Chapter 2
Differential Equations of Order 1

Instant gratification

Very often, when I am doing mathematical problems, such as integrating something, or solving a differential equation, I find that during the course of the calculation I have occasion to do some routine algebra, such as solve two simultaneous equations:

$$\begin{aligned} 2x + 3y + 4 &= 0 \\ 3x - \ \ y + 5 &= 0 \end{aligned}$$

or solve a quadratic equation:

$$6x^2 + 7x - 20 = 0$$

or I need to know whether a quadratic expression such as $78x^2 - 127x - 5$ can easily be written as the product of two linear factors.

I know how to do it, but it's a bit of a nuisance, it is not very interesting, I am liable to make a mistake, and, above all, it interrupts the flow of thought that I am trying to maintain when trying to solve the main problem. What I need is instant gratification. I need the answers, and I need them now, without interrupting my train of thought.

A long time ago, I wrote a series of very short computer programs for solving the most frequent two dozen or so routine algebra and trigonometry problems that I commonly come across - such as, for example solving two simultaneous linear equations, or solving a quadratic equation. All I do is to type in the coefficients and I instantly get the correct answers. I highly commend this to you. If you do this, it will save you enormous grief later on.

In any case, if, in the course of the following notes, I encounter a need for the instantaneous solution of some routine algebraic equation, I shall just write down the solutions with no further explanation.

Differential equations

The following equations are examples of differential equations:

$$\sin x \frac{dy}{dx} + 2y\cos x = \cos x$$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + (x^2 + y)z = 0.$$

The first two of these have only two variables - x and y - and consequently no partial derivatives. They are called ***ordinary*** differential equations, in contrast to the third, which has more than two variables, and consequently some partial derivatives. The third is a ***partial*** differential equation, called by those who like using abbreviations a PDE. When you integrate an ordinary differential equation, you must always remember to add an arbitrary constant. When you integrate a partial differential equation with respect to one of the variables, you must add an arbitrary function of all the other variables. Seen in this light, you may imagine that partial differential equations can be more difficult than many ordinary equations.

In these notes, at least for the time being, I shall discuss only ordinary differential equations, so you can forget about PDEs just now.

The object of solving an ordinary differential equation, such as the first two above, is to integrate it and hence find a relation between y and x, with no derivatives.

The first equation is an ordinary DE of ***order unity***, or order 1, because it has no derivatives of order higher than 1. The second equation is an ordinary DE of ***order two***, since it contains derivatives of order as high as 2, but no higher.

In this file we shall deal only with **ordinary DEs of order one**. Ordinary DEs of order two may come in a separate file.

In what follows, I shall assume that you are already skilled at integration, or that you don't mind using Wolfram to integrate for you. What we shall be doing, therefore, is to rearrange each differential equation so that all you have to do is to integrate something in the usual way. I shall not go through how to perform the integration - that skill is already assumed.

There are several types of ordinary differential equation of order unity, and several corresponding ways of dealing with them. We shall see that many (perhaps not all) ordinary DEs of order one can be classified into a few types, and part of the skill (which comes with practice) in solving them is to recognize which type a particular equation is, or to see whether it might be manipulated by algebra, or by a change of variable ("Brilliant Substitution"), into one of these types. The types I describe below are

Equations with separable variables.
Homogeneous equations.
Equations that can easily be made homogeneous.
Equations that require a Brilliant Substitution.
Equations solvable by means of an integrating factor.
Bernoulli-type equations.
Exact equations.

Equations with separable variables

The first equation cited above, $\sin x\frac{dy}{dx} + 2y\cos x = \cos x$, is one with *separable variables*. Such equations are fairly common in practice, but are rare in examinations. The reason that they are rare in examinations is that they are regarded as the easiest type of differential equation to solve, and hence too easy to put on an exam!

By saying that the variables are separable, I mean that it is fairly easy, by routine algebraic manipulation, to put all the *x*'s and the differential *dx* on one side of the equation, and all the *y*'s and the differential *dy* on the other side, and then you can just integrate each side in a routine manner that is familiar to you.

Thus the equation $$\sin x\frac{dy}{dx} + 2y\cos x = \cos x$$

can easily be written in the form $$\int\frac{dy}{1-2y} = \int\cot x dx .$$

I shan't show how to integrate these here, since you probably know how to do it anyway - or you can use Wolfram Alpha for instant gratification if you like - but the answer is

$$\underline{\underline{y = \tfrac{1}{2} + C\csc^2 x}}$$

You can check this by substitution in the original equation - which you always do anyway, whenever you solve an equation.

Here's another one:

$$(1 - x^2)\frac{dy}{dx} - xy = xy^2$$

The answer is

$$\underline{\underline{y^2(1-x^2) = C(y+1)^2}}$$

Now try this one: $$\frac{dy}{dx} = \cos(x + y)$$

This looks more difficult, because the variables aren't obviously separable. Sometimes, however, you can make a change of variable which results in an equation with separable variables.

For example, let $z = x + y$ so that $\frac{dy}{dx} = \frac{dz}{dx} - 1.$

The equation then becomes

$$\frac{dz}{dx} - 1 = \cos z$$

The variables are now separable:

$$\int \frac{dz}{1 + \cos z} = \int dx$$

and away you go. (See integral 5 in the integration examples if you get stuck with this).

Answer: $\underline{\underline{\tan \tfrac{1}{2}(x + y) = x + C}}$

Here's another of a similar type:

$$(x + y)^2 \frac{dy}{dx} = 9$$

Answer: $\underline{\underline{y = 3\tan^{-1}\left(\frac{x + y}{3}\right) + C}}$

Now try this one:

$$\left(\frac{dy}{dx}\right)^2 - (x + y - 1)\frac{dy}{dx} + x(y - 1) = 0$$

It looks horrible, but there's nothing to fear. It's just a quadratic equation in $\frac{dy}{dx}$, so it's really two separate equations, $\frac{dy}{dx} = x$ and $\frac{dy}{dx} = y - 1$.

Answers: $\underline{\underline{y = \tfrac{1}{2}x^2 + C}}$ and $\underline{\underline{y = 1 + Ce^x}}$

Already it's getting easier. After 200 or so of these, you'll be able to write down the solutions on sight.

Homogeneous equations

Now let's look at $$(y + 2x)\frac{dy}{dx} + x - 3y = 0$$

$$(x^2 - xy)\frac{dy}{dx} = 2xy + y^2$$

$$(x^3 + xy^2)\frac{dy}{dx} = 2y^3$$

These are all *homogeneous* equations.

The first is homogeneous of degree 1. That is, all terms are of degree 1. There are no terms of degree 2, such as x^2 or xy, and no constants.

The second is homogeneous of degree 2. There are no constant terms, and no terms of degree 1, such as x or y.

The third is homogeneous of degree 3. There are no constant terms, and no terms of degree 1, such as x or y, or of degree 2, such as y^2 or xy.

You will see that an obvious brilliant substitution to make is: Let $y/x = u$. That is $y = ux$.
Then $\frac{dy}{dx} = \frac{du}{dx} + 1$. The three equations become

$$(u + 2)\left(\frac{du}{dx} + 1\right) + 1 - 3u = 0$$

$$(1 - u)\left(\frac{du}{dx} + 1\right) = 2u + u^2$$

$$(1 + u^2)\left(\frac{du}{dx} + 1\right) = 2u^3$$

The variables are separable, and the three equations can now easily be reorganized:

$$\int \frac{(u + 2)du}{u^2 - u + 1} = -\int \frac{dx}{x}$$

$$\int \frac{u - 1}{2u^2 + u} du + \int \frac{dx}{x} = 0$$

$$\int \frac{u^2 + 1}{u^3 - u} du = \int \frac{dx}{x}$$

We have now got the equations in a form in which they can be integrated. We've broken the back of the problems as differential equations. It's now just an exercise in integration. How to integrate them is up to you. They may need a bit of patience. (If you are really, really stuck, you can email me. Or use Wolfram.) The solutions are:

$$\sqrt{3}\ln(y^2 - xy + x^2) + 10\tan^{-1}\left(\frac{2y - x}{\sqrt{3}x}\right) = C$$

$$(2y + x)^3 x = Cy^2$$

$$x^2 - y^2 = Cx^2 y$$

Now look at the following:

$$\left(2\sqrt{xy} - x\right)\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$$

$$xy^3\frac{dy}{dx} = x^4 + 2y^4$$

$$x\frac{dy}{dx} + \sqrt{x^2 + y^2} = y$$

$$\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$$

Are any of them homogeneous? Yes! The first four are all homogeneous! If you try $y = ux$ you will be able to separate the variables u and x and you then have just an exercise in integration. (They are not too bad.) Here are the answers to the first four. I think so, anyway. If you think I may have got any of them wrong, let me know. jtatum at uvic dot ca

$$\sqrt{\frac{x}{y}} + \ln y = C$$

$$\tan\left(\frac{y}{2x}\right) = Cx$$

$$\underline{\underline{y + \sqrt{x^2 + y^2} = C}}$$

$$\underline{\underline{(5y - 7)^2 + (5x - 1)(5y - 7) - (5x - 1)^2 = C}}$$

Equations that can be made homogeneous

But the last (fifth) one is not homogeneous. We need a bit more help. We can make a change of variables thus: Let $x = X + h$ and $y = Y + k$. Then

$$\frac{dY}{dX} = \frac{2X - Y + 2h - k + 1}{X + 2Y + h + 2k - 3}.$$

If we choose h and k to be such that

$$2h - k + 1 = 0$$

and $$h + 2k - 3 = 0,$$

that is, $$h = \tfrac{1}{5}, \quad k = \tfrac{7}{5},$$

the original equation becomes $$\frac{dY}{dX} = \frac{2X - Y}{X + 2Y},$$

which is homogeneous in X and Y, and we now know what to do. Let $Y = UX$, $\frac{dY}{dX} = X\frac{dU}{dX} + U$, and after a little bit of algebra the variables become separable, and we arrive at

$$\int \frac{2U + 1}{U^2 + U + 1} dU + \int \frac{2dX}{X} = 0$$

Now we can integrate: $$(U^2 + U - 1)X^2 = \text{constant}$$

Back to Y and X: $$Y^2 + XY - X^2 = \text{constant}$$

Back to y and x: $$(5y - 7)^2 + (5x - 1)(5y - 7) - (5x - 1)^2 = \text{constant},$$

which simplifies to $$\underline{\underline{x^2 - xy + x - y^2 + 3y + C = 0.}}$$

Try this one: $$\frac{dy}{dx} = \frac{x - 2y + 3}{2x - y + 3}$$

The answer is

$$\underline{\underline{(y-1)^2 - 4(x+1)(y-1) + (x+1)^2 = \text{constant}}}$$

In these last two examples, each equation had two linear expressions of the form $ax + by + c$, and it was easy to change the equations to homogeneous form by means of the substitutions $x = X + h$ and $y = Y + k$. And of course once we have the equations in homogeneous form, we can do a further substitution $Y = UX$ so that the variables become separable.

Equations that require a Brilliant Substitution

Here is a little set of differential equations which don't quite fit any of the patterns so far.

$$y^3 \frac{dy}{dx} + x + y^2 = 0$$

$$\frac{dy}{dx} = \frac{3xy}{2(x^2 - xy^2 + y^4)}$$

$$xy + (y^4 - 3x^2)\frac{dy}{dx} = 0$$

By suitable change of variable, they can all be put into forms such that the variables are separable. I can't think of any general advice for finding suitable substitutions. Sometimes they just come by inspiration. Sometimes I try one after another, discarding numerous sheets of paper before I find one that works.

Let's try the first one:

$$y^3 \frac{dy}{dx} + x + y^2 = 0 \quad \text{or} \quad 2y^3 \frac{dy}{dx} + 2(x + y^2) = 0$$

Let's try the Brilliant Substitution $w = x + y^2$. Not a very sophisticated try, but I can't think of anything else, and, as it happens, it does help. If $w = x + y^2$, then $\frac{dw}{dx} = 1 + 2y\frac{dy}{dx}$. The equation then takes the form $$y^2 \times \left(\frac{dw}{dx} - 1\right) + 2w = 0 \ ,$$

or $$(w - x) \times \left(\frac{dw}{dx} - 1\right) + 2w = 0 \ .$$

This looks a little better, but we still need to try something further. Not very difficult, because the equation is now homogeneous.

Let's try $w = ux$, $\frac{dw}{dx} = x\frac{du}{dx} + u$

Then:

$$(u-1)\left(x\frac{du}{dx} + u - 1\right) + 2u = 0.$$

After some modest manipulation this becomes

$$x\frac{du}{dx} + \frac{u^2+1}{u-1} = 0.$$

The variables are now separable, so we can integrate:

$$\int \frac{u-1}{u^2+1}du + \int \frac{dx}{x} = 0$$

$$\ln(u^2+1) - 2\tan^{-1}u + 2\ln x = C$$

$$\underline{\underline{\ln(y^4 + 2xy^2 + 2x^2) - 2\tan^{-1}\left(\frac{y^2 + x}{x}\right) = C}}$$

We used two Brilliant Substitutions: $w = x + y^2$ followed by $u = w/x$. Now in hindsight we see that we could have done it all in one fell swoop by means of the single Extra Brilliant Substitution, $u = 1 + y^2/x$. Try it and see if it works. The advantage is that no one would imagine how you ever came up with such a Brilliant Substitution.

Now try this one:

$$\frac{dy}{dx} = \frac{3xy}{2(x^2 - xy^2 + y^4)}$$

After a few unsuccessful tries, I tried letting $w = y^2$, which resulted in a homogeneous equation. I then followed up with the usual Let $w = ux$. In hindsight I suppose I could have tried a single substitution Let $u = y^2/x$.

The answer is $$\underline{\underline{\frac{(2x - y^2)(y^2 + x)^2}{y^2} = C}}$$

Here's another one: $$xy + (y^4 - 3x^2)\frac{dy}{dx} = 0$$

See if you can find a Brilliant Substitution that will make the equation homogeneous.

The answer is $$y^6 = C(x^2 - y^4)$$

Equations that need an integrating factor

The following is a rather common type of differential equation:

$$\frac{dy}{dx} + f(x)y = g(x).$$

Equations of this type can be integrated fairly readily. The skill required is to recognize that a given equation is of this type. Her are some examples:

$$\frac{dy}{dx} + \left(\frac{1-x}{1+x}\right)y = \frac{1}{1+x}$$
$$x\ln x\frac{dy}{dx} + y = x$$
$$x\frac{dy}{dx} + 3y = x + 1$$
$$\frac{dy}{dx}\cos x + y = \sin^2 x$$

The first of these is obviously of the form $\frac{dy}{dx} + f(x)y = g(y)$. The others can each be written in that form, can they not? Let's re-write them:

$$\frac{dy}{dx} + \left(\frac{1-x}{1+x}\right)y = \frac{1}{1+x}$$
$$\frac{dy}{dx} + \left(\frac{1}{x\ln x}\right)y = \frac{1}{\ln x}$$
$$\frac{dy}{dx} + \left(\frac{3}{x}\right)y = \frac{x+1}{x}$$
$$\frac{dy}{dx} + (\sec x)y = \sin x\tan x$$

To solve these equations, I would multiply

both sides of the first equation by $(x+1)^2 e^{-x}$,
both sides of the second equation by $\ln x$,
both sides of the third equation by x^3,
both sides of the fourth equation by $\sec x + \tan x$.

At this point you will be wondering, what on Earth made me think of multiplying these equations by these *integrating factors*? And, when I have done so, why do I assert that I can now solve the equations?

In fact, once you have been told the secret, it becomes very routine, and you don't have to think very hard at all.

The integrating factor is in fact

$$e^{\int f(x)dx}$$

We'll work them out now, and explain later. Use Wolfram if you like, so as not to interrupt your flow of thought.

$f(x)$	$\int f(x)dx$	$e^{\int f(x)dx}$
$\dfrac{1-x}{1+x}$	$\ln(1+x)^2 - x$	$(1+x)^2 e^{-x}$
$\dfrac{1}{x\ln x}$	$\ln(\ln x)$	$\ln x$
$\dfrac{3}{x}$	$3\ln x = \ln x^3$	x^3
$\sec x$	$\ln(\sec x + \tan x)$	$\sec x + \tan x$

You'll notice that I have allowed myself the rare liberty of not adding " +*C*" to the integral – or, what amounts to the same thing, I have taken C to be 0. It will be clear why I can do this as we go on.

So − why do these integrating factors work? The explanation follows. You may find this explanation difficult to follow, but don't despair. Once you know what to do, you will find that equations of the form $\dfrac{dy}{dx} + f(x)y = g(x)$ are *easy, routine and straightforward.*

Consider $\frac{dy}{dx} + f(x)y$. Multiply by $e^{\int f(x)dx}$:

$$e^{\int f(x)dx}\frac{dy}{dx} + ye^{\int f(x)dx}f(x)$$

Next question: What is the derivative of $e^{\int f(x)dx}$? I.e., what is $\frac{d}{dx}\left(e^{\int f(x)dx}\right)$?
Think of the integrand as a function of a function. Thus you'll agree that

$$\frac{d}{dx}\left(e^{\int f(x)dx}\right) = e^{\int f(x)dx} \times \frac{d}{dx}\left(\int f(x)dx\right) = e^{\int f(x)dx} f(x)$$

Thus the expression $e^{\int f(x)dx}\frac{dy}{dx} + ye^{\int f(x)dx}f(x)$ is now $e^{\int f(x)dx}\frac{dy}{dx} + y\frac{d}{dx}e^{\int f(x)dx}$.
That is to say it is $\text{IF}\frac{dy}{dx} + y\frac{d}{dx}\text{IF}$, where IF is the integrating factor. In other words it is $\frac{d}{dx}(y \times \text{IF})$. The right hand side is $g(x)\times$ IF. Incidentally you can now see why we could ignore the "+ *C*" in calculating the IF, or, rather, take *C* to be zero. If we add the "+ *C*", when we come to multiplying both sides of the DE by the IF, we would just be multiplying the equation by an extra e^C throughout.

Anyway, the equation is now in the form $\frac{d}{dx}(y \times \text{IF}) = g(x)\times \text{IF}$. and we can now integrate it:

$$y \times \text{IF} = \int g(x)\times \text{IF}dx.$$

So, if you see an equation of the form

$$\frac{dy}{dx} + f(x)y = g(x),$$

go for it! First, calculate the IF:

$$e^{\int f(x)dx}$$

(and don't bother with a "+ *C*" in the integral.

Then the solution to the DE is found from

$$y \times \text{IF} = \int g(x) \times \text{IF}\,dx.$$

That's all!

Let's now go through the four examples we started with. The first one was

$$\frac{dy}{dx} + \left(\frac{1-x}{1+x}\right)y = \frac{1}{1+x}$$

We already worked out that the IF is $(1+x)^2 e^{-x}$.

The solution to the DE can now be found from

$$y \times (1+x)^2 e^{-x} = \int \frac{1}{1+x} \times (1+x)^2 e^{-x} dx = \int (1+x) e^{-x} dx$$

$$y(1+x)^2 e^{-x} = -e^{-x}(2+x) + C$$

$$\underline{\underline{y(1+x)^2 + 2 + x = Ce^{x}}}$$

Now the second one:

$$\frac{dy}{dx} + \frac{1}{x\ln x} y = \frac{1}{\ln x}$$

$$\text{IF} = \ln x$$

DE:

$$y \ln x = \int \frac{1}{\ln x} \ln x\,dx = \int dx$$

$$\underline{\underline{y \ln x = x + C}}$$

The next two:

$$\frac{dy}{dx} + \frac{3}{x} y = \frac{x+1}{x}$$

$$\frac{dy}{dx} + (\sec x) y = \sin x \tan x$$

should now be easy. The answers are

$$\underline{\underline{y = \tfrac{1}{4} x + \tfrac{1}{3} + Cx^{-3}}}$$

$$\underline{\underline{y(\sec x + \tan x) - \sec x - \cos x - \tan x + x = C}}$$

Here's an interesting one:

$$\frac{dy}{dx} = \frac{3x - 2y}{x + 5}$$

You can do this by making it homogeneous by means of $x = X + h,\ \ y = Y + k$, with $h = -5,\ k = -\frac{15}{2}$, followed by $Y = UX$ in the now familiar manner, or by re-writing the equation as $\frac{dy}{dx} + \frac{2}{x+5}y = \frac{3x}{x+5}$ and determining the IF. Try it both ways. I found the second to be faster. By either method, I got $\underline{\underline{2y(x+5)^2 = 2x^3 + 15x^2 + C}}$

Bernoulli's equations

These are equations of the form

$$\frac{dy}{dx} + f(x)y = g(x)y^n,$$

where n is a number greater than 1. If $n = 0$, the equation is of the form that we have just dealt with. If $n = 1$, the variables are separable, because $f(x) - g(x)$ is now just a single function of x. In this section, therefore, we shall deal with cases where n is a number other than 0 or 1.

An equation such as

$$\frac{dy}{dx} - 2y\tan x = y^2\tan^2 x$$

is obviously of the Bernoulli form. It might not be immediately obvious that the equation

$$(x - x^3)\frac{dy}{dx} + x^2 y = y^3 \ln x$$

is also of Bernouilli form, but of course it is, and the first step is to recognize it as potentially Bernoulli and then to re-write it in exactly Bernoulli form:

$$\frac{dy}{dx} + \frac{xy}{1 - x^2} = \frac{y^3 \ln x}{x - x^3}$$

Once you have recognized that your equation is a Bernoulli and have written it exactly in Bernoulli form, there are two steps to do:

i. Divide the equation through by y^n. In practice I nearly always divide it by $-y^n$. That is, I change the sign of each term as well as dividing by y^n, and I recommend doing this.

ii. Make the Brilliant Substitution $u = \frac{1}{y^{n-1}}$.

When you have done this, you will find yourself on familiar ground.

We'll do three examples, with $n = 2, 3, \frac{3}{2}$.

First, an example with $n = 2$:

$$\frac{dy}{dx} - 2y\tan x = y^2 \tan^2 x$$

i. Divide by $-y^2$:

$$-\frac{1}{y^2}\frac{dy}{dx} + \frac{2\tan x}{y} = -\tan^2 x$$

ii. Make the Brilliant Substitution $u = \frac{1}{y}$, which means that $\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$.

The equation now becomes

$$\frac{du}{dx} + 2u\tan x = -\tan^2 x$$

We are now on familiar ground. This can be solved by means of an integrating factor (IF), so there is no need for me to go further. However, I'll do so anyway. The integrating factor is

$$\exp(2\int \tan x dx) = \exp(2\ln\sec x) = \exp(\ln\sec^2 x) = \sec^2 x.$$

After doing a few of these it will become evident that often the integrating factor can be written down on sight. In any case, the solution is now found, as in all integrating factor examples, from

$$u \times \text{IF} = \int g(x) \times \text{IF} dx.$$

That is,

$$u\sec^2 x \;=\; -\int \sec^2 x\tan^2 x dx \;=\; -\tfrac{1}{3}\tan^3 x + C\,,$$

so the solution is

$$y \;=\; \frac{\sec^2 x}{C \;-\; \tfrac{1}{3}\tan^3 x}$$

<u>*Now, an example with*</u> $n = 3$:

$$(x - x^3)\frac{dy}{dx} \;+\; x^2 y \;-\; y^3 \ln x$$

i. Divide by $-y^3$:

$$-\frac{1}{y^3}\frac{dy}{dx} \;-\; \frac{x}{(1\;-x^2)\,y^2} \;=\; -\frac{\ln x}{x - x^3}$$

ii. Make the Brilliant Substitution $u = \dfrac{1}{y^2}$, which means that $\dfrac{du}{dx} \;=\; -\dfrac{2}{y^3}\dfrac{dy}{dx}$.

The equation now becomes

$$\frac{1}{2}\frac{du}{dx} \;-\; \frac{xu}{(1\;-x^2)} \;=\; -\frac{\ln x}{x(1-x^2)}$$

I might mention that I sometimes go wrong and forget the $\frac{1}{2}$ here. Make sure you don't! In any case, for convenience, now multiply throughout by 2:

$$\frac{du}{dx} \;-\; \frac{2xu}{(1\;-x^2)} \;=\; -\frac{2\ln x}{x(1-x^2)}$$

We are now on familiar ground again. The integrating factor is $-(1-x^2)$, and away we go:

$$-u(1-x^2) \;=\; 2\int \frac{\ln x}{x}\,dx$$

$$\frac{1-x^2}{y^2} \;=\; C \;-\; (\ln x)^2$$

<u>*Lastly, an example with*</u> $n = \tfrac{3}{2}$:

$$\frac{dy}{dx} \;-\; 4y \;=\; xy^{3/2}$$

i. Divide by $-y^{3/2}$:

$$-\frac{1}{y^{3/2}}\frac{dy}{dx} \;+\; \frac{4}{y^{1/2}} \;=\; -x$$

ii. Make the Brilliant Substitution $u = \dfrac{1}{y^{1/2}}$, which means that $\dfrac{du}{dx} \;=\; -\dfrac{1}{2y^{3/2}}\dfrac{dy}{dx}$.

$$\frac{du}{dx} + 2u \;=\; -\tfrac{1}{2}x$$

The integrating factor is $\exp(\int 2dx) \;=\; e^{2x}$, and the solution is

$$\frac{1}{y} \;=\; \left(Ae^{-2x} \;-\; \tfrac{1}{4}x \;+\; \tfrac{1}{8}\right)^2$$

Exact equations

How easy is it to solve

$$\frac{dy}{dx} \;=\; \frac{f(x,y)}{g(x,y)}?$$

The answer to that question doubtless depends on what the functions are! It could be very easy, or it could be very difficult. However, if the functions are such that

$\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$ [that is to say if $\frac{\partial}{\partial y}$(numerator) = $-\frac{\partial}{\partial x}$(denominator)] the equation $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ is called an *exact* equation, and there is a neat way of solving it. You may think it rather unlikely that you will ever come across a differential equation in which this condition for exactness is satisfied, but in reality exact equations are surprisingly common. They occur with conservative forces in classical mechanics, with electrostatic fields in electromagnetism, with functions of state in thermodynamics, and in examinations (because exact differential equations are fairly easy for an examiner to concoct). For that reason, it may well be worth checking to see whether an equation is exact before trying to solve it.

For example

$$\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$$

is exact, because $\frac{\partial}{\partial y}$(numerator) is -1 and $\frac{\partial}{\partial x}$(denominator) is $+1$,

but

$$\frac{dy}{dx} = \frac{2x + 4y + 8}{x - y - 2}$$

is not exact.

In fact we can see that any differential equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{dx + ey + h}$$

is exact if $b + d = 0$ regardless of the values of the other coefficients.

Let us look at a general case, $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, which we suppose to be exact in the sense that $\frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$. This may be difficult to follow, so we'll then apply it to the particular case $\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$ to see how it works.

Let us imagine that the solution to the equation $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ is $H(x, y) = 0$, where H is a *well-behaved* function in the sense that it is everywhere finite and single-valued and that there are no discontinuities in the function or its first derivatives. In that case its mixed second derivatives satisfy $\frac{\partial^2 H}{\partial y \partial x} = \frac{\partial^2 H}{\partial x \partial y}$. (Those who are rusty or unfamiliar with partial derivatives can find more - especially about exact differentials - in my thermodynamics notes at http://orca.phys.uvic.ca/~tatum/thermod/thermod02.pdf, especially Section 2.5)

To form a first-order differential equation, we can calculate the total derivative $\frac{dy}{dx} = -\frac{\partial H / \partial x}{\partial H / \partial y}$ from an elementary theorem in differential calculus. Thus, if we have an exact differential equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ we can assume that the solution is a well-behaved function of the form $H(x, y) = 0$, where $\partial H / \partial x = f(x, y)$ and $\partial H / \partial y = -g(x, y)$.

Integrate the equation $\partial H / \partial x = f(x, y)$ with respect to *x*, treating *y* as a constant.
We obtain not

$H(x, y) = \int f(x, y) dx = F(x, y)$ plus an arbitrary constant, but
$H(x, y) = \int f(x, y) dx = F(x, y)$ plus an arbitrary function of *y*.
That is:

$$H(x, y) = \int f(x, y) dx = F(x, y) + \phi(y)$$

We can determine $\phi(y)$ as follows. Differentiate this with respect to *y*:

$$\frac{d\phi}{dy} = \frac{\partial H}{\partial y} - \frac{\partial F}{\partial y} = -g(x, y) - \frac{\partial F}{\partial y}$$

and, by integrating this with respect to *y* we can find $\phi(y)$. Put this into equation (1) and this gives us the solution to the differential equation.

Let us apply this to the particular example

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} = \frac{2x - y + 1}{x + 2y - 3}.$$

The solution is

$$H(x, y) = \int f(x, y) dx = \int (2x - y + 1) dx = x^2 - xy + x + \phi(y) = F(x, y) + \phi(y),$$

where $\phi(y)$ is to be found from

$$\frac{d\phi}{dy} = -g(x,y) - \frac{\partial F}{\partial y} = -x - 2y + 3 - \frac{\partial}{\partial y}(x^2 - xy + x) = -x - 2y + 3 + x = -2y + 3.$$

That is, $$\phi = -y^2 + 3y + C.$$

The solution, $H(x,y) = 0$, is therefore

$$\underline{\underline{x^2 - xy + x - y^2 + 3y + C = 0.}}$$

Compare this method with the method we used for the same differential equation on page 7, and see which one you prefer.

I have been trying to concoct an exact differential equation which I can solve by this method, but which I cannot solve by any of the more familiar methods, but so far I haven't found one. If any viewer can suggest one, (remember, we are looking for an equation that we cannot solve by other methods) let me know: jtatum at uvic dot ca. It's a bit like Hamiltonian mechanics - it's all very interesting, but I cannot find a problem in mechanics which I can solve by Hamiltonian methods but cannot solve by Newtonian or Lagrangian methods. So just think of this method as an alternative to other methods. Just make sure that the differential equation is exact:

$$\frac{\partial}{\partial y}(\text{numerator}) = -\frac{\partial}{\partial x}(\text{denominator})$$

before you use it.

If the differential equation is of the form

$$\frac{dy}{dx} = \frac{\text{linear function of } x \text{ and } y}{\text{another linear function of } x \text{ and } y}$$

and if it is exact, it can always be written in the form

$$\frac{dy}{dx} = \frac{ax + y + b}{-x + cy + d}.$$

This can be made homogeneous by a change of variables, but, since it is exact, let us solve it by the "exact" method.

The solution is

$$H(x,y) = \int f(x,y)dx = \int(ax + y + b)dx = \tfrac{1}{2}ax^2 + xy + bx + \phi(y) = F(x,y) + \phi(y),$$

where $\phi(y)$ is to be found from

$$\frac{d\phi}{dy} = -g(x,y) - \frac{\partial F}{\partial y} = x - cy - d - \frac{\partial}{\partial y}(\tfrac{1}{2}ax^2 + xy + bx) = x - cy - d + x = -cy - d.$$

That is, $$\phi = -\tfrac{1}{2}cy^2 - dy + \text{constant}.$$

The solution, $H(x,y) = 0$, is therefore

$$\underline{\underline{ax^2 + 2xy - cy^2 + 2bx + 2dy = C.}}$$

Or, to change the notation a bit, if the differential equation is

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f},$$

the solution is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

which is the general equation for a conic section.

For example, the exact equation

$$\frac{dy}{dx} = \frac{3x - y + 1}{x - 5y + 3} = -\frac{3x - y + 1}{-x + 5y - 3}$$

has $a = 3$, $h = -1$, $g = 1$, $b = 5$, $f = -3$

and its solution is

$$3x^2 - 2xy + 5y^2 + 2x - 6y + c = 0.$$

If $c < \frac{13}{7} = 1.857$, this is a family of ellipses with centre at $(-\frac{1}{7}, \frac{4}{7})$

If $c = \frac{13}{7}$ the equation is satisfied only by the point $(-\frac{1}{7}, \frac{4}{7})$.

If $c > \frac{13}{7}$, no real points satisfy the equation.

[In case you are wondering how I knew this, it is only because I happen to be a lover of conic sections. I mentioned at the top of this file, under “Instant Gratification”, that a long time ago I wrote a series of computer programs for solving the most frequent two dozen or so routine algebra and trigonometry problems that I commonly come across. One of them was to get the computer to look at any quadratic expression in x and y and to tell me instantly what sort of conic section it is, and where is its centre. You can see how to do this if you look at my Celestial Mechanics notes http://orca.phys.uvic.ca/~tatum/celmechs/celm2.pdf especially page 49, where I provide a key to the conic sections.]

This one

$$\frac{dy}{dx} + \frac{3x + 3y - 1}{2x + 2y} = 0$$

looks as though it might be similar, but it isn't exact $\left(\frac{\partial f}{\partial y} \neq -\frac{\partial g}{\partial x}\right)$, so you can't treat it as if it were. You'll have to do it some other way. If you make the Brilliant (though rather obvious) Substitution $u = x + y$, you'll find that the variables are separable, and you'll soon arrive at the solution $\underline{\underline{3x + 2y + 2\ln(x + y - 1) = C}}$. This is not a "well-behaved" function, because the function goes to $-\infty$ all along the line $x + y = 1$.