain $\overline{\mathrm{T}}$, and then calculate the inverse Fourier transform to obtain the true profile. This procedure is well known in radio astronomy, in which the observed map of a sky region is the convolution of the true map with the beam of the radio telescope, though, unlike the one-dimensional spectroscopic problem the corresponding radio astronomy problem is two-dimensional.

10.9 *Other Line-broadening mechanisms*

I just briefly mention here one or two additional sources of line-broadening.

Lines may be broadened by unresolved or smeared Zeeman splitting, particularly for lines involving levels with large Landé *g*-factors. By "smeared" I mean the situation that arises if there is a large range of magnetic field strength through the line of sight or because (as is always the case with stars other than the Sun) you are looking at a whole-disc spectrum. Since the splitting depends on the field strength, the lines will obviously be smeared rather than cleanly divided into a number of discrete Zeeman components. Zeeman smearing is often large in the spectrum of white dwarf stars, where magnetic fields can be large and the observer looks through a large range of magnetic field strength.

Different Zeeman components are plane or circularly polarized according to the direction of the magnetic field. Thus in principle one should be able to recognize Zeeman effect, even if smeared or not fully resolved, by its changing appearance in different polarization directions. However, this will be true only if the magnetic field is uniform in direction, as it may mostly be in, for example, a sunspot. For a whole-disc spectrum there will be a variety of different directions of the magnetic field, and so the polarization information will be lost.

Broad lines are sometimes the result of unresolved hyperfine structure in elements with a large nuclear spin such as vanadium, or unresolved isotopic lines in elements with several isotopes of comparable abundance such as tin, copper or chlorine.

Another source of line broadening is autoionization (in absorption spectra) or dielectronic recombination (in emission spectra) in elements such as copper. These mechanisms were described in section 8.8.

One last remark might be made, namely that line broadening, whether instrumental, thermal, rotational, etc., does not change the equivalent width of a line, provided that the line is everywhere optically thin. This does not apply, however, if the line is not everywhere optically thin.

APPENDIX A

Convolution of Gaussian and Lorentzian Functions

Equation 10.5.6 is

$$G(x) = G_1(x) * G_2(x) = \frac{1}{g_1 g_2}\frac{\ln 2}{\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2 \ln 2}{g_1^2}\right).\exp\left(-\frac{(\xi - x)^2 \ln 2}{g_2^2}\right).d\xi\,. \qquad 10.A.1$$

The integration is straightforward, if taken slowly and carefully, provided you know the integral $\int_{-\infty}^{\infty} \exp(-kx^2)dx = \sqrt{\frac{\pi}{k}}$. It goes thus:

$$G(x) = \frac{1}{g_1 g_2}\frac{\ln 2}{\pi}\int_{-\infty}^{\infty} \exp[-(a\xi^2 + b\xi + c)]\,d\xi, \qquad 10.A.2$$

where $a = \frac{(g_1^2 + g_2^2)\ln 2}{g_1^2 g_2^2}, \quad b = -\frac{x \ln 4}{g_2^2}, \quad c = \frac{x^2 \ln 2}{g_2^2}$.

$$G(x) = \frac{1}{g_1 g_2}\frac{\ln 2}{\pi}\int_{-\infty}^{\infty} \exp[-a(\xi^2 + 2B\xi + C)]\,d\xi, \qquad 10.A.3$$

where $B = b/(2a), \quad C = c/a$.

$$G(x) = \frac{1}{g_1 g_2}\frac{\ln 2}{\pi}\int_{-\infty}^{\infty} \exp[-a\{(\xi + B)^2 + C - B^2\}]\,d\xi \qquad 10.A.4$$

$$= \frac{1}{g_1 g_2}\frac{\ln 2}{\pi}\int_{-\infty}^{\infty} \exp[-a(\zeta^2 + C - B^2)]d\zeta \qquad 10.A.5$$

$$= \frac{K\ln 2}{\pi g_1 g_2}\int_{-\infty}^{\infty}\exp(-a\zeta^2)\,d\zeta \;=\; \frac{K\ln 2}{g_1 g_2\sqrt{\pi a}}, \qquad 10.A.6$$

where $K = \exp[-a(C - B^2)]$.

We have now completed the integration, except that we now have to remember what a, C and B were. When we do this, after a bit more careful algebra we arrive at the result

$$G(x) = G_1(x) * G_2(x) = \frac{1}{g}\cdot\sqrt{\frac{\ln 2}{\pi}}\exp\left(-\frac{x^2\ln 2}{g^2}\right). \qquad 10.A.7$$

In a similar manner, equation 10.5.10 is

$$L(x) = L_1(x) * L_2(x) = \frac{l_1 l_2}{\pi^2}\int_{-\infty}^{\infty}\frac{1}{\xi^2 + l_1^2}\frac{1}{(\xi - x)^2 + l_2^2}d\xi. \qquad 10.A.8$$

Resolve the integrand into partial fractions:

$$\frac{1}{\xi^2 + l_1^2}\frac{1}{(\xi - x)^2 + l_2^2} = \frac{A\xi}{\xi^2 + l_1^2} + \frac{B}{\xi^2 + l_1^2} + \frac{C(\xi - x)}{(\xi - x)^2 + l_2^2} + \frac{D}{(\xi - x)^2 + l_2^2}\cdot \qquad 10.A.9$$

Evaluation of the constants is straightforward, if slightly tedious, by the usual method of partial fractions:

$$\begin{aligned} A &= -C = 2x\alpha, \\ B &= (x^2 + l_2^2 - l_1^2)\alpha, \\ D &= (x^2 - l_2^2 + l_1^2)\alpha, \\ &\text{where } \alpha = 1/[(x^2 + l_2^2 + l_1^2)^2 - 4l_1^2 l_2^2]. \end{aligned} \qquad 10.A.10$$

Now

$$L(x) = \frac{l_1 l_2}{\pi^2}\left(A\int_{-\infty}^{\infty}\frac{\xi d\xi}{\xi^2 + l_1^2} + B\int_{-\infty}^{\infty}\frac{d\xi}{\xi^2 + l_1^2} + C\int_{-\infty}^{\infty}\frac{(\xi - x)d\xi}{(\xi - x)^2 + l_2^2} + D\int_{-\infty}^{\infty}\frac{d\xi}{(\xi - x)^2 + l_2^2}\right).$$

10.A.11

From symmetry considerations, this is:

$$L(x) = \frac{2l_1 l_2}{\pi^2}\left(B\int_0^{\infty}\frac{d\xi}{\xi^2 + l_1^2} + D\int_0^{\infty}\frac{d\zeta}{\zeta^2 + l_2^2}\right). \qquad 10.A.12$$

$$\therefore \qquad L(x) = \frac{2l_1 l_2}{\pi^2}\left(\frac{\pi B}{2l_1} + \frac{\pi D}{2l_2}\right) = (l_2 B + l_1 D)/\pi. \qquad 10.A.13$$

We have now completed the integration, except that we now have to remember what *B* and *D* were. When we do this, after a bit more careful algebra we arrive at the result

$$L(x) = \frac{l}{\pi}\cdot\frac{1}{x^2 + l^2}, \qquad 10.A.14$$

where $l = l_1 + l_2$.

The Voigt profile is given by equation 10.5.14:

$$V(x) = \frac{l}{g}\sqrt{\frac{\ln 2}{\pi^3}}\int_{-\infty}^{\infty}\frac{\exp\left(-[(\xi - x)^2 \ln 2]/g^2\right)}{\xi^2 + l^2}d\xi. \qquad 10.A.15$$

For short, I am going to write the ratio l/g as a. The relation between this ratio and the gaussian fraction k_G is $a = (1 - k_G)/\, k_G$, $k_G = 1/(1 + a)$. In the above equation, $x = \lambda - \lambda_0$, and I am going to choose a wavelength scale such that $g = 1$; in other words wavelength interval is to be expressed in units of g. Thus I shall write the equation as

$$V(x) = a\sqrt{\frac{\ln 2}{\pi^3}}\int_{-\infty}^{\infty} \frac{\exp\left(-(\xi - x)^2 \ln 2\right)}{\xi^2 + a^2} d\xi. \qquad \text{10.A.16}$$

The integration has to be done numerically, and there is a problem in that the limits are infinite. We can deal with this with the change of variable $\xi = a \tan \theta$, when the integral becomes

$$V(x) = \sqrt{\frac{\ln 2}{\pi^3}}\int_{-\pi/2}^{\pi/2} \exp[-(a \tan \theta - x)^2 \ln 2]\, d\theta. \qquad \text{10.A.17}$$

The limits are now finite, and the integrand is zero at each limit. Computing time will be much diminished by the further substitution $t = \tan\frac{1}{2}\theta$, when the expression becomes

$$V(x) = \sqrt{\frac{\ln 16}{\pi^3}}\int_{-1}^{1} \frac{\exp[-\{2at/(1-t^2) - x\}^2 \ln 2]}{1 + t^2} dt \qquad \text{10.A.18}$$

This is faster than the previous expression because one avoids having to compute the trigonometric function tan. It could also have been arrived at in one step by means of the substitution $\xi = \dfrac{2at}{1 - t^2}$, though such a substitution may not have been immediately obvious. Like the previous expression, the limits are finite, and the integrand is zero at each end. Numerical integration would now seem to be straightforward, although there may yet be some difficulty. Suppose one is integrating, for example, by Simpson's method. A question might arise as to how many intervals should be used. Simpson's method is often very effective with a remarkably small number of intervals, but, for high precision, one may nevertheless wish to use a fine interval. If one uses a fine interval, however, as one approaches either limit, the expression $t/(1 - t^2)$ becomes very large, and, even though the integrand then becomes small, a computer may be reluctant to return a value for the exp function, and it may deliver an error message. The best way to deal with that difficulty is to set the integrand equal to zero whenever the absolute value of the argument of the exp function exceeds some value below which the computer is happy.

One might be tempted to reduce the amount of computation by saying that $\int_{-1}^{1} = 2\int_{0}^{1}$, but this is not correct, for, while the Voigt profile is symmetric about $x = 1$, the integrand is not symmetric about $t = 0$. However, if

$$V(x) = \int_{-1}^{1}, \text{ and } V_1(x) = \int_{-1}^{0}, \text{ and } V_2(x) = \int_{0}^{1},$$

it is true that $V(x) = V_1(x) + V_2(x)$ and $V_1(x) = V_2(-x)$, and hence that $V(x) = V_1(x) + V_1(-x)$, and this can be used to economise to a small extent. It is still necessary to calculate $V_1(x)$ for all values of x, both positive and negative, but the number of integration steps for each point can be halved.

APPENDIX B

Radiation Damping as Functions of Angular Frequency, Frequency and Wavelength

It occurred to me while preparing this Chapter as well as the preceding and following ones, that sometimes I have been using angular frequency as argument, sometimes frequency, and sometimes wavelength. In this Appendix, I bring together the salient formulas for radiation damping in terms of $\Delta\omega = \omega - \omega_0$, $\Delta\nu = \nu - \nu_0$ and $\Delta\lambda = \lambda - \lambda_0$. I reproduce equation 10.2.11 for the absorption coefficient for a set of forced, damped oscillators, except that I replace n, the number per unit volume of oscillators with n_1f_{12}, the effective number of atoms per unit volume in the lower level of a line, and I replace the classical damping constant γ with Γ, which may include a pressure broadening component.

$$\alpha = \frac{n_1 f_{12}\Gamma e^2\omega^2}{m\varepsilon_0 c[(\omega^2-\omega_0^2)^2+\Gamma^2\omega^2]} \qquad \text{m}^{-1}. \qquad 10.B.1$$

You should check that the dimensions of this expression are L^{-1}, which is appropriate for linear absorption coefficient. You may note that $[e^2/\varepsilon_0] \equiv \text{ML}^3\text{T}^{-2}$ and $[\Gamma] \equiv \text{T}^{-1}$. Indeed check the dimensions of all expressions that follow, at each stage.

We can write $\omega^2 - \omega_0^2 = (\omega-\omega_0)(\omega+\omega_0) = \Delta\omega(2\omega_0+\Delta\omega)$, and the equation becomes

$$\alpha = \frac{n_1 f_{12}\Gamma e^2(\omega_0+\Delta\omega)^2}{m\varepsilon_0 c[(\Delta\omega)^2(2\omega_0+\Delta\omega)^2+\Gamma^2(\omega_0+\Delta\omega)^2]} \qquad \text{m}^{-1}. \qquad 10.B.2$$

Now I think it will be owned that the width of a spectrum line is very, very much smaller than its actual wavelength, except perhaps for extremely Stark-broadened hydrogen lines, so that, in the immediate vicinity of a line, $\Delta\omega$ can be neglected compared with ω_0; and a very long way from the line, where this might not be so, the expression is close to zero anyway. (Note that you can neglect $\Delta\omega$ *only with respect to* ω; you cannot just put $\Delta\omega = 0$ where it lies alone in the denominator!) In any case, I have no compunction at all in making the approximation

$$\alpha(\Delta\omega) = \frac{n_1 f_{12}\Gamma e^2}{4m\varepsilon_0 c[(\Delta\omega)^2 + (\tfrac{1}{2}\Gamma)^2]} \qquad \text{m}^{-1}. \qquad 10.B.3$$

The maximum of the $\alpha(\Delta\omega)$ curve is

$$\alpha(0) = \frac{e^2 n_1 f_{12}}{m\varepsilon_0 c\Gamma} \qquad \mathrm{m}^{-1}. \qquad 10.\mathrm{B}.4$$

The optical thickness at the line centre (whether or not the line is optically thin) is

$$\tau(0) = \frac{e^2 N_1 f_{12}}{m\varepsilon_0 c\Gamma}\,. \qquad 10.\mathrm{B}.5$$

N_1 is the number of atoms in level 1 per unit area in the line of sight, whereas n_1 is the number per unit volume.

The HWHM of $\alpha(\Delta\omega)$ curve is

$$\mathrm{HWHM} = \tfrac{1}{2}\Gamma \qquad \mathrm{rad\ s}^{-1}. \qquad 10.\mathrm{B}.6$$

The area under the $\alpha(\Delta\omega)$ curve is $\quad \mathrm{Area} = \dfrac{\pi e^2 n_1 f_{12}}{2m\varepsilon_0 c} \qquad \mathrm{m}^{-1}\ \mathrm{rad\ s}^{-1}. \qquad 10.\mathrm{B}.7$

As expected, the area does not depend upon Γ.

To express the absorption coefficient as a function of *frequency*, we note that $\omega = 2\pi\nu$, and we obtain

$$\alpha(\Delta\nu) = \frac{n_1 f_{12}\Gamma e^2}{16\pi^2 m\varepsilon_0 c[(\Delta\nu)^2 + (\frac{\Gamma}{4\pi})^2]} \qquad \mathrm{m}^{-1}. \qquad 10.\mathrm{B}.8$$

The maximum of this is (of course) the same as equation 10.B.4.

The HWHM of the $\alpha(\Delta\nu)$ curve is

$$\mathrm{HWHM} = \Gamma/(4\pi) \qquad \mathrm{s}^{-1}. \qquad 10.\mathrm{B}.9$$

The area under the $\alpha(\Delta\nu)$ curve is

$$\mathrm{Area} = \frac{e^2 n_1 f_{12}}{4m\varepsilon_0 c} \qquad \mathrm{m}^{-1}\ \mathrm{s}^{-1}. \qquad 10.\mathrm{B}.10$$

To express the absorption coefficient as a function of *wavelength*, we can start from equation 10.B.8 and use $\nu = c/\lambda$, but, just to avoid any possible doubt, let's start from equation 10.B.1 and put $\omega = 2\pi c/\lambda$. This gives

$$\alpha = \frac{n_1 f_{12}\Gamma e^2}{m\varepsilon_0 c}\cdot\left(\frac{\lambda^2\lambda_0^4}{4\pi^2 c^2\left(\lambda_0^2-\lambda^2\right)^2 + \lambda^2\lambda_0^4\Gamma^2}\right) \qquad \mathrm{m}^{-1}. \qquad 10.B.11$$

In a manner similar to our procedure following equation 10.B.12, we write $\lambda_0^2 - \lambda^2 = (\lambda_0 - \lambda)(\lambda_0 + \lambda)$, and $\lambda = \lambda_0 + \Delta\lambda$, and neglect $\Delta\lambda$ with respect to λ_0, and we obtain:

$$\alpha(\Delta\lambda) = \frac{n_1 f_{12}\Gamma e^2}{16\pi^2 m\varepsilon_0 c^3}\cdot\left(\frac{\lambda_0^4}{(\Delta\lambda)^2 + \frac{\lambda_0^4\Gamma^2}{16\pi^2c^2}}\right) \qquad \mathrm{m}^{-1}. \qquad 10.B.12$$

The maximum of this is (of course) the same as equation 10.B.4. (Verifying this will serve as a check on the algebra.)

The HWHM of the $\alpha(\Delta\lambda)$ curve is

$$\mathrm{HWHM} = \frac{\lambda_0^2\Gamma}{4\pi c} \qquad \mathrm{m}. \qquad 10.B.13$$

The area under the $\alpha(\Delta\lambda)$ curve is

$$\mathrm{Area} = \frac{\lambda_0^2 e^2 n_1 f_{12}}{4m\varepsilon_0 c^2}\,. \qquad 10.B.14$$

Did I forget to write down the units after this equation?

These results for α might be useful in tabular form. For τ, replace n_1 by N_1.

	$\Delta\omega$	$\Delta\nu$	$\Delta\lambda$
	$\frac{\Gamma e^2 n_1 f_{12}}{4m\varepsilon_0 c[(\Delta\omega)^2 + (\frac{1}{2}\Gamma)^2]}$	$\frac{\Gamma e^2 n_1 f_{12}}{16\pi^2 m\varepsilon_0 c[(\Delta\nu)^2 + (\frac{\Gamma}{4\pi})^2]}$	$\frac{\Gamma e^2\lambda_0^4 n_1 f_{12}}{16\pi^2 m\varepsilon_0 c^3\left[(\Delta\lambda)^2 + \frac{\lambda_0^4\Gamma^2}{16\pi^2c^2}\right]}$
Height	$\frac{e^2 n_1 f_{12}}{m\varepsilon_0 c\Gamma}$	$\frac{e^2 n_1 f_{12}}{m\varepsilon_0 c\Gamma}$	$\frac{e^2 n_1 f_{12}}{m\varepsilon_0 c\Gamma}$

Area	$\dfrac{\pi e^2 n_1 f_{12}}{2m\varepsilon_0 c}$	$\dfrac{e^2 n_1 f_{12}}{4m\varepsilon_0 c}$	$\dfrac{\lambda_0^2 e^2 n_1 f_{12}}{4m\varepsilon_0 c^2}$
HWMH	$\frac{1}{2}\Gamma$	$\Gamma/(4\pi)$	$\dfrac{\lambda_0^2 \Gamma}{4\pi c}$

It is to be noted that if the radiation damping profile is thermally broadened, the height of the absorption coefficient curve diminishes, while the area is unaltered provided that the line is optically thin. The optically thick situation is dealt with in the following chapter. It might also be useful to note that a gaussian profile of the form

$$\alpha(\Delta\lambda) \; = \; \alpha(0)\exp\left(-\frac{c^2(\Delta\lambda)}{V_\mathrm{m}^2\lambda_0^2}\right) \qquad \text{10.B.15}$$

has an area of $\dfrac{\lambda_0^2 e^2 n_1 f_{12}}{4m\varepsilon_0 c^2}$ if

$$\alpha(0) \; = \; \frac{\lambda_0 e^2 n_1 f_{12}}{4\sqrt{\pi} m\varepsilon_0 c V_\mathrm{m}}\cdot \qquad \text{10B.16}$$

APPENDIX C

Optical Thinness, Homogeneity and Thermodynamic Equilibrium

It has also occurred to me while preparing these chapters that some of the equations are valid only under certain conditions, such as that the gas is *optically thin*, or is *homogeneous* or is in *thermodynamic equilibrium*, or some combination of these, or none of them. It would be tedious to spell out all of the conditions after each equation. Yet it is important to know under what conditions each is valid. In this Appendix I try to give some guidance. For example, most of the equations in this Chapter deal with line profiles in an optically thin gas, whereas in the next Chapter the gas is no longer optically thin. In the end, however, the only way of being sure of what conditions apply to each equation is to understand the basic physics behind each rather than attempting to memorize which conditions apply to which equations.

The linear absorption coefficient α at a point within a gas is proportional to the local number density n_1 of absorbers. (The subscript 1 refers to “atoms in the lower level of the line concerned”.)The optical thickness of a slab of gas of thickness D is related to the absorption coefficient (which may or may not vary throughout the slab) by $\tau = \int_0^D \alpha(x)\,dx$. This is so whether or not the gas is optically thin or whether it is homogeneous. Likewise, the column density $\mathcal{N}_1$ of absorbers is related to the number density by $\mathcal{N}_1 = \int_0^D n_1(x)\,dx$. If the gas is homogeneous in the sense that n_1 is not a function of x, and consequently α is not a function of x either, then these equations become simply $\tau = \alpha D$ and $\mathcal{N}_1 = n_1 D$, and this is so whether or not the gas is optically thin.

Whether optically thin or thick, and whether homogeneous or not, the optical thickness is proportional to the column density $\mathcal{N}$, just as the absorption coefficient is proportional to n_1.

If a layer of gas of thickness D is not homogeneous, the optical thickness is related to the absorption coefficient and the thickness of the gas by $\tau = \int_0^D \alpha(x)\,dx$. If the gas is homogeneous so that α is independent of x, then the relation is merely $\tau = \alpha D$. Neither of these equations requires the gas to be optically thin. That is, they are valid whether the gas is optically thin or thick. The absorption coefficient at a point within the gas is proportional to the local density (number of absorbers per unit volume there.) The optical thickness is proportional to the column density of absorbers along the line of sight, whether or not the gas is optically thin and whether or not it is homogeneous.

However, the *equivalent width* and *central depth* of an absorption line, or the *intensity* or *radiance*, or *central intensity or radiance per unit wavelength interval* of an emission line, are proportional to the column density of atoms *only if the gas is optically thin*. Indeed this simple proportionality can serve as a good definition of what is meant by being optically thin.

The equivalent width of an absorption line is given by $W = \int [1 - \exp\{-\tau(\lambda)\}]\,d\lambda$. If the gas is homogeneous, this becomes $W = \int [1 - \exp\{-D\alpha(\lambda)\}]\,d\lambda$. If, in addition, the gas is optically thin at all wavelengths within the line, this becomes (by Maclaurin expansion), merely $W = D\int \alpha(\lambda)\,d\lambda$. Note that, if λ and D are expressed in m and if α is expressed in m^{-1}, the equivalent width will be in m. If, however, you choose to express wavelengths in angstroms and the thickness of a cloud in parsecs, that is your problem, and you are on your own.

Any equations in which we have gone from n, the total number of atoms per unit volume in all levels to n_1 *via* Boltzmann's equation, implies an assumption of thermodynamic equilibrium. An example would be going from equation 9.2.4 (which does not imply thermodynamic equilibrium) to equations 9.2.6-10 (which do imply thermodynamic equilibrium). If a gas is truly in thermodynamic equilibrium, this implies that the gas will be at a single, homogenous temperature – otherwise there will be heat flow and no equilibrium. It is doubtful if anything in the Universe is truly in thermodynamical equilibrium in the very strictest use of the term. However, even in an atmosphere in which the temperature is different from point to point, we may still have *local* thermodynamic equilibrium (LTE), in the sense that, at any point, it is all right to calculate the distribution of atoms among their energy levels by Boltzmann's equation, or the degree of ionization by Saha's equation, or the atomic speeds by the Maxwell-Boltzmann equation, or the radiation energy density by Planck's equation – and you may even be able to use the same temperature for each. This may be all right within a small volume of an atmosphere; only when considered over large ranges of space and time will it be evident that the atmosphere is not in true thermodynamic equilibrium.