Chapter 3
Second Order Differential Equations
Preamble.
The Operator *D*

Preamble

Second order differential equations, or differential equations of order two, include second derivatives $\frac{d^2y}{dx^2}$. Examples are

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x).$$

These examples are *linear equations with constant coefficients*. They are linear because there are no terms such as $\left(\frac{dy}{dx}\right)^2$. The coefficients are constants rather than functions of *x* or *y*. To begin with we shall deal only with linear equations with constant coefficients. Within this constraint, however, we may find that we need not limit ourselves to equations of order two. We may find also that we are able to solve equations of third or higher order, such as

$$a\frac{d^3y}{dx^3} + b\frac{d^2y}{dx^2} + c\frac{dy}{dx} + ey = f(x).$$

We shall find that the *general solution* (GS) of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

can be written as the sum of two functions, known as the particular integral (PI) and the complementary function (CF):

$$\text{GS} = \text{CF} + \text{PI},$$

in which the CF is the solution of

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Thus, if we can solve $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, we are already halfway to solving $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$.

Some people like, for brevity, to use the "prime" notation y' to denote $\frac{dy}{dx}$. This is usually pronounced "*y*-primed" in North America, and "*y*-dashed" in Britain. The equation then looks like

$$ay'' + by' + cy = f(x).$$

In practical applications, very often the independent variable is the time *t*, so we are solving equations such as

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t).$$

In that case you may like to use the "dot" notation, in which $\dot{y}$ means $\frac{dy}{dt}$, and the equation looks like

$$a\ddot{y} + b\dot{y} + cy = f(t).$$

The equation

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

is obviously (with change of notation) of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$. It is of enormous importance in classical mechanics - it is the equation that describes *damped oscillatory motion*.

Likewise the equations $m\ddot{x} + b\dot{x} + kx = \hat{F}\cos\omega t$ and $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \hat{f}\cos\omega t$

are also of immense importance - they describe *forced, damped oscillations.*
Similar equations occur in the theory of alternating currents. Thus the equation $LC\ddot{Q} + RC\dot{Q} + Q = \hat{V}C\cos\omega t$ is of huge importance in the study of electricity. It describes how the current in a circuit containing resistance, inductance and capacitance varies when you apply an alternating voltage across it. If you have an analogue radio, when you turn a knob to tune it to a particular station at a given a resonant frequency, you

are changing the capacitance of the circuit, and the radio responds according to the solution of the above equation.

In spite of the huge importance of equations of the type $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \hat{f}\cos\omega t$ and $LC\ddot{Q} + RC\dot{Q} + Q = \hat{V}C\cos\omega t$, I shan't give them an undue amount of importance in this file. Instead I refer you to Chapters 11 and 12 of
http://orca.phys.uvic.ca/~tatum/classmechs.html

and Chapters 13 and 14 of
http://orca.phys.uvic.ca/~tatum/elmag.html

where the applications and solutions of these equations are dealt with in exquisite detail.

You'll find it yet again in spectroscopy. See, for example, Chapter 9 Section 9.2 of
http://orca.phys.uvic.ca/~tatum/stellatm.html
where the theory is used to relate the strengths of spectrum lines to the number of atoms producing them in a stellar atmosphere. It's all the same equation.

In addition to the "prime" notation y' for $\frac{dy}{dx}$ and the "dot" notation $\dot{y}$ for $\frac{dy}{dt}$, I shall use, in this file, the notation D to denote the operator $\frac{d}{dx}$. This notation will be convenient in the context of this file, though if you use it in some other context, you cannot assume that your readers will know what you mean by it, and you will have to define it. The equation to be solved, then, may be written

$$(aD^2 + bD + c)y = f(x).$$

You may find yourself greatly disconcerted or alarmed, when I tell you that the solution to this equation is

$$y = \frac{f(x)}{aD^2 + bD + c}$$

and even more alarmed when you see me bandying the symbol D around with careless abandon as though it were just another variable rather than an operator with special meaning. For example, if $b^2 > 4ac$, I may take even further liberties, and factorize the denominator into $a(D - \alpha)(D - \beta)$, and I may even go outrageously further and split

$\frac{1}{(D-\alpha)(D-\beta)}$ into partial fractions: $\frac{1}{\alpha-\beta}\left[\frac{1}{D-\alpha}-\frac{1}{D-\beta}\right]$. Obviously such liberties will have to be justified, and we shall have to have a clear understanding of exactly what is meant by an operation such as $\frac{1}{D-\alpha}$.

Alarmed or not, or whatever notation I may be using, you will probably not be surprised to find that the nature of the solution, or the procedure for solving it, may depend upon whether $b^2 - 4ac$ is negative, zero or positive.

How much preparatory background mathematical knowledge are you going to need before embarking further into the theory of second order differential equations with constant coefficients? Well, of course you do need the basic algebra, trigonometry and calculus that you almost certainly already have if you are reading this page. As I write this, I anticipate two topics that you will need some familiarity with.

One is the decomposition of *partial fractions*. For example, if we come across an expression such as

$$\frac{x^2 + 5x + 7}{2x^3 - 5x^2 + 3x - 2}$$

I shall expect the viewer to be able to decompose it into

$$\frac{3}{x-2} - \frac{5x+2}{2x^2 - x + 1}.$$

I wouldn't call this easy or difficult, but something in between, and I wouldn't expect the viewer to write down the answer instantly on sight (though with lots of practice it's not impossible). It takes a few minutes of algebra. However, I do assume that the viewer knows how to do it, so, if I encounter such an expression in the text that follows, I shall write down the decomposition without further explanation

You will need to be familiar with the basic algebra and arithmetic of complex numbers. (I don't anticipate, at least just now, that you will need to be familiar with "functions of a complex variable".) Perhaps the most important thing to know is that, if you ever (not only in the present context) find yourself faced with a complicated fraction with lots of algebra in the numerator and in the denominator, and there is a *complex number in the denominator*:

$$\frac{\text{ABCD}}{\text{EF}(x+iy)\text{GH}}$$

in which each capital letter represents some function of x or y or both, such as $\sqrt{x^2 - 1}$ or $\ln(x/y)$ or $\cos(5y + 2x - 3)$, and there is a complex number in the denominator, you never need hesitate for an instant as to what to do next. Without giving it a moment's thought, you multiply top and bottom by the complex conjugate to obtain

$$\frac{\mathsf{ABCD}(x - iy)}{\mathsf{EFGH}(x^2 + y^2)}.$$

If I come across such an expression in the text that follows, I shall always do that without a word of explanation.

Return to the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x).$$

If the function $f(x)$ is a *periodic* function such as $\cos kx$, we may well find *complex numbers* to be useful in solving the equation. If, however, $f(x)$ is some other sort of function, not periodic, complex numbers may be less useful, and we may find that we can solve the equation by means of *Laplace Transforms*. At the time of writing this, I am not going to assume that viewers are familiar with Laplace transforms, so that you can certainly read on if you are not familiar with them. If I find, later on, that we have need of them, I'll try to give a brief explanation.

The Operator *D*

We shall use the symbol D to mean $\frac{d}{dx}$, and D^2 to mean $\frac{d^2}{dx^2}$, etc. We mentioned in the preamble that we shall be taking certain liberties with this operator, and that we shall need to justify such liberties and we shall need to become familiar with some of its properties. For example, if f and g are functions of x, the viewer will have no difficulty in understanding that $D(f + g) = Df + Dg$ or that $D^m D^n f = D^{m+n} f$ or that $(D + a)f = Df + af$. The viewer may have to spend a little time to convince him- or herself that $(D + a)(D + b)f = (D^2 + aD + bD + ab)f$ or that $(D - a)(D + a)f = (D^2 - a^2)f$ or conversely that $(D^2 - a^2)f = (D - a)(D + a)f$. It is worth spending a moment or two to convince yourself of these. You will then be well on your way to dealing confidently with the operator often in just the same way that you treat other variables. Not always, of course. For example $D(fg) = fDg + gDf$. But you already know that, and are unlikely to get it wrong.

A useful thing to know is what if one of the functions, say f, is of the form e^{ax}. Then what is $D(e^{ax}g)$? A moment or two will show you that

$$D(e^{ax}g) = e^{ax}(D + a)g \tag{1}$$

If you feel like spending the time to differentiate a second time, you will eventually arrive at

$$D^2(e^{ax}g) = e^{ax}(D + a)^2 g \tag{2}$$

[This looks interesting, so, in case you didn't do it, here's a start:

$D^2(e^{ax}g) = D[D(e^{ax}g)] = D[e^{ax}(D + a)g] = ae^{ax}(D + a)g + e^{ax}(D^2 + aD)g$, etc.]

I wonder if it is generally true that

$$D^n(e^{ax}g) = e^{ax}(D + a)^n g \tag{3}$$

If you differentiate this again, you will find (in a similar manner to the way in which we derived equation (2) that

$$D^{n+1}(e^{ax}g) = e^{ax}(D + a)^{n+1} g \tag{4}$$

And since we know that equation (3) is indeed true for $n = 1$, it must be true for all positive integral n.

The Operator $\frac{1}{D}$ or D^{-1}

We define this such that $DD^{-1}f = f$. In other words, D^{-1} does just the opposite to what D does. In other words $D^{-1}f = \int f dx$ – except that we take the arbitrary constant of integration to be zero. Readers of this site will already know how to use the operator D^{-1} on almost any function, but we need to know the meaning of

The Operator $\frac{1}{D - a}$ or $(D - a)^{-1}$

and how to use it. Believe it or not, but it means an operation such that

$$(D-a)^{-1}f = e^{ax}D^{-1}(fe^{-ax}). \tag{5}$$

We'll explain why in a moment, but first an example. Suppose $f = x^2$. Then

$$(D-a)^{-1}x^2 = e^{ax}D^{-1}(e^{-ax}x^2) = e^{ax}\int x^2e^{-ax}dx = -\frac{(ax+1)^2+1}{a^3}.$$

That's all very well, but why does $(D-a)^{-1}f = e^{ax}D^{-1}(e^{-ax}f)$?
$(D-a)^{-1}f$ has to be defined so that $(D-a)[(D-a)^{-1}f] = f$. That is to say, $(D-a)^{-1}f$ has to be defined so that $(D-a)g = f$, where $g = (D-a)^{-1}f$. That is to say $\frac{dg}{dx} - ag = f(x)$. Since you have by now thoroughly mastered the chapter on differential equations of the first order, you immediately recognize this as a first order DE, which can be solved by means of the IF e^{-ax}, and that it has the solution

$$g = e^{ax}D^{-1}(fe^{-ax}). \tag{6}$$

There will now be a long pause while you absorb that lot.

LONG PAUSE

Having got so far, we may need another pause, a shorter one, to see now why its it that the operation $(D-a)^{-1}$ has to mean what equation (5) says it does.

SHORT PAUSE

Now that we are convinced of equation (5), let us do $(D-a)^{-1}$ to both sides of it. We soon arrive at

$$(D-a)^{-2}f = e^{ax}D^{-2}(fe^{-ax}). \tag{7}$$

This leads us to speculate that maybe

$$(D-a)^{-n} f \;=\; e^{ax} D^{-n}(fe^{-ax}). \tag{8}$$

If you now do $(D-a)^{-1}$ to both sides of equation (8), you almost immediately get

$$(D-a)^{-(n+1)} f \;=\; e^{ax} D^{-(n+1)}(fe^{-ax}). \tag{9}$$

And since equation (8) is true for $n = 1$, it is true for any positive integer.

Examples

Show that

1. $(D-2)^{-1}2 \;=\; -1$

2. $(D-2)^{-2}2 \;=\; \frac{1}{2}$

3. $(D-a)^{-1}e^{ax} \;=\; xe^{ax}$

4. $(D-a)^{-1}C = -C/a$

5. $(D-a)^{-1}x = -\left(\dfrac{ax+1}{a^2}\right)$

6. $(D-a)^{-1}x^2 \;=\; -\dfrac{(ax+1)^2+1}{a^3}$

7. $(D-a)^{-1}(x^2+Bx+C) \;=\; -\left[\dfrac{a^2x^2+a(2+Ba)x+2+Ba+Ca^2}{a^3}\right]$

Most readers will be comfortable with the equivalence of the operators $(D-a)(D-b)$ and $D^2-(a+b)D+ab$. That is $(D-a)(D-b) \equiv D^2-(a+b)D+ab$, or $(D-a)(D-b)f = \left(D^2-(a+b)D+ab\right)f$.

Most will probably also go along with

$$\frac{1}{D^2 - (a+b)D + ab} \equiv \frac{1}{(D-a)(D-b)}$$

Are we also willing to go along with the next step - i.e. to declare that either of these operators is equivalent to the operator

$$\frac{1}{a-b}\left(\frac{1}{D-a} - \frac{1}{D-b}\right)?$$

If indeed $\frac{1}{a-b}\left(\frac{1}{D-a} - \frac{1}{D-b}\right) \equiv \frac{1}{(D-a)(D-b)}$ then the operator $(D-a)(D-b)$ (which is the inverse of the operator $\frac{1}{(D-a)(D-b)}$) must also be the inverse of the operator $\frac{1}{a-b}\left(\frac{1}{D-a} - \frac{1}{D-b}\right)$. That is to say

$(D-a)(D-b)\left[\frac{1}{a-b}\left(\frac{1}{D-a} - \frac{1}{D-b}\right)f\right]$ must equal *f*.

Let us do the calculation:

$$(D-a)(D-b)\left[\frac{1}{a-b}\left(\frac{1}{D-a} - \frac{1}{D-b}\right)f\right] = \frac{1}{a-b}(D-a)\left[\left(\frac{D-b}{D-a} - 1\right)f\right]$$

$$= \frac{1}{a-b}\left[(D-b) - (D-a)\right]f = \frac{a-b}{a-b}f = f.$$

This, then, justifies spitting up an operator of the form $\frac{1}{D^2 - (a+b)D + ab}$ into partial fractions, just as if *D* were an ordinary variable rather than an operator.