Chapter 14
Functions of a Complex Variable

1. *Function of a Complex Variable*

Let z be a complex variable, which can be written either as $x + iy$ or as $re^{i\theta}$.

A function $f(z)$ (such as, for example z^2 or $\sin z$) will result in a new complex number, which we'll call w, which can be written either as $u + iv$ or as $\rho e^{i\phi}$.

It should be possible, for a given function, to express u and v in terms of x and y, and it should be possible to express ρ and ϕ in terms of r and θ.

Let us do an example. Thus, suppose that $w = f(z) = \sin z$.

That is:

$$w = \sin(x + iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$

Thus $u = \sin x \cosh y$ and $v = \cos x \sinh y$

Can we work in polar coordinates, and express ρ and ϕ in terms of r and θ? Yes, we can, although this particular example is a slightly difficult and messy one. Other simple functions may be easier.

$$w = \sin re^{i\theta} =$$
$$\sin(r\cos\theta + ir\sin\theta) = \sin(r\cos\theta)\cosh(r\sin\theta) + i\cos(r\cos\theta)\sinh(r\sin\theta)$$

We now have u and v in terms of r and θ and we can calculate ρ from $\rho^2 = u^2 + v^2$ and ϕ from $\tan\phi = y/x$. We find, after a little algebra, that

$$\rho^2 = \cosh^2(r\sin\theta) - \cos^2(r\cos\theta)$$

and

$$\tan\phi = \cot(r\cos\theta)\tanh(r\sin\theta)$$

2. *Some Simple Functions*

We shall look at the following simple functions:

$$w = z^2, \quad 1/z, \quad \sqrt{z}, \quad \ln z, \quad \sin z, \quad \cos z, \quad e^z.$$

Exercise: For each of the above seven functions, express u and v in terms of x and y¸ and express ρ and ϕ in terms of r and θ. The function cos z will be similar to sin z. The others will be easier.

I show below the answers that I get. Please let me know if you think there are any mistakes. tatumjb352 at gmail dot com

$w = z^2 \quad u = x^2 - y^2 \quad v = 2xy \quad \rho = r^2 \quad \phi = 2\theta$

$w = \frac{1}{z} \quad u = \frac{x}{x^2 + y^2} \quad v = -\frac{y}{x^2 + y^2} \quad \rho = \frac{1}{r} \quad \phi = -\theta$

$w = \sqrt{z} \quad u^2 = \frac{\sqrt{x^2 + y^2} + x}{2} \quad v^2 = \frac{\sqrt{x^2 + y^2} - x}{2} \quad \rho^2 = r \quad \phi = \frac{1}{2}\theta$

$w = \ln z \quad u = \frac{1}{2}\ln(x^2 + y^2) \quad \tan v = \frac{y}{x}$

$\rho^2 = (\ln r)^2 + \theta^2 \quad \tan\phi = \frac{\theta}{\ln r}$

$w = \sin z \quad u = \sin x \cosh y \quad v = \cos x \sinh y$

$\rho^2 = \cosh^2(r\sin\theta) - \cos^2(r\cos\theta) \quad \tan\phi = \cot(r\cos\theta)\tanh(r\sin\theta)$

$w = \cos z \quad u = \cos x \cosh y \quad v = -\sin x \sinh y$

$\rho^2 = \cosh^2(r\sin\theta) + \cos^2(r\cos\theta) \quad \tan\phi = -\tan(r\cos\theta)\tanh(r\sin\theta)$

$w = e^z \quad u = e^x\cos y \quad v = e^x \sin y \quad \rho = e^{2r\cos\theta} \quad \phi = r\sin\theta$

2. *Mapping*

Given a function *f(z)*, to any point $z = x + iy$ in the z-plane there will be a corresponding point $w = u + iv$ in the w-plane. And if the point z is constrained to move in the z-plane along some curve $F(x, y) = 0$, the point w will move along some curve $G(u, v) = 0$ in the w-plane. We can say that the function $f(z)$ maps the curve $F(x, y) = 0$ in the z-plane on to the curve $G(u, v) = 0$ in the w-plane. This is the sense in which we use the word "mapping" in the title to this section.

In what follows I am going to try to map first a circle and then a square in the *z*-plane using several different *f* (*z*). Each of these functions maps the circle and the square on to some most interesting and unexpected loci in the *w*-plane. It is great fun. I hope (but don't guarantee) that I have done them all correctly. I hope viewers will do them themselves – and let me know if I've got any wrong. tatumjb352 at gmail dot com

2a. *Mapping a circle*

Let us suppose that we have a point $z = x + iy = re^{i\theta}$ in the *z*-plane, and that the point lies upon the circle $x^2 + y^2 = 1$. That is to say, in polar coordinates, it lies upon the circle *r* = 1. How do each of the functions

$$w = z^2, \quad 1/z, \quad \sqrt{z}, \quad \ln z, \quad \sin z, \quad \cos z, \quad e^z$$

map the circle on to the *w*-plane?

They are fairly easy to calculate. Vary θ from 0° to 360° in steps of one degree. Since the circle is of unit radius, *x* and *y* are just cos θ and sin θ respectively. Then calculate *u* and *v* from the formulas given in the previous section, and plot a graph of *v* versus *u*.

$w = z^2$

$z = re^{i\theta}$, therefore $w = r^2 e^{2i\theta}$. Since *z* lies on the circle *r* = 1, *w* also lies on a unit circle, but when the argument of *z* is θ, the argument of *w* is 2θ. As *z* moves around its circle in its plane, *w* moves around a similar circle in its plane, but at twice the angular speed. By the time that *z* has moved completely round its circle, *w* has moved around its circle twice.

$w = \sqrt{z}$

As *z* moves around its circle in its plane, *w* moves around a similar circle in its plane, but at half the angular speed. By the time that *z* has moved completely round its circle, *w* has moved only through a semicircle.

$w = 1/z$

As *z* moves around its circle in a counterclockwise direction, *w* moves in a similar circle at the same angular speed, but in the clockwise direction.

$w = \ln z$

$z = re^{i\theta}$, or, while it is on its unit circle, $z = e^{i\theta}$, so $w = i\theta$.
While *z* moves around its circle from θ = 0 to 2π, *w* moves in a straight line up the imaginary axis from *v* = 0 to 6.28*i*.

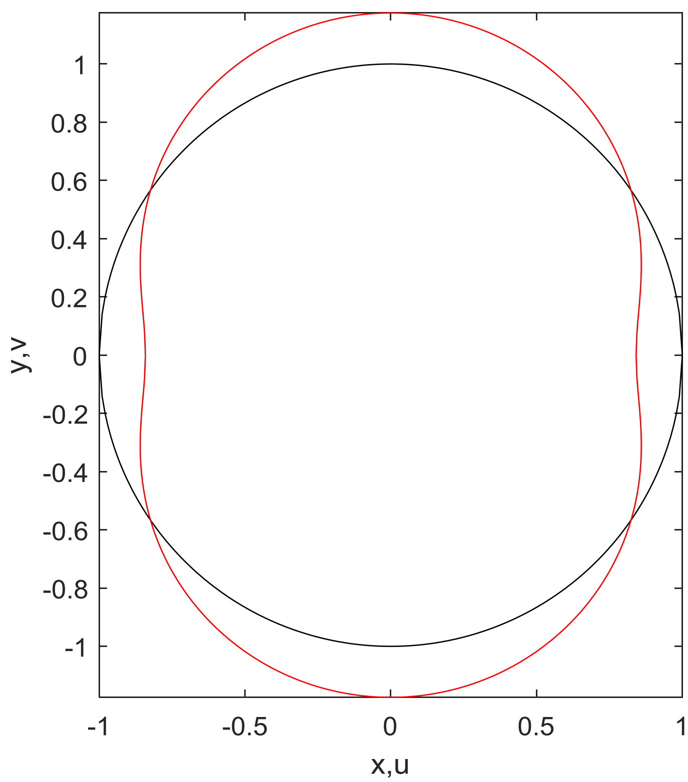
$\underline{w \;=\; \sin z}$

Recall (Chapter 13, for example) that

$$w \;=\; \sin z \qquad u \;=\; \sin x \cosh y \qquad v \;=\; \cos x \,\sinh y$$

$$\rho^2 \;=\; \cosh^2(r \sin \theta) - \cos^2(r \cos \theta) \qquad \tan \phi \;=\; \cot(r \cos \theta) \tanh(r \sin \theta)$$

Imagine z to move counterclockwise around its unit circle ($r = 1$) one degree at a time starting at $\theta = 0$, we (or our computer) can calculate in turn x (= cos θ), y (= sin θ), u, v, ρ and ϕ, and so it is straightforward to plot the progress of w in its plane. I show below z (in black, in its x, y plane) and w (in red, in its u , v plane).



On the real axis, $u \;=\; \pm \sin 1 \;=\; \pm 0.8415$

On the imaginary axis, $v \;=\; \pm\, i \sinh 1 \;=\; \pm 1.1752 i$

$\underline{w \;=\; \cos z}$

It turns out that cos z looks surprisingly different from sin z

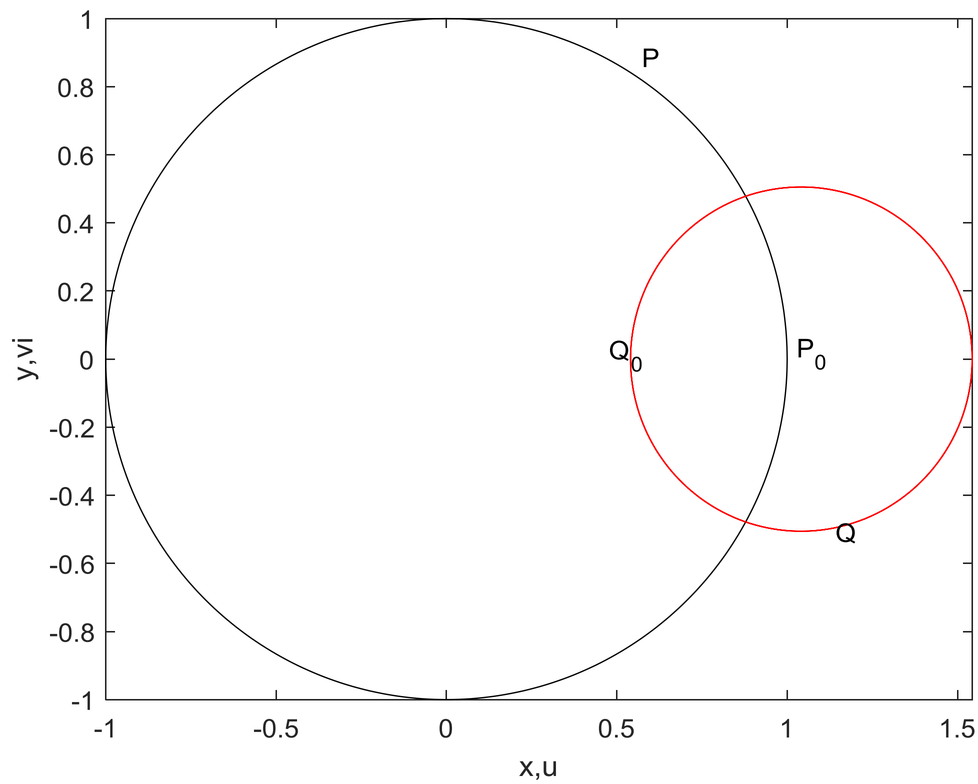
Recall that

$$w = \cos z \qquad u = \cos x \cosh y \qquad v = -\sin x \ \sinh y$$

$$\rho^2 = \cosh^2(r\sin\theta) + \cos^2(r\cos\theta) \qquad \tan\phi = -\tan(r\cos\theta)\tanh(r\sin\theta)$$

and go through the same procedure as with sin *z*.

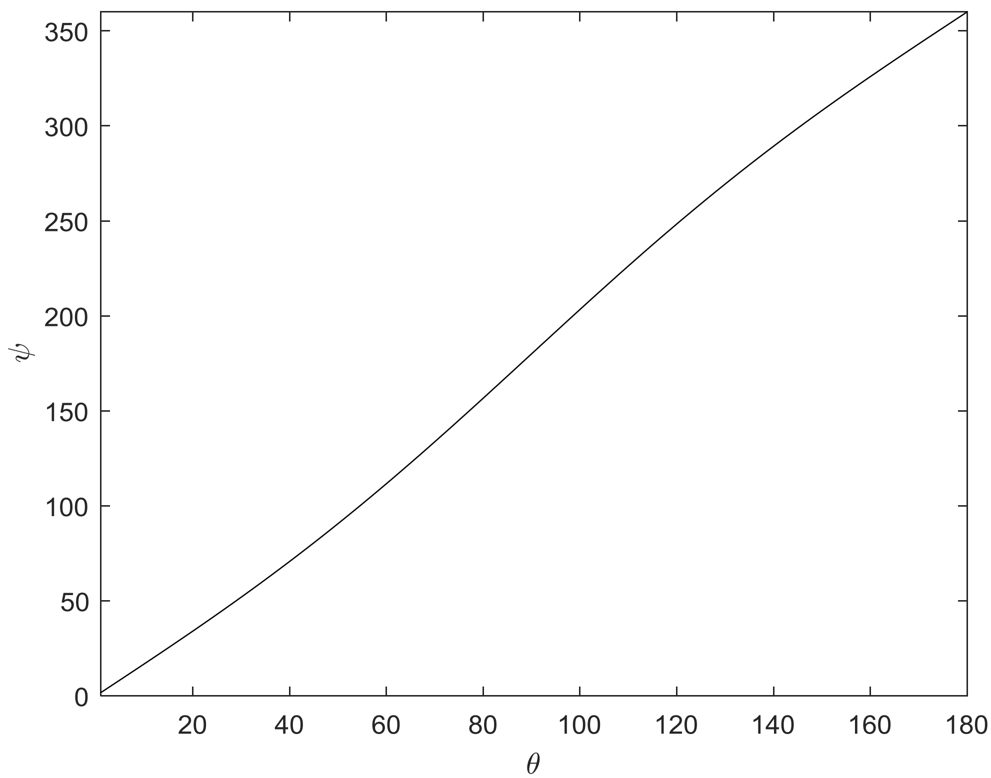
. I show below what I get for *z* (in black, in its *x* , *y* plane) and *w* (in red, in its *u* , *v* plane).



The red locus of *w* in its plane looks like a circle, and in fact is very close to a circle, although not exactly so. As *z* moves counterclockwise from P_0 through an angle θ to P, *w* moves also counterclockwise from Q_0 through an angle ψ to Q. If *z* moves at a uniform angular speed, the angular speed of *w* is not quite uniform, but on average is twice the angular speed of *z*, so, as *z* goes round its black circel once, *w* goes round its red circle twice. Shown below are a table and a graph of ψ versus θ.

θ°	ψ°
0	0
15	25
30	51

45	80
60	111
75	145
90	180
105	214
120	248
135	279
150	308
165	334
180	360



On the real axis, *u* has the values cos 1 = 0.5403 and cosh 1 = 1.5431, so that the horizontal diameter of the red quasicircle is cosh1 – cos1 = 1.002778, and the mid-point of the quasicircle is at *u* = 0.5(cosh1 + cos 1) = 1.0417

The values of *v* on the imaginary axis (and hence the vertical diameter of the quasicircle) are slightly less easy to compute. We start from $v = -\sin x \sinh y = -\sin x \ \sinh(1 - x^2)^{1/2}$.

Some differential calculus shows that the greatest and least values of v occur where

$$y \tanh y = x \tan x$$

in which $$y = (1 - x^2)^{1/2}.$$

The solution to these simultaneous equations is

$x = 0.647421 \quad y = \pm 0.762133$

corresponding to $v = \pm 0.505476$.

The viewer might ask if the quasicircle is an ellipse, and, if it is, what is its eccentricity. The present answer to the first question is that, at the moment, I don't know. However, if it *is* an ellipse, its eccentricity is 0 0186.

The mappings of the unit circle by sin x and by cos x seem surprisingly different. Perhaps some enterprising viewer might try mappings of the unit circle by sin ($z + \alpha$), and see how the peanut morphs into the quasicircle as α goes from 0 to $\pi/2$. Maybe even make a movie of it, and share it with us on the Web.

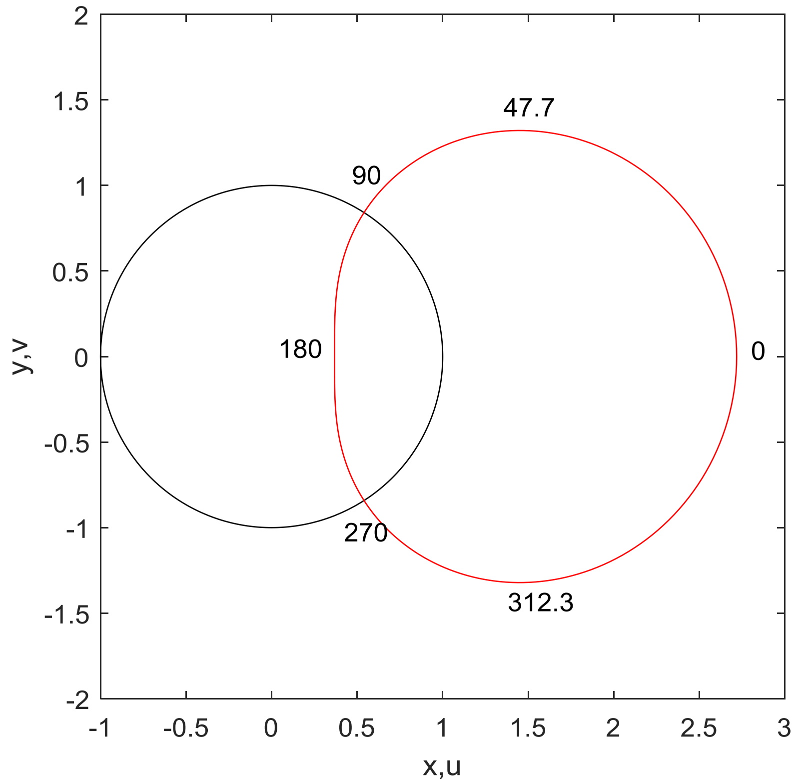
$\underline{w = e^z}$

Recall that

$$w = e^z \qquad u = e^x \cos y \qquad v = e^x \sin y \qquad \rho = e^{2r\cos\theta} \qquad \phi = r \sin\theta$$

and go through the same procedure as with sin z.

. I show below what I get for z (in black, in its plane) and w (in red, in its plane).



If we start at $x = 1$, $y = 0$ on the black circle, and move counterclockwise by θ around the circle, then in the *w*-plane, we start at the right hand side of the red "bean" and move counterclockwise. I show the value of θ in degrees at several points around the bean.

For $v = 0$ on the bean, u has the values $e^{-1} = 0.3679$ and $e = 2.7183$.

The maximum and minimum values of v can be found by putting the derivative of v to zero. This results in $x = y \tan y$. Combined with $x^2 + y^2 = 1$ this results in

$x = 0.073612$ $y = 0.739085$., which corresponds to $u = 1.449574$ $v = \pm 1.321161$.

2a. *Mapping a square*

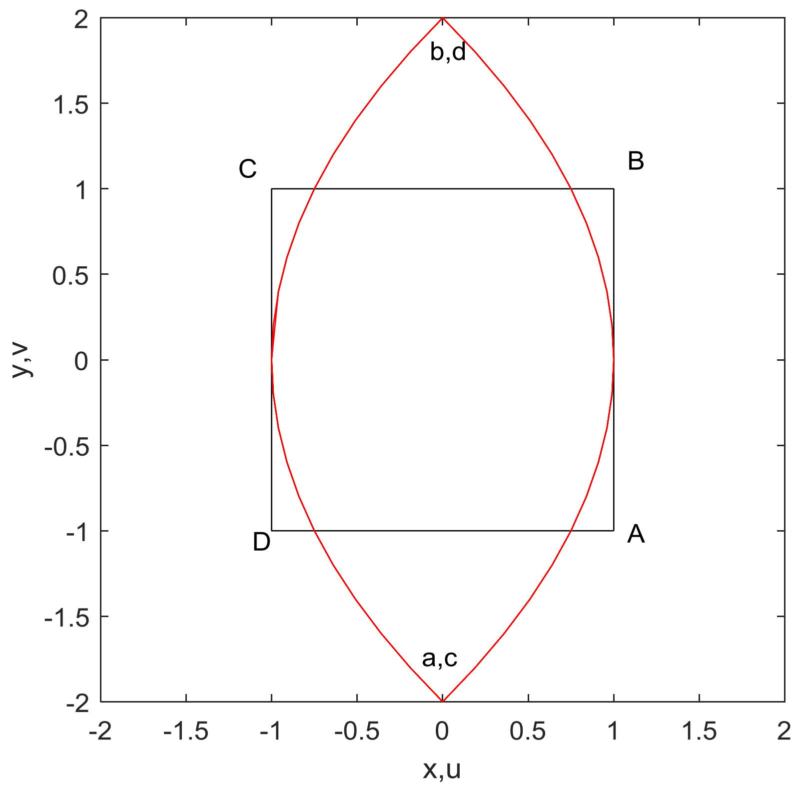
Now let us use the same seven functions

$$w = z^2\,, \quad 1/z\,, \quad \sqrt{z}\,, \quad \ln z\,, \quad \sin z\,, \quad \cos z\,, \quad e^z$$

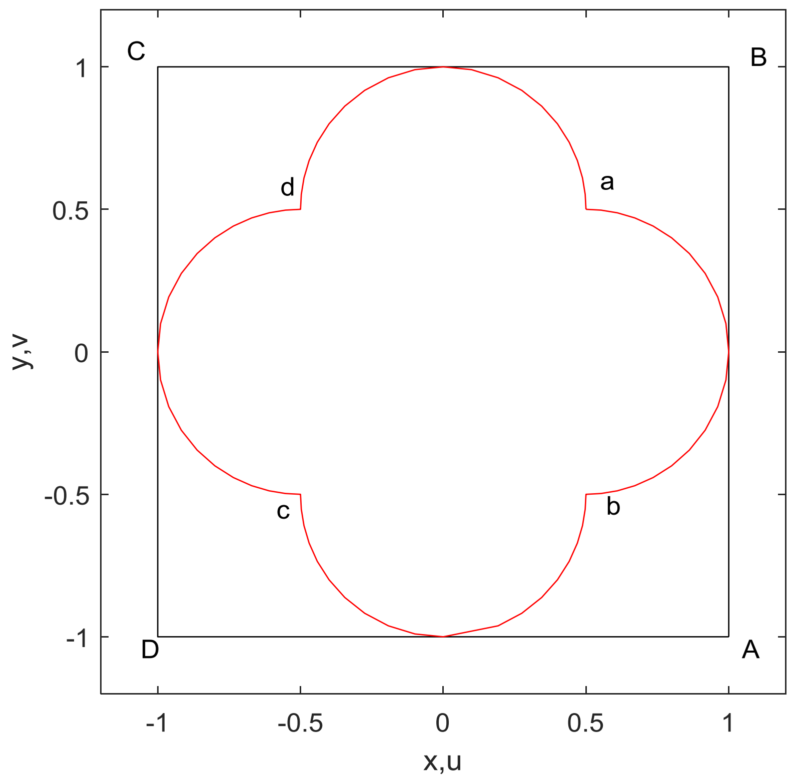
to map a square in the *z*-plane on to the *w*-plane. We choose the square to be bounded by the lines $x = \pm 1$, $y = \pm 1$. It is easy to generate numbers (x,y) that delineate the square. Then, for each (x,y) we calculate u and v, and hence draw the locus of w in the *w*-plane. Here are the results that I get.

$\underline{w = z^2}$

The square in the *z*-plane maps on to a lens-shaped figure in the *w*-plane. As we go round the square once in the *z*-plane, we go round the lens twice in the *w*-plane. In the figure, I have labelled the four corners of the square A, B, C, D. The small letters a, b, c, d show the corresponding points in the lens.

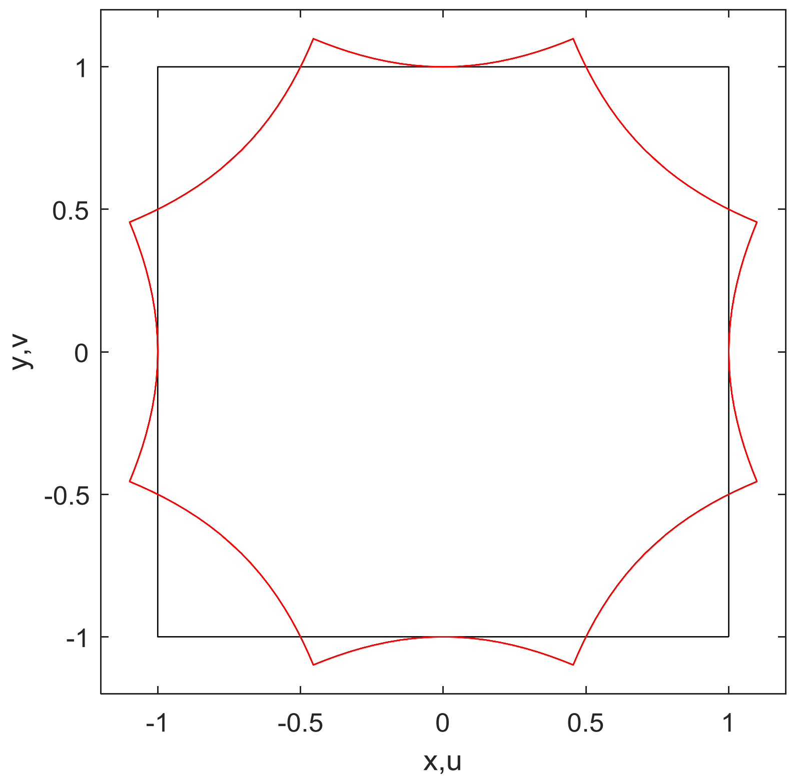


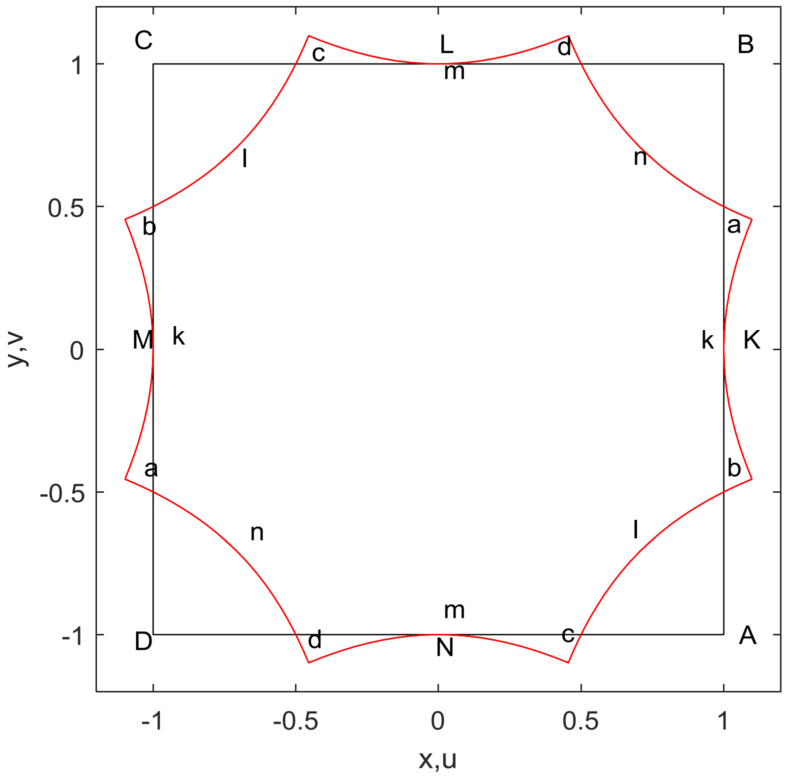
$\underline{w = 1/z}$



$\underline{w = z^{1/2}}$

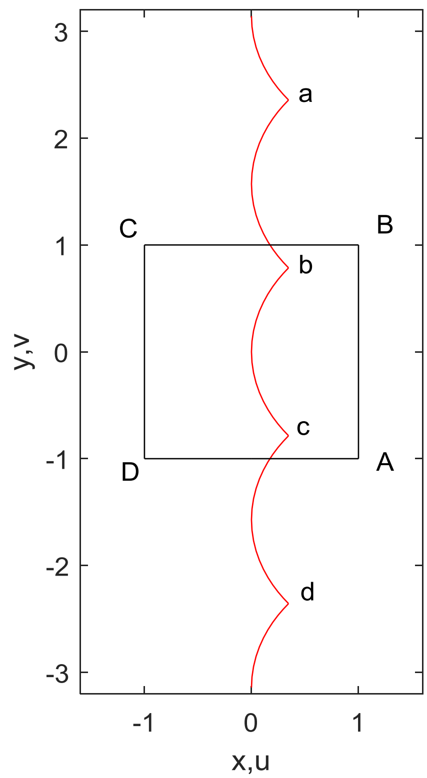
In preparing the figures below, I have taken account of the positive and negative values of the square roots. The first figure is uncluttered with letters. In the second figure I have labelled, outside the black square, in capital letters, key points on the squuare. I have labelled, inside the red star, in small letters, correspondng points on the star. As *z* goes counterclockwise once around the square, *w* goes twice, clockwise, around the star.


1
0.5
0
-0.5
-1
y,v
-1
-0.5
0
0.5
1
x,u



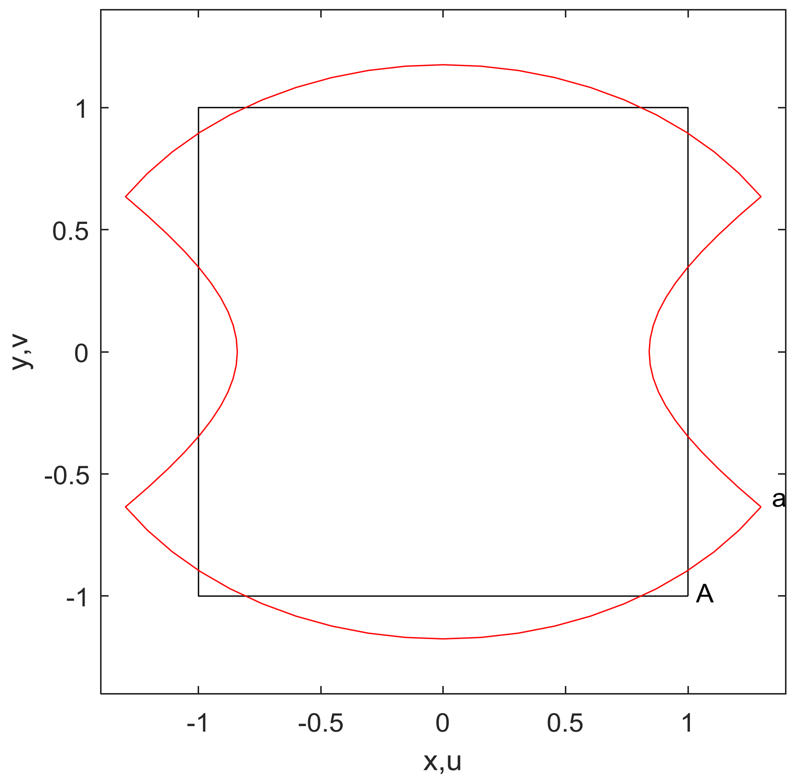
$\underline{w = \ln z}$

The mapping of the square (black) in the z-plane on to the w-plane (red) is shown below. The real part of w, namely u, is restricted between 0 and $\frac{1}{2}\ln 2 = 0.3466$. The cusps are at ±45° and ±135°.



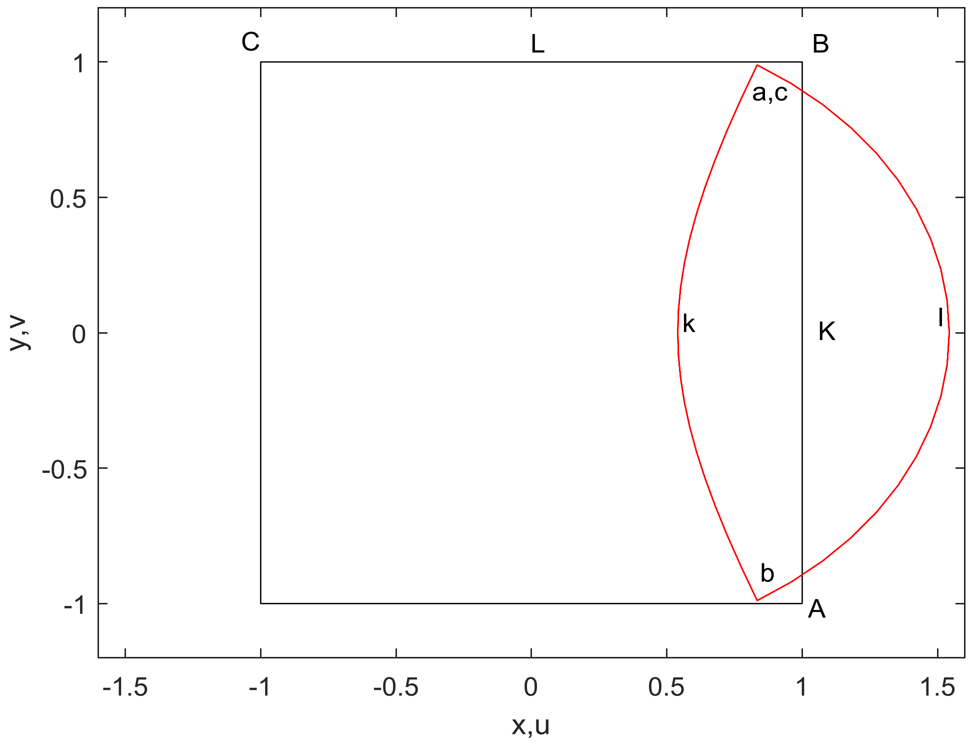
$w = \sin z$

If z starts at A in the figure below, and then goes counterclockwise around the square, w starts at a in the figure below, and goes counterclockwise round the red path.



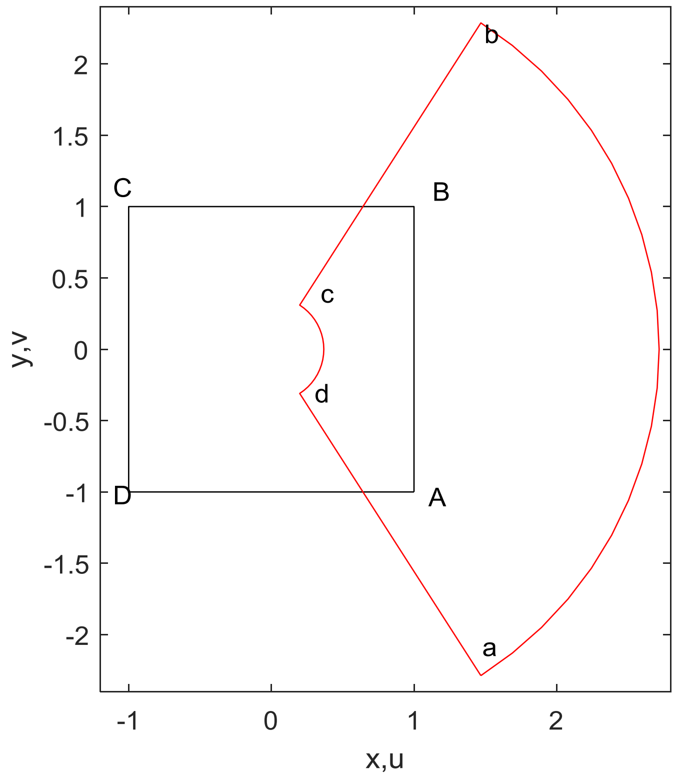
<u>*w* = cos *z*</u>

If *z* starts at A in the figure below and proceeds counterclockwise around the black square, *w* starts at a on the red path, and goes twice counterclockwise round the red path as *z* goes round he square once.



$\underline{w = e^z}$

If z starts at A in the figure below and proceeds couterclockwise around the black square, w starts at a on the red path, and goes counterclockwise round the red path.



As in the case of mapping the circle, the paths in the *w*-plane are remarkably different for the sine and cosine functions, and it might be interesting for an enterprising viewer to try mapping through the function $\sin(z + \alpha)$ as α goes from 0 to $\pi/2$