CHAPTER 5
General Quadratic Equation, Part I
Translation and Rotation of Axes
$\Delta_3 \neq 0$ Central quadrics

5.1 *Introduction*

In writing this chapter, I have assumed that the reader has read the Preamble to these notes and that, before s/he reads another word of this chapter, s/he has available computer programs that will instantly evaluate 3×3 determinants and 4×4 determinants, and solve quadratic equations. Without this, this chapter will be impossibly tedious to work through. With it, it should be relatively easy.

The general quadratic equation in three variables is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \qquad 5.1.1$$

It represents a quadric surface such as an ellipsoid or a paraboloid, or perhaps a pair of planes, or possibly something else. If we know the values of the coefficients, can we tell which type of surface it represents?

We shall shortly learn that, in order to answer this question, we shall have to evaluate the following two determinants. Indeed, on seeing an equation of the form of 5.1.1, evaluating these determinants is the very first thing we must do. We should not hesitate for a moment, wondering what to do.

$$\Delta_3 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \qquad 5.1.2$$

$$\Delta_4 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} \qquad 5.1.3$$

I shall be using the notation that $\bar{a}$ is the cofactor of a in Δ_3, and $\hat{a}$ is the cofactor of a in Δ_4. With this notation, it will be noted that $\hat{d} = \Delta_3$.

Here are a few examples:

$$\bar{a} = +\begin{vmatrix} b & f \\ f & c \end{vmatrix} \quad \bar{h} = -\begin{vmatrix} h & f \\ g & c \end{vmatrix} \quad \hat{a} = +\begin{vmatrix} b & f & v \\ f & c & w \\ v & w & d \end{vmatrix} \quad \hat{h} = -\begin{vmatrix} h & f & v \\ g & c & w \\ u & w & d \end{vmatrix} \quad \hat{d} = \Delta_3 \qquad 5.1.4$$

This chapter will be exclusively concerned with examples in which $\Delta_3 \neq 0$. Examples in which $\Delta_3 = 0$ will be discussed in subsequent chapters. We shall learn that if $\Delta_3 \neq 0$ and $\Delta_4 \neq 0$ equation 5.1.1 represents an ellipsoid or a hyperboloid. If, however, $\Delta_3 \neq 0$ and $\Delta_4 = 0$, it represents a cone.

We shall first ask ourselves, is it possible to translate, without rotation, the coordinate axes such that, when referred to the new axes, equation 5.1.1 takes the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0 \quad ? \qquad 5.1.5$$

That is to say, the coefficients of x, y, z are zero. Note that, if in equation 5.1.5 we change the signs of x, y, z, the equation is unaltered. That is to say, the surface represented by equation 5.1.5 has a *centre of symmetry* at the origin of coordinates. It is unaltered by reflection through the origin of coordinates. That is to say, equation 5.1.5 will usually represent an ellipsoid, a hyperboloid (of one or two sheets) or a cone (elliptic or hyperbolic). I say "usually". Note, however, that the equation contains no terms of odd degree. This means that if all the coefficients are positive (or all negative) and if d = 0, equation 5.1.5 can be satisfied only by the single point (0, 0, 0). It does not represent a paraboloid or a pair of planes. If all the coefficents as well as d are positive, there is no point (x, y, z) that satisifes the equation.

If we succeed, by translation, in obtaining an equation of the form of equation 5.1.5, we shall then try to rotate (without translation) the axes of coordinates so that the coefficients of the mixed quadratic terms become zero, and the equation now becomes of the form

$$ax^2 + by^2 + cz^2 + d = 0 \qquad 5.1.6$$

and we are now on familiar ground. This is a central quadric in which the centre of symmetry is at the origin of coordinates, and the coordinate axes coincide with the symmetry axes of the surface.

5.2 *Translation of Coordinate Axes*

In this section I am going to refer the surface to a new set of axes, xyz (that's *Bookman Old Style italic* font on my computer), whose origin is at $x = x_0$, $y = y_0$, $z = z_0$, so that $x = x_0 + x$, $y = y_0 + y$, $z = z_0 + z$.

If we make these substitutions in equation 5.1.1, we find that, referred to the new coordinate system, the equation to the surface becomes

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \qquad 5.2.1$$

where the coefficients $a,\ b,\ c,\ f,\ g,\ h$ are the same as the old coefficients $a,\ b,\ c,\ f,\ g,\ h$, but:

$$u = ax_0 + hy_0 + gz_0 + u \qquad 5.2.2$$

$$v = hx_0 + by_0 + fz_0 + v \qquad 5.2.3$$

$$w = gx_0 + fy_0 + cz_0 + w \qquad 5.2.4$$

$$d = ax_0^2 + by_0^2 + cy_0^2 + 2fy_0z_0 + 2gz_0x_0 + 2hx_0y_0 + 2ux_0 + 2vy_0 + 2wz_0 + d. \qquad 5.2.5$$

Some algebraic manipulation of the equations will show that the new value of the constant term can also be written as

$$d = \frac{\Delta_4}{\Delta_3}, \qquad 5.2.6$$

which may or may not be helpful.

Now our aim is to see if we can find some $x_0,\ y_0,\ z_0$ such that the new coefficients of x, y, z are zero. If we can do this, we shall have found a *central quadric*, because, with no terms in odd powers of x, y *or* z, the equation is invariant upon changing the signs of x, y and z. All (!) we have to do is to solve the equations

$$ax_0 + hy_0 + gz_0 + u = 0, \qquad 5.2.7$$

$$hx_0 + b\bar{y}_0 + fz_0 + v = 0, \qquad 5.2.8$$

and

$$gx_0 + fy_0 + cz_0 + w = 0. \qquad 5.2.9$$

The algebra is straightforward, if slightly tedious, and we arrive at

$$x_0 = \frac{chv + b(gw - cu) + f(fu - gv - hw)}{\text{Bottom}} \qquad 5.2.10$$

$$y_0 = \frac{afw + c(hu - av) + g(gv - hw - fu)}{\text{Bottom}}, \qquad 5.2.11$$

$$z_0 = \frac{bgu + a(fv - bw) + h(hw - fu - gv)}{\text{Bottom}} \qquad 5.2.12$$

where

$$\text{Bottom} = abc + 2fgh - af^2 - bg^2 - ch^2 \quad . \qquad 5.2.13$$

f we refer back to Section 5.1 for the notation used for determinants and cofactors, we see that equations 5.2.10 - 5.2.12 can be written

$$x_0 = \frac{\hat{u}}{\hat{d}} \qquad y_0 = \frac{\hat{v}}{\hat{d}} \qquad z_0 = \frac{\hat{w}}{\hat{d}} \qquad \text{5.2.14a,b,c}$$

provided that $\hat{d}$, which is also equal to Δ_3 , is not zero.

Again, these abbreviated forms may or may not be helpful.

Thus we have shown that equation 5.1.1 represents a *central quadric* whose centre has coordinates given by equation 5.2.10 - 13 (or equations 5.2.14) *provided that* $\Delta_3 \neq 0$. If this determinant is zero, the surface is not a central quadric.

For example, consider the following equation:

$$3x^2 + y^2 - 4z^2 + yz - 5zx + 9xy - 2x - 6y + 5z + 4 = 0. \qquad 5.2.15$$

The first thing to do, without pausing for thought, is to evaluate the two determinants. We find that:

The determinant Δ_3 is not zero, and equations 5.2.10 - 12 show us that the centre is at

$$x_0 = \tfrac{39}{58} = 0.6724 \qquad y_0 = -\tfrac{7}{58} = -0.1207 \qquad z_0 = \tfrac{11}{58} = 0.1897 \qquad 5.2.16$$

Now that we know the coordinates of the centre, we can calculate d from equation 5.2.5 or 5.2.6 (or from both, as a check). We find that $d = \tfrac{483}{116} = 4.1638$. Hence, referred to the centre as origin, the equation to the surface is

$$3x^2 + y^2 - 4z^2 + yz - 5zx + 9xy + \tfrac{483}{116} = 0. \qquad 5.2.17$$

(This is similar to the original equation, except for the absence of the terms in x, y and z, and a different constant.)

On multiplication by 116, to remove the fraction for convenience, it becomes.

$$348x^2 + 116y^2 - 464z^2 + 116yz - 580zx + 1104xy + 483 = 0\,. \qquad 5.2.28$$

Our next task will be to rotate the coordinate axes to make them parallel to the symmetry axis of the surface. By the time we have done this we shall know what type of surface it is (i.e. ellipsoid or hyperboloid or other central quadric).

5.3 *Rotation of Coordinate Axes. Ellipsoids and Hyperboloids*

In the previous Section, we had succeeded, by means of a translation of the coordinate axes, without rotation, in reducing the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad 5.3.1$$

to an equation of simpler form:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0. \qquad 5.3.2$$

[If the new value of the constant, d, is zero, we have a problem. See Section 5.4.] In this form, the origin of coordinates is at the centre of the quadric surface represented by the equation.

We are now going to try to refer the surface to another set of coordinate axes, rotated with respect the current set, so that the equation becomes of the form

$$ax^2 + by^2 + cz^2 + d = 0. \qquad 5.3.3$$

If we can do this, we can determine what type of quadric surface it is, and can find any further properties of interest. When we have found a general way of doing this, we shall look at three particular examples, namely

$$248x^2 + 179y^2 + 342z^2 + 157yz + 129zx + 15xy - 3600 = 0$$
$$3x^2 + y^2 - 4z^2 + yz - 5zx + 9xy + \tfrac{483}{116} = 0$$
$$18x^2 + 25y^2 - 4z^2 - 39yz - 3zx - 13xy + 105 = 0$$

All are central quadrics, as is shown by evaluating the determinant Δ_3 (equation 5.1.2), which is not zero in any of them. These will turn out to be (although we don't know this yet) respectively an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. The second example is the one we left at the end of Section 5.2.

Let $(x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta)$ be a point P on the quadric surface represented by equation 5.3.2. I have drawn this in figure V.1 in which I suppose that the quadric surface is an ellipsoid.

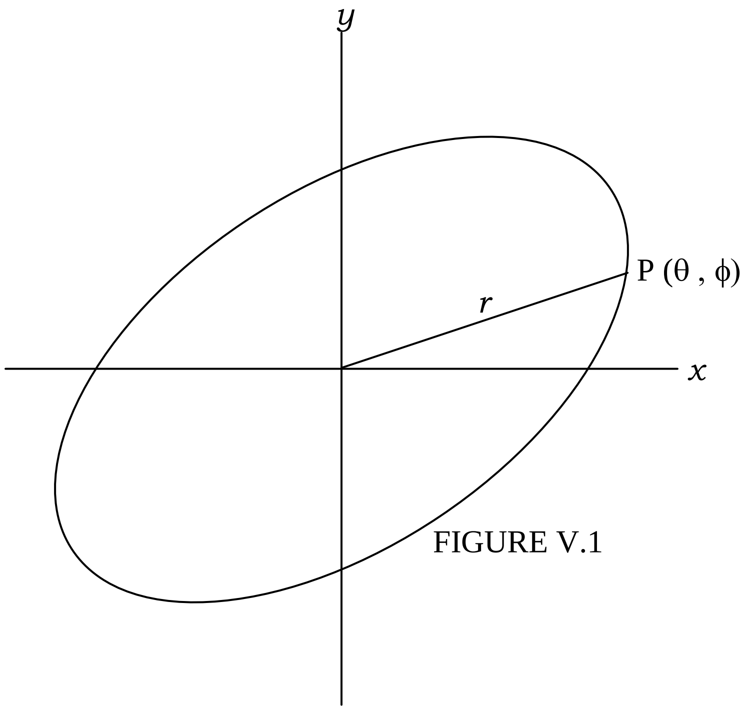


FIGURE V.1

Then

$$r^2(a\sin^2\theta\cos^2\phi + b\sin^2\theta\sin^2\phi + c\cos^2\theta + 2f\sin\theta\cos\theta\sin\phi + 2g\sin\theta\cos\theta\cos\phi + 2h\sin^2\theta\sin\phi\cos\phi) + d = 0 \qquad 5.3.4$$

This enables us to calculate the length of r in a given direction (θ, ϕ).

If we move the point P around on the surface until we find that r is maximum or a minimum or a saddle point, we shall have found an axis of the quadric surface. Thus what we have to do is to find where $\frac{\partial r}{\partial \theta}$ and $\frac{\partial r}{\partial \phi}$ are both zero. This will give us two equations in θ and φ, and hence we can calculate the direction cosines and the lengths of the axes of the quadric surface.

If the quadric surface is a triaxial ellipsoid (as drawn) the reader will, I think, agree that s/he will be able to find three mutually orthogonal directions where $\frac{\partial r}{\partial \theta}$ and $\frac{\partial r}{\partial \phi}$ are both zero. One of these will be a maximum, one will be a minimum, and one will be a saddle point.

I now ask the reader to replace, in his/her mind, the ellipsoid in figure V.1 with a hyperboloid of one sheet. I think s/he will now agree that there will be only two mutually orthogonal such directions. One will be a maximum and one will be a minimum. And if we now replace the surface with a hyperboloid of two sheets, there will be only one such direction, and it will be a minimum.

One look at equation 5.3.4 will tell us that calculating the derivatives $\frac{\partial r}{\partial \theta}$ and $\frac{\partial r}{\partial \phi}$ would be a quite irritable experience. However, if we let

$$\begin{aligned} R = {} & a \sin^2\theta\cos^2\phi + b\sin^2\theta\sin^2\phi + c\cos^2\theta \\ & + 2f\sin\theta\cos\theta\sin\phi + 2g\sin\theta\cos\theta\cos\phi + 2h\sin^2\theta\sin\phi\cos\phi \end{aligned} \qquad 5.3.5$$

then we just have to find the θ and φ where $\frac{\partial R}{\partial \theta}$ and $\frac{\partial R}{\partial \phi}$ are both zero, which, although not quite trivial, would be much less tedious than $\frac{\partial r}{\partial \theta}$ and $\frac{\partial r}{\partial \phi}$. In preparation for the calculation of these derivatives, it may be convenient to re-write equation 5.3.5 in the forms

$$R = (a\cos^2\phi + b\sin^2\phi + h\sin 2\phi - c)\mathbf{\sin^2\theta} + (f\sin\phi + g\cos\phi)\mathbf{\sin 2\theta} + c \qquad 5.3.6$$

or

$$\begin{aligned} R = {} & (a - b)\sin^2\theta\mathbf{\cos^2\phi} + h\sin^2\theta\mathbf{\sin 2\phi} \\ & + f\sin 2\theta\mathbf{\sin\phi} + g\sin 2\theta\mathbf{\cos\phi} + b\sin^2\theta + c\cos^2\theta \end{aligned} \qquad 5.3.7$$

(Many other trigonometric identities are, of course, possible - I found these two to be particularly convenient for calculating the derivatives .)

On setting the partial derivatives to zero, we find that we have to solve the following two equations for the θ and φ of the principal axes of the surface.

$$F(\theta,\phi) = [c - \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b)\cos 2\phi - h\sin 2\phi]\tan 2\theta - 2f\sin\phi - 2g\cos\phi = 0 \qquad 5.3.8$$

$$G(\theta,\phi) = [(b-a)\sin 2\phi + 2h\cos 2\phi]\tan\theta + 2f\cos\phi - 2g\sin\phi = 0 \qquad 5.3.9$$

(Again, there are several ways of writing these. I find the above versions convenient.)

In order not to interrupt the flow of thought, I assume at this point that we can solve these equations for θ and φ, by pressing a button or waving a wand, and I look at three particular examples. **I'll look at the actual mechanics of solving the equations in an Appendix to this Chapter.** I remark only that we are looking for solutions for θ in the range 0 to π and solutions for φ in the range 0 to 2π. Having found θ and φ we can set up a new coordinate system xyz so that the equation to the surface, referred to the new coordinate system, has no terms in x, y or z.

Example 1

$$248x^2 + 179y^2 + 342z^2 + 157yz + 129zx + 15xy - 3600 = 0 \qquad 5.3.10$$

That is, $a = 248 \quad b = 179 \quad c = 342 \quad 2f = 157 \quad 2g = 129 \quad 2h = 15 \quad d = -3600$

For this example, equations 5.3.8 and 5.3.9 become

$$F(\theta, \phi) = (128.5 - 34.5\cos 2\phi - 7.5\sin 2\phi)\tan 2\theta - 157\sin\phi - 129\cos\phi = 0 \qquad 5.3\ 11$$

$$G(\theta, \phi) = [-69\sin 2\phi + 15\cos 2\phi]\tan\theta + 157\cos\phi - 129\sin\phi = 0 \qquad 5.3.12$$

By waving a wand, or pressing a button, or reading the Appendix to this chapter, we find three pairs of oppositely-directed solutions in the range θ from 0 to π, and φ from 0 to 2π. These are the directions of the three axes of the quadric surface. Here they are, in radians and degrees, as follows, together with the corresponding direction cosines calculated from $l = \sin\theta\cos\phi$, $m = \sin\theta\sin\phi$ $n = \cos\theta$:

θ rad θ deg	φ rad φ deg	l	m	n
1.142510 65.460989	4.498167 257.725967	-0.193386	-0.888885	0.415313
1.999083 114.539011	1.356574 77.725967	0.193386	0.888885	-0.415313
1.853778 106.213660	5.935812 340.096987	0.902873	-0.326889	-0.279220
1.287815 73.786340	2.794220 160.096987	-0.902873	0.326889	0.279220
0.524116 30.029662	0.696295 39.894764	0.383956	0.320977	0.865766
2.617476 149.970338	3.837888 219.894764	-0.383956	-0.320977	-0.865766

Each lmn represents the direction ratios of one axis of the quadric surface. For reference, I'll use subscript notation for these direction ratios as follows

$$\begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} = \begin{pmatrix} -0.193386 & -0.888885 & 0.415313 \\ 0.902873 & -0.326889 & -0.279220 \\ 0.383956 & 0.320977 & 0.865766 \end{pmatrix}$$

At this stage it is very necessary to verify that the matrix is orthonormal. (It is.)

We can now use any (or all) of equations 5.3.5, 5.3.6, 5.3.7 to calculate R for each (θ, ϕ), and then equation 5.3.4 to calculate r^2 and r. Real values of the latter are the lengths of the semi-axes of the quadric surface. This is what I obtain:

R	r^2	r
143.954251881	25.007944906	5.000794427
225.337497578	15.976036118	3.997003392
399.708250541	9.006569154	3.001094659

We have three unequal real values of r, the semi axes of the quadric surface, which is, therefore, a **triaxial ellipsoid**.

The original equation (5.3.10) was

$$248x^2 + 179y^2 + 342z^2 + 157yz + 129zx + 15xy - 3600 = 0 \qquad 5.3.10$$

That is, it was of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0. \qquad 5.3.13$$

We are now going to rotate the axes of the coordinate system so that they coincide with the symmetry axes of our quadric surface. We shall do this by substituting

$$\begin{array}{ll} l_1x + l_2y + l_3z & \text{for } x \\ m_1x + m_2y + m_3z & \text{for } y \\ n_1x + n_2y + n_3z & \text{for } z \end{array} \qquad 5.3.14$$

On making these substitutions into equation 5.3.10, we find that the equation to the ellipsoid referred to its own axes as the axes of the coordinate system is:

$$
\begin{aligned}
&(al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gn_1l_1 + 2hl_1m_1)x^2 \\
&+ (al_2^2 + bm_2^2 + cn_2^2 + 2fm_2n_2 + 2gn_2l_2 + 2hl_2m_2)y^2 \\
&+ (al_3^2 + bm_3^2 + cn_3^2 + 2fm_3n_3 + 2gn_3l_3 + 2hl_3m_3)z^2 \\
&+ (2al_2l_3 + 2bm_2m_3 + 2cn_2n_3 + 2f(m_2n_3 + n_2m_3) + 2g(n_2l_3 + l_2n_3) + 2h(l_2m_3 + m_2l_3))yz \\
&+ (2al_3l_1 + 2bm_3m_1 + 2cn_3n_1 + 2f(m_3n_1 + n_3m_1) + 2g(n_3l_1 + l_3n_1) + 2h(l_3m_1 + m_3l_1))zx \\
&+ (2al_1l_2 + 2bm_1m_2 + 2cn_1n_2 + 2f(m_1n_2 + n_1m_2) + 2g(n_1l_2 + l_1n_2) + 2h(l_1m_2 + m_1l_2))xy \\
&+ d = 0.
\end{aligned}
\qquad 5.3.15
$$

On substitution of the numerical values of the direction cosines and the constants $abcfgh$, (trivially easy if you are sitting in front of your computer; impossibly difficult if you are not) we find, to our ineffable relief, that the coefficients of yz, zx and xy are indeed all zero, and the equation to the ellipsoid becomes

$$143.9543x^2 + 225.3375y^2 + 399.7083z^2 = 3600\,. \qquad 5.3.16$$

We have now succeeded in what we set out to do at the beginning of this Section 5.3. I repeat here the opening sentences of this Chapter:

In this and the previous Section (5.2), we succeeded, by means of first a translation of the coordinate axes, and then a rotation, in reducing the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad 5.3.1$$

first to the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0 \qquad 5.3.2$$

and then to the form

$$ax^2 + by^2 + cz^2 + d = 0\,. \qquad 5.3.3$$

We have done this. But have we made any mistakes?

When we made these transformations, all we did was to translate and rotate the coordinate axes sysgtems; we did not change the size of the ellipsoid in any way. Consequently we should expect that $a + b + c = a + b + c = a + b + c$ and we observe that this is indeed the case – all are equal to 769. In technical terms, in rotating the axes of coordinates, we were making an *orthogonal transformation*, and it is a well known theory of matrices that the trace of a matrix (in this case the matrix (Δ_3)) is invariant under an orthogonal transformation.

On division of 5.3.16 by 3600 and rearrangement, this becomes

$$\frac{x^2}{(5.000\,794)^2} + \frac{y^2}{(3.997\,003)^2} + \frac{z^2}{(3.001095)^2} = 1, \qquad 5.3.17$$

agreeing exactly with the figures we had earlier obtained for the semi-axes.

It may be argued that, as soon as we had, earlier, determined the lengths of the semi-axes, we knew that this was going to be the equation of the surface referred to is own axes as coordinate axes, and there was no need to carry out any of the subsequent lengthy computations. This I vigorously deny. The full calculations are necessary in order to check for mistakes. At the time when we had first calculated the lengths of the semi-axes, we had no idea at all whether we had made any mistakes in the calculations. By the time we had finished the full calculation and had obtained the same answer as earlier for the semi-axes, we are certain that we have made none.

Example 2

$$3x^2 + y^2 - 4z^2 + yz - 5zx + 9xy + \tfrac{483}{116} = 0 \ . \qquad 5.3.18$$

That is, $a = 3 \quad b = 1 \quad c = -4 \quad f = 0.5 \quad g = -2.5 \quad h = 4.5 \quad d = \tfrac{483}{116}$

This is the example that we saw in Section 5.2. For this example, the equations to be solved (equations 5.3.8 and 5.3.9) are

$$(6 + 2\cos 2\phi + 4.5\sin 2\phi)\tan 2\theta + \sin\phi - 5\cos\phi = 0 \qquad 5.3.19$$

$$(2\cos 2\phi - 9\sin 2\phi)\tan\theta - \cos\phi - 5\sin\phi = 0 \qquad 5.3.20$$

The solutions I find are:

θ rad	φ rad	l	m	n
θ deg	φ deg			
0.997553	2.113461	-0.433868	0.719447	0.542360
57.155593	121.092406			
2.144039	5.255054	0.433868	-0.719447	-0.542360
122.844407	301.092406			
0.599152	5.587266	0.432806	-0.361538	0.825814

34.328852	320.126768			
2.542441	2.445673	-0.432806	0.361538	-0.825814
145.671148	140.126768			
1.725939	0.643801	0.790214	0.593031	-0.154521
98.889008	36.887096			
1.415654	3.785394	-0.790214	-0.593031	0.154521
81.110992	216.887096			

The matrix of the direction cosines checks for orthonormality, so all is well so far.

We can now use any (or all) of equations 5.3.5, 5.3.6, 5.3.7 to calculate R for each (θ, ϕ), and then equation 5.3.4 to calculate r^2 and r. Real values of the latter are the lengths of the semi-axes of the quadric surface. This is what I obtain:

R	r^2	r
-1.336832	2.624578	1.620055
-5.529137	0.634569	0.796599
6.865969	-0.511016	0.714854 i

There are two unequal real values of r, and one imaginary value. This tells us that we have an **elliptical hyperboloid of one sheet**.

As in Example 1, we now substitute into the original equation

$$\begin{array}{ll} l_1x + l_2y + l_3z & \text{for } x \\ m_1x + m_2y + m_3z & \text{for } y \\ n_1x + n_2y + n_3z & \text{for } z \end{array}$$

On substitution of the numerical values of the direction cosines, we find, again to our immense relief, that the coefficients of yz, zx and xy are indeed all zero, and the equation to the hyperboloid (by this time we are certain that that is what it is) is

$$-1.336832x^2 - 5.529137y^2 + 6.865969z^2 + \tfrac{483}{116} = 0 \qquad 5.3.21$$

Is the trace unchanged? That is, does $a + b + c = a + b + c$? It does indeed.

Equation 5.3.21 becomes on rearrangement

$$\frac{x^2}{(1.764843)^2} + \frac{y^2}{(0.867793)^2} - \frac{z^2}{(0.778742)^2} = 1, \qquad 5.3.22$$

which is the equation to an elliptic hyperboloid of one sheet.

Example 3

$$18x^2 + 25y^2 - 4z^2 - 39yz - 3zx - 13xy + 105 = 0 \quad . \qquad 5.3.23$$

The equations to be solved (equations 5.3.8 and 5.3.9) are

$$(-25.5 + 3.5\cos 2\phi + 6.5\sin 2\phi)\tan 2\theta + 39\sin\phi + 3\cos\phi = 0 \qquad 5.3.24$$
$$(7\sin 2\phi - 13\cos 2\phi)\tan\theta - 39\cos\phi + 3\sin\phi = 0 \qquad 5.3.25$$

The solutions I find are:

θ rad θ deg	φ rad φ deg	l	m	n
0.496381 28.440544	1.288647 73.834025	0.132597	0.457415	0.879312
2.645212 151.559456	4.430239 253.834025	-0.132597	-0.457415	-0.879312
1.814469 103.961426	0.194746 11.158140	0.952114	0.187801	-0.241269
1.327123 76.038574	3.336339 191.158140	-0.952114	-0.187801	0.241269
1.993919 114.243135	1.877734 107.586232	-0.275495	0.869196	-0.410610
1.147674 65.756865	5.019327 287.586232	0.275495	-0.869196	0.410610

The matrix of the direction cosines is orthonormal.
The values of *R* and *r* are as follows:

R	r^2	r
-14.370036	7.306871	2.703122
17.098004	-6.141068	2.478118 i
36.272032	-2.894792	1.701409 i

There is just one real value of *r*. This tells us that we have a hyperboloid of two sheets.

As in Example 1, we now substitute into the original equation

$$\begin{array}{ll} l_1x + l_2y + l_3z & \text{for } x \\ m_1x + m_2y + m_3z & \text{for } y \\ n_1x + n_2y + n_3z & \text{for } z \end{array} \qquad 5.3.26$$

We are again relieved to find that the new coefficients of yz, zx and xy are all zero, and the new equation becomes:

$$-14.370036x^2 + 17.098004y^2 + 36.272031z^2 + 105 = 0. \qquad 5.3.27$$

The trace is preserved (it is 19).

On division by 105 and rearrangement, this becomes

$$\frac{x^2}{(2.702122)^2} - \frac{y^2}{(2.478118)^2} - \frac{z^2}{(1.701409)^2} = 1.$$

This is an **elliptic hyperboloid of two sheets**. The denominators agree exactly with our previous calculations of *r*.

5.4 *Cones. General Homogeneous Quadratic Equation in Three Variables*

Most of us are familiar with a circular cone, in which the cross-section is a circle. However, the cross-section need not be a circle, and it is easy to imagine a cone with an elliptical cross-section - or indeed a cross-section in the form of any closed curve. Slightly more difficult to imagine, or perhaps slightly less familiar, is a cone with a cross-section of a curve that is not necessarily closed, such as a hyperbola. We shall be dealing in this section with cones.

The general quadratic equation in three variables is an equation of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0. \qquad 5.4.1$$

It represents a quadric surface.

We showed in Section 5.2 how to translate the surface to another set of coordinate axes parallel to the original coordinate axes, so that the equation, referred to these new coordinate axes is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d = 0, \qquad 5.4.2$$

provided that $\Delta_3 \neq 0$,

and we showed how the new coefficients were related to the old ones. We pointed out that, since there are no terms in x, y, or z, reversing the signs of x, y and z does not change the equation, and that therefore equation 5.4.2 represents a surface that has a centre of symmetry at the origin of coordinates.

By rotating the axis of coordinates, we were able to reduce the terms in yz, zx and xy to zero, and hence we were able to see easily what sort of quadric surface was represented. We tried three numerical examples, which we showed represented, respectively, a triaxial ellipsoid, an elliptical hyperboloid of one sheet, and an elliptical hyperboloid of two sheet.

In what follows in this section we suppose not only that $\Delta_3 \neq 0$, but also that $d = 0$, so that the equation is of the form of a general homogeneous quadratic equation in three variables:

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \qquad 5.4.3$$

This, like equation 5.4.2, represents a surface with a centre of symmetry at the origin of coordinates. However, it is also seen that $x = 0,\ y = 0,\ z = 0$ satisfies equation 5.4.3. In other words the origin of coordinates is a point on the surface. From this we see that equation 5.4.3 represents a *cone*.

We recall from Section 5.1 that, if equation 5.1.2

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad 5.1.2$$

is to represent a central quadric, the determinant Δ_3 (defined by) equation 5.1.3) must not be zero. And if equation 5.4.2 is to represent a cone, d must be zero. Recall also from equation 5.2.6 that $d = \dfrac{\Delta_4}{\Delta_3}$. Thus the general equation 5.1.2 represent a cone iff $\Delta_4 = 0$ and $\Delta_3 \neq 0$.

Let us try two numerical examples:

Example 4

$$4x^2 + 7y^2 - 6z^2 - 8yz + 3zx + 2xy = 0 \qquad 5.4.4$$

$\Delta_3 = \begin{vmatrix} 4 & 1 & 1.5 \\ 1 & 7 & -4 \\ 1.5 & -4 & -6 \end{vmatrix} \neq 0$, (it is –253.75) so we can safely deal with it in this section.

To try to visualize this, let us see what the cross-sections in the planes $x = 0,\ \ y = 0,\ z = 0$ look like:

$x\ =\ 0$: $\qquad 7y^2 - 8yz\ -\ 6z^2\ =\ 0$ $\qquad$ 5.4.5

$y\ =\ 0$: $\qquad 6z^2 - 3zx\ -\ 4x^2\ =\ 0$ $\qquad$ 5.4.6

$z\ =\ 0$: $\qquad 4x^2\ +\ 2xy\ +\ 7y^2\ =\ 0$ $\qquad$ 5.4.7

We are, I hope, sitting in front of our computers and can instantly solve quadratic equations, to obtain:

$x\ =\ 0$: $\qquad (z + 1.9360y)(z - 0.6026y)\ = 0$ $\qquad$ 5.4.8

$y\ =\ 0$: $\qquad (z + 0.6039x)(z - 1.1039x)\ = 0$ $\qquad$ 5.4.9

$z\ =\ 0$: $\qquad x = 0,\ \ y = 0$ $\qquad$ 5.4 10

I'll let the reader sketch these.

It will also help to visualize the surface if we look at the cross-sections in the planes $x = 1,\ \ y = 1,\ z = 1$. Well start with the plane $z\ =\ 1$.

$z\ = 1$: $\qquad 4x^2\ +\ 2xy\ +\ 7y^2\ + 3x\ -\ 8y\ -\ 6\ =\ 0$ $\qquad$ 5.4.11

We can determine what sort of curve this is by making use of the table on page 50 of orca.phys.uvic.ca/~tatum/celmechs/celm2.pdf. This may take a minute or two, though I admit that I long ago programmed the table into my computer, so that I can tell instantly that this is an ellipse whose centre is at the point $(-0.5\overline{370},\ +0.6\overline{481},\ \ 1)$. So we now know that equation 5.4.4 represents an **elliptical cone**, whose vertex is at the origin of coordinates, and the direction cosines of its axis are $(-0.411,\ +0.496,\ +0.765)$.

Now let us look at the cross-sections in the planes $x\ =\ 1$ and $y =\ 1$:

$x\ = 1$: $\qquad 7y^2\ -\ 8yz\ -\ 6z^2\ + 2y\ +\ 3z\ +\ 4\ =\ 0$ $\qquad$ 5.4.12

$y\ = 1$: $\qquad 4x^2\ +\ 3xz\ -\ 7z^2\ + 2x\ -\ 8z\ +\ 7\ =\ 0$ $\qquad$ 5.4.13

Unsurprisingly, these are both hyperbolas. The centre of 5.4.12 is at $(1,\ 0,\ -\frac{2}{3})$ and the centre of 5.4.13 is at $(0, 1,\ \frac{1}{4})$.

Let us try another example:

Example 5

$$2x^2 - 3y^2 +\ z^2 + 8yz + 6zx + 4xy\ =\ 0. \qquad 5.4.14$$

$$\Delta_3 = \begin{vmatrix} 2 & 2 & 3 \\ 2 & -3 & 4 \\ 3 & 4 & 1 \end{vmatrix} \neq 0,$$ (it is 33) so we can safely deal with it in this section.

To try to visualize this, let us see what the cross-sections in the planes $x = 0, \ y = 0, \ z = 0$ look like:

$x = 0$: $\quad -3y^2 + 8yz + z^2 = 0$ 5.4.15

$y = 0$: $\quad 2x^2 + 4xz + z^2 = 0$ 5.4.16

$z = 0$: $\quad 2x^2 + 4xy - 3y^2 = 0$ 5.4.17

That is to say:

$x = 0$: $\quad (y + 0.1196z)(y - 2.7863z) = 0.$ 5.4.18

$y = 0$: $\quad x = 0, \ z = 0$ 5.4 19

$z = 0$: $\quad (y + 0.3874x)(y - 1.7208x) = 0$ 5.4.20

I'll let the reader sketch these, and by this time it is clear that we are dealing with a cone whose vertex is at the origin of coordinates.

To visualize further what the surface looks like, we'll look at the cross-sections in the planes $x = 1, \ y = 1, \ z = 1.$ These are

$x = 1$: $\quad -3y^2 + 8yz + z^2 + 4y + 6z + 2 = 0$ 5.4.21

$y = 1$: $\quad 2x^2 + 6xz + z^2 + 4x + 8z - 3 = 0$ 5.4.22

$z = 1$: $\quad 2x^2 + 4xy - 3y^2 + 6x - 8y + 1 = 0$ 5.4.23

These are all hyperbolas. Equation 5.4.14 represents a **hyperbolic cone**.

APPENDIX 5A

Earlier in the chapter, we encountered, as equations 5.3.8 and 5.3.9. two simultaneous equations in θ and φ. Here I re-label them as 5A.1 and 5A.2:

$$F(\theta, \phi) = [c - \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b)\cos 2\phi - h\sin 2\phi]\tan 2\theta - 2f\sin\phi - 2g\cos\phi = 0$$ 5A.1

$$G(\theta, \phi) = [(b - a)\sin 2\phi + 2h\cos 2\phi]\tan\theta + 2f\cos\phi - 2g\sin\phi = 0$$ 5A.2

There are probably several approaches to solving such pairs of non-linear equations. I offer two methods here. The equations, and their solutions below, may look fearsome – but programming their solutions is totally straightforward (no conditionals or other complications), and the actual numerical solution is apparently instantaneous.

For a numerical example we shall use the **Example 1** above (equation 5.3.10), in which

$$a = 248 \quad b = 179 \quad c = 342 \quad 2f = 157 \quad 2g = 129 \quad 2h = 15$$

First Method

It is easily possible to eliminate θ between the two equations 5A.1 and 5A.2, thus obtaining a single equation in the single variable φ.

The two equations 5A.2 and 5A.1 respectively can be written

$$\tan\theta = \frac{-2f\cos\phi + 2g\sin\phi}{(b-a)\sin 2\phi + 2h\cos 2\phi} = A(\phi), \qquad 5A.3$$

$$\tan 2\theta = \frac{2f\sin\phi + 2g\cos\phi}{c - \frac{1}{2}(a+b) - \frac{1}{2}(a-b)\cos 2\phi - h\sin 2\phi} = B(\phi) \qquad 5A.4$$

To make them look slightly simpler, introduce

$$p = c - \tfrac{1}{2}(a+b), \qquad q = \tfrac{1}{2}(a-b), \qquad 5A.5a,b$$

so that the equations become

$$\tan\theta = \frac{-f\cos\phi + g\sin\phi}{-q\sin 2\phi + h\cos 2\phi} = A(\phi), \qquad 5A.6$$

$$\tan 2\theta = \frac{2f\sin\phi + 2g\cos\phi}{p - q\cos 2\phi - h\sin 2\phi} = B(\phi). \qquad 5A.7$$

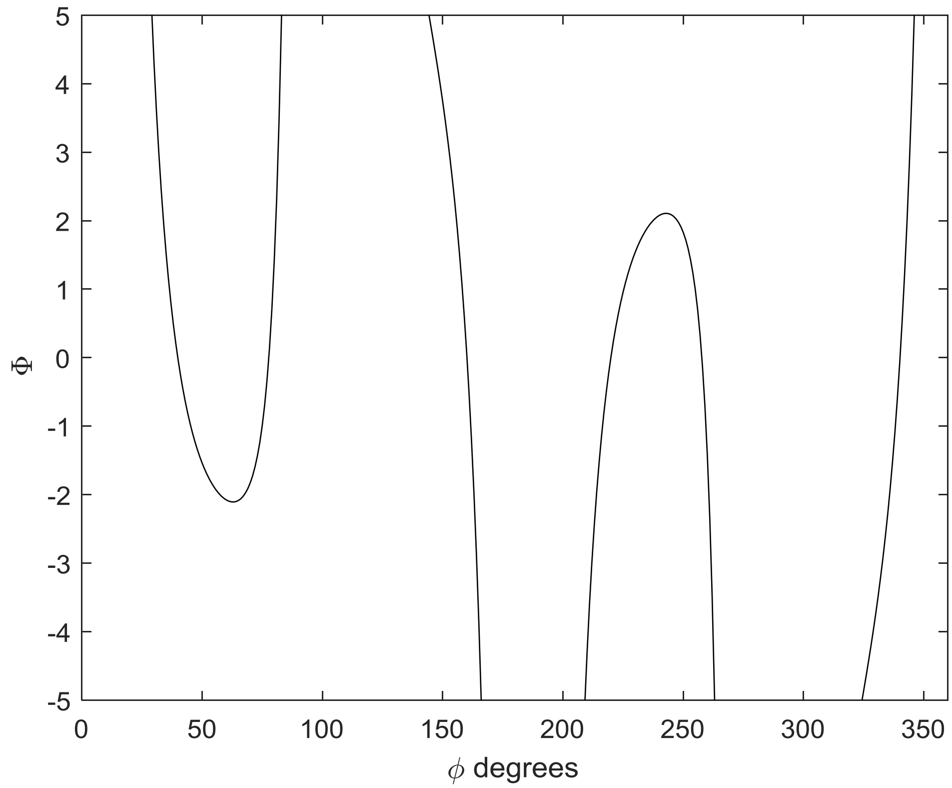
and we then immediately have, as the θ-eliminant between these equations,

$$B = \frac{2A}{1 - A^2}, \qquad 5A.8$$

and so the promised single equation to be solved for the single variable φ is

$$\Phi(\phi) = A^2 B + 2A - B = 0. \qquad 5A.9$$

It is straightforward to find approximate solutions for this equation, merely by plotting a graph of Φ *versus* ϕ. Although Φ is a complicated funtion of ϕ, it is straightofroward to program, and on my computer the calculation for 361 values of ϕ from 0 to 360 degrees was apparently instantaneous.



We see that there are solutions near to

ϕ = 40 78 160 220 258 340 degrees

From equation 5A.3 or 5A.4 (or from both, as a check) we find that these correspond to

θ = 30 114 74 150 65 106 degrees

These give the directions (positive and negative) of three orthogonal axes.

To refine the solutions to equation 5A.9, we use the Newton-Raphson process, which is:

$$\phi = \phi - \frac{\Phi}{\Phi'} = \frac{A(AB + 2) - B}{A(2A'B + AB') + 2A' - B'} \,, \qquad 5A.10$$

in which the prime denotes the derivative with respect to φ. The derivatives are:

$$A'(\phi) = \frac{\sin\phi(s_1 \sin 2\phi + s_2 \cos 2\phi) + \cos\phi[s_3 \sin 2\phi - s_4 \cos 2\phi]}{(-q\sin 2\phi + h\cos 2\phi)^2} \qquad 5A.11$$

$$B'(\phi) = \frac{2[\sin\phi(-pg + s_4 \sin 2\phi - s_3 \cos 2\phi) + \cos\phi(pf - s_2 \sin 2\phi + s_1 \cos 2\phi)]}{(p - q\cos 2\phi - h\sin 2\phi)^2} \qquad 5A.12$$

in which

$$s_1 = -qf + 2gh \qquad 5A.13$$
$$s_2 = hf + 2qg \qquad 5A.14$$
$$s_3 = -qg - 2hf \qquad 5A.15$$
$$s_4 = gh - 2qf \qquad 5A.16$$

Although this may look formidable, it is perfectly straightforward to code, with no conditionals or other complications, and computation time is instantaneous. Thus, here is the procedure:

Enter the constants $a \;\; b \;\; c \;\; f \;\; g \;\; h \;\; d$ and the first guess for φ. (It need not be a particularly good guess, but we already have good guesses from our graph, so we might as well use them.)

Calculate and store one after the other, the following quantities:

$\sin\phi \quad \cos\phi \quad \sin 2\phi \quad \cos 2\phi \quad p \quad q \quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad A \quad B \quad A' \quad B' \quad \Phi \quad \Phi'$

Subtract Φ/Φ' from φ to obtain an improved φ and repeat until convergence is achieved.

The second spherical angle, θ, can then by calculated from equation 5A.5 or 5A.6 (preferably from both, as a check).

With a first guess of 40 degrees, I achieved convergence to 12 significant figures in four iterations, the final solution appearning on the computer screen almost instantaneously.

Second Method

In the first method, we took advantage of the fact that one of the two variables, θ, could easily be eliminated between the two equations, leaving us with a single equation in one variable, which could then be solved by a standard Newton-Raphson procedure. The second method can be used to solve two simultaneous equations in two variables even in cases where one of the variables cannot be eliminated. The method is merely an extension to two variables of the Newton-Raphson procedure.

Before I describe it in detail, I warn in advance: 1. We are going to have to make an initial guess at the solutions for both θ and ϕ, and 2. We are going to need the four partial derivatives $\frac{\partial F}{\partial \theta}$, $\frac{\partial F}{\partial \phi}$, $\frac{\partial G}{\partial \theta}$, $\frac{\partial G}{\partial \phi}$.

1. As for an initial guess, you probably have no idea. However, as you vary θ and ϕ over the surface of the quadric, looking for a spot where *r* is an extremum, there are not likely to be any sudden rapid changes in *r*. That is, *F* and *G* are "well-behaved" functions. That being the case, you don't have to make a particularly intelligent initial guess. Since we are looking for solutions for θ in the range 0 to π, and for ϕ in the range 0 to 2π, you might be tempted to try $\theta = \pi/2$ and $\phi = \pi$. However, I'd advise against this - you don't want to risk asking your computer to divide something by zero, so zero first guesses are best avoided. Note that there must be six solutions for (θ, ϕ) − there are three mutually orthogonal axes, each of which has two ends. I'll describe how I made my own guesses a little further on.

2. I warned that we are going to need the partial derivatives, so let's start by listing them here. (You can probably get them from Wolfram if you don't trust yourself.)

$$\frac{\partial F}{\partial \theta} = [2c - a - b - (a - b)\cos 2\phi - 2h\sin 2\phi]\sec^2 2\theta \qquad \text{5A.17}$$

$$\frac{\partial F}{\partial \phi} = [(a - b)\sin 2\phi - 2h\cos 2\phi]\tan 2\theta - 2f\cos\phi + 2g\sin\phi \qquad \text{5A.18}$$

$$\frac{\partial G}{\partial \theta} = [(b - a)\sin 2\phi + 2h\cos 2\phi]\sec^2\theta \qquad \text{5A.19}$$

$$\frac{\partial G}{\partial \phi} = [2(b - a)\cos 2\phi - 4h\sin 2\phi]\tan\theta - 2f\sin\phi - 2g\cos\phi \qquad \text{5A.20}$$

Here then, is the method. You make a guess. Suppose that your guesses are $\theta + \Delta\theta$ and $\phi + \Delta\phi$. You would like to know your errors $\Delta\theta$ and $\Delta\phi$, so that you can subtract them from your original, erroneous, guesses. If you substitute your original guesses in the original equations, you will find, alas, that neither $F(\theta + \Delta\theta, \phi + \Delta\phi)$ nor $G(\theta + \Delta\theta, \phi + \Delta\phi)$ are zero. However, at least to a first approximation,

$$F(\theta + \Delta\theta, \phi + \Delta\phi) = F(\theta, \phi) + \frac{\partial F}{\partial \theta}\Delta\theta + \frac{\partial F}{\partial \phi}\Delta\phi \qquad 5A.21$$

and

$$G(\theta + \Delta\theta, \phi + \Delta\phi) = G(\theta, \phi) + \frac{\partial G}{\partial \theta}\Delta\theta + \frac{\partial G}{\partial \phi}\Delta\phi\ . \qquad 5A.22$$

(No advanced mathematics here - just common sense.) And of course we are looking for solutions with $F(\theta, \phi) = 0$ and $G(\theta, \phi) = 0$, so equations 5A.21 and 5A.22 are just

$$F(\theta + \Delta\theta, \phi + \Delta\phi) = \frac{\partial F}{\partial \theta}\Delta\theta + \frac{\partial F}{\partial \phi}\Delta\phi \qquad 5A.23$$

and

$$G(\theta + \Delta\theta, \phi + \Delta\phi) = \frac{\partial G}{\partial \theta}\Delta\theta + \frac{\partial G}{\partial \phi}\Delta\phi. \qquad 5A.24$$

So now all you have to do is to evaluate *F*, *G* and their derivatives with your guessed solutions, and solve equations 5A.9 and 5A.10 for $\Delta\theta$ and $\Delta\phi$. These solutions are

$$\Delta\theta = (G_\phi F - F_\phi G)/(F_\theta G_\phi - F_\phi G_\theta) \qquad 5A.25$$

$$\Delta\phi = (F_\theta G - G_\theta F)/(F_\theta G_\phi - F_\phi G_\theta) \qquad 5A.26$$

As with Method 1, it looks awfully complicated, but it is very straghtforward to program, and the actual calculation is instantaneous.

Here is the procedure:

Enter the constants $a \;\; b \;\; c \;\; f \;\; g \;\; h \;\; d$ and the first guesses for θ and ϕ.
Calculate and store cos 2θ tan θ tan 2θ
Calculate *F*, *G* and the four partial derivatives.
Calculate $\Delta\theta$ and $\Delta\phi$

Subtract these from your first guesses, to get a new guess much closer to the truth.
Repeat this until θ and ϕ change by as little as you like.

There are going to be six solutions for (θ, ϕ), corresponding to the two ends of the three orthogonal axes. Making a first guess is a potential problem. However, we have argued that the equations are "well-behaved", so that a particularly good first guess is not essential. For my first guess I tried the completely arbitrary values of θ and ϕ each equal to one radian. It was not a good guess, and it took 13 iterations to converge to a precision of 10^{-12} degree – although the calculation was practically instantaneous. To six decimal places, the solution I got, in degrees, was (θ, ϕ) = (30.029662. 39.894764). For the other end of this axis, the solution has to be (180 − θ, 180 + ϕ).

For a second axis, I didn't really know what to so, but I arbitrarily tried θ = 60 and ϕ = 120 degrees. This was apparently a lucky guess, and the computer came up, in seven iterations, with θ = 73.786340 and ϕ = 160.096987 radians. The other end of this axis is, as before, (180 − θ, 180 + ϕ).

For a third axis, there are two methods. One is to try another guess until you come to the third solution. The other is to convert θ, ϕ to *l, m, m*, and note that the three directions must be orthogonal, so that

$$l_3 = m_1 n_2 - n_1 m_2, \quad m_3 = n_1 l_2 - l_1 n_2, \; n_3 = l_1 m_2 - m_1 l_2 \quad . \qquad 5A.27$$

Both methods should be used as a check against mistakes.