Chapter 5
Second Order Differential Equations 4
Equations of the form $(aD^2 + bD + c)y = f(x)$

Equations of the form $(aD^2 + bD + c)y = f(x)$ or $ay'' + by' + cy = f(x)$ (including the case where $f(x) = 0$, dealt with in the last chapter) are called linear equations with constant coefficients. Linear, because there are no terms in y^2 or y'^2 or y''^2, and constant coefficients because the coefficients *a*, *b* and *c* are just numbers, and not functions of *x* or of *y*.

We can write the solution symbolically as

$$y = \left(\frac{1}{aD^2 + bD + c}\right) f(x) + \ldots$$

plus not a constant, but plus another function of *x*, which we'll shortly see is not quite arbitrary. So, we'll write the general solution as

$$y = y_1 + y_2,$$

where $y_1 = \left(\frac{1}{aD^2 + bD + c}\right) f(x)$ is called the *particular integral*, and y_2 is called the *complementary function.*

Thus we could write the original equation as

$$(aD^2 + bD + c)y_1 + (aD^2 + bD + c)y_2 = f(x) \text{ .}$$

Since y_1 already satisfies $(aD^2 + bD + c)y_1 = f(x)$ **,** it follows that the complementary function satisfies $(aD^2 + bD + c)y_2 = 0$ **.**

In Summary:

The solution to an equation of the form $(aD^2 + bD + c)y = f(x)$ can be written as the sum of two parts: $y = y_1 + y_2$, where the *complementary function* y_2 is the solution of $(aD^2 + bD + c)y_2 = 0$, which we already know how to do (from Second Order Differential Equations 2 (SODE2)), and the *particular integral*

$$y_1 = \left(\frac{1}{aD^2 + bD + c}\right) f(x),$$

which we are going to be able to do, using the properties of the operator D described in SODE3.
I'll give one example each of the cases $b^2 > 4ac$, $b^2 < 4ac$, $b^2 = 4ac$.

$b^2 > 4ac$

$$y'' - 5y' + 6y = x^3 e^x$$

First, we'll find the *complementary function* y_2, which is the solution to the equation

$$y_2'' - 5y_2' + 6y_2 = 0.$$

The auxiliary equation is

$k^2 - 5k + 6$, with solutions $k_1 = 2$, $k_2 = 3$.

The solution is given explicitly in SODE2. It is

$$\underline{y_2 = Ae^{2x} + Be^{3x}}$$

Now for the *particular integral* y_2 , which is the answer to

$$y_1 = \left(\frac{1}{D^2 - 5D + 6}\right)x^3e^x = \left(\frac{1}{(D-2)(D-3)}\right)x^3e^x = \left(\frac{1}{D-3} - \frac{1}{D-2}\right)x^3e^x$$

We now refer to SODE3 to remind ourselves what is meant by $\dfrac{1}{D-a}$. We see there that $(D-a)^{-1}f = e^{ax}D^{-1}(fe^{-ax})$.

That is to say

$$\left(\frac{1}{D-3}\right)x^3e^x = e^{3x}D^{-1}(x^3e^xe^{-3x}) = e^{3x}\int x^3e^{-2x}dx$$
$$= e^{3x}\left[-\tfrac{1}{8}e^{-2x}(4x^3+6x^2+6x+3)\right] = -\tfrac{1}{8}e^x(4x^3+6x^2+6x+3)$$

(Remember from SODE3 that we can take the arbitrary constant to be zero. We already have the necessary and sufficient two arbitrary constants in the complementary function.)

Similarly

$$\left(\frac{1}{D-2}\right)x^3e^x = e^{2x}D^{-1}(x^3e^xe^{-2x}) = e^{2x}\int x^3e^{-x}dx$$

$$= e^{2x}\left[-e^{-x}(x^3+3x^2+6x+6)\right] = -e^x(x^3+3x^2+6x+6)$$

Thus

$$y_1 = e^x(x^3 + 3x^2 + 6x + 6 - \tfrac{1}{2} - \tfrac{3}{4}x^2 - \tfrac{3}{4}x - \tfrac{3}{8})$$

$$= \tfrac{1}{8}e^x(4x^3 + 18x^2 + 42x + 45)$$

The general solution is then

$$\underline{\underline{y = \tfrac{1}{8}(4x^3 + 18x^2 + 42x + 45)e^x + Ae^{2x} + Be^{3x}}}$$

Equations of the form $(x^2D^2 + bxD + c)y = f(x)$ **.**

That is, $x^2\dfrac{d^2y}{dx^2} + bx\dfrac{dy}{dx} + cy = f(x)$

An equation such as $(3x^2D^2 - 5xD + 6)y = \sin x$ is immediately made into the above form by dividing throughout by 3. Therefore, in describing this type of differential equation, I shall always assume that the coefficient of x^2D^2y is 1. The method for solving this type of equation can also be extended to equations of higher order, such as

$$(x^3D^3 + bx^2D^2 + cxD + g)y = f(x) \text{ .}$$

These equations are easily solved by means of the Brilliant Substitution

$$x = e^u,$$

from which it is easy to show (do it!) that

$$x\frac{dy}{dx} = \frac{dy}{du}$$

and slightly less easy to show (persist!) that

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}.$$

The original differential equation then immediately transforms to

$$\frac{d^2y}{du^2} + (b-1)\frac{dy}{du} + cy = F(u),$$

where $F(u) = f(e^u)$. We are now on familiar ground.

Before doing an example, we might wonder how this can be extended to equations of higher order. For example, what is $x^3\dfrac{d^3y}{dx^3}$? And $x^4\dfrac{d^4y}{dx^4}$? And so on... You may skip this and go straight to *Example 1* if you wish. Let's use the notation:

$$D \equiv \frac{d}{dx} \quad \text{and} \quad \mathcal{D} \equiv \frac{d}{du}$$

Thus the original equation is

$$(x^2D^2 + bxD + c)y = f(x)$$

and the transformed equation is

$$\left(\mathcal{D}^2 + (b-1)\mathcal{D} + c\right)y = F(u).$$

We have established that

$$xD \equiv \mathcal{D}$$

and that

$$x^2D^2 \equiv \mathcal{D}(\mathcal{D} - 1).$$

Further differentiation will establish that

$$x^3D^3 \equiv \mathcal{D}(\mathcal{D} - 1)(\mathcal{D} - 2)$$

and yet again that

$$x^4D^4 \equiv \mathcal{D}(\mathcal{D} - 1)(\mathcal{D} - 2)(\mathcal{D} - 3)$$

and so on.

Example 1

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = x^2.$$

That is:

$$(x^2D^2 + xD - 4)y = x^2.$$

The transformed equation becomes simply

$$(\mathcal{D}^2 - 4)y = e^{2u}.$$

The *particular integral* y_2 is the solution of $(\mathcal{D}^2 - 4)y = 0$, or $\frac{d^2y}{du^2} = 4y$, which is

$$y_2 = Ae^{2u} + Be^{-2u} = \underline{Ax^2 + \frac{B}{x^2}}.$$

The *complementary function* y_1 is given by

$$
\begin{aligned}
y_1 &= \frac{e^{2u}}{(\mathcal{D}^2 - 4)} = \frac{1}{4}\left[\frac{e^{2u}}{\mathcal{D}-2} - \frac{e^{2u}}{\mathcal{D}+2}\right] \\
&= \frac{1}{4}\left[(\mathcal{D}-2)^{-1}e^{2u} - (\mathcal{D}+2)^{-1}e^{2u}\right] \\
&= \frac{1}{4}\left[e^{2u}\mathcal{D}^{-1}(e^{2u}.e^{-2u}) - e^{-2u}\mathcal{D}^{-1}(e^{2u}.e^{2u})\right] \\
&= \frac{1}{4}\left[e^{2u}\mathcal{D}^{-1}(1) - e^{-2u}\mathcal{D}^{-1}(e^{4u})\right] \\
&= \frac{1}{4}\left[e^{2u}u - \tfrac{1}{4}e^{-2u}e^{4u}\right] \\
&= \underline{\frac{1}{4}\left[x^2 \ln x - \tfrac{1}{4}x^2\right]}
\end{aligned}
$$

When we add the complementary function y_1 to the particular integral y_2 to obtain the general solution y, we can absorb the term $-\frac{1}{16}x^2$ of y_1 into the term Ax^2 of y_2, so that the general solution is

$$\underline{\underline{y = Ax^2 + \frac{B}{x^2} + \tfrac{1}{4}x^2 \ln x.}}$$

Example 2

$$x^4 \frac{d^2 y}{dx^2} + 2x^3 \frac{dy}{dx} + 4 = 0.$$

This can be written:

$$(x^2 D^2 + 2xD)y = -\frac{4}{x^2}.$$

This transforms to $$\mathcal{D}(\mathcal{D}+1)y = -4e^{-2u}.$$

The *particular integral* is found by solution of

$$\frac{d^2 y}{du^2} + \frac{dy}{du} = 0.$$

The solution of this can be found by elementary means (e.g. Let $z = \dfrac{dy}{du}$)

$$y = C + Be^{-u} = \underline{C + \frac{B}{x}}.$$

(Don't fret if you find $y = C - \frac{B}{x}$. After all, B is just an *arbitrary* constant.)

The *complementary function* is found from

$$y = -4\frac{1}{\mathcal{D}(\mathcal{D}+1)}e^{-2u} = 4\left[\frac{1}{\mathcal{D}+1} - \frac{1}{\mathcal{D}}\right]e^{-2u}$$

It is left to the reader to find that $y = \underline{-2e^{-2u}}$, so that the general solution is

$$\underline{\underline{y = C + \frac{B}{x} - \frac{2}{x^2}.}}$$