CHAPTER 22
DIMENSIONS

22.1 *Mass, Length and Time*

Any mechanical quantity can be expressed in terms of three fundamental quantities, *mass*, *length* and *time*. For example, speed is a length divided by time. Force is mass times acceleration, and is therefore a mass times a distance divided by the square of a time.

We therefore say that [Force] = MLT^{-2}. The square brackets mean: "The dimensions of the quantity within". The equations indicate how force depends on mass, length and time. We use the symbols MLT (not in *italics*) to indicate the fundamental dimensions of mass, length and time. In the above equation, MLT^{-2} are *not* enclosed within square brackets; it would make no sense to do so.

We distinguish between the *dimensions* of a physical quantity and the *units* in which it is expressed. In the case of MKS units (which are a subset of SI units), the units of mass, length and time are the kg, the m and the s. Thus we could say that the *units* in which force is expressed are kg m s^{-2}, while its *dimensions* are MLT^{-2}.

For electromagnetic quantities we need a fourth fundamental quantity. We could choose, for example, quantity of electricity Q, in which case the dimensions of current are QT^{-1}. We do not deal further with the dimensions of electromagnetic quantities here. Further details are to be found in my notes on Electricity and Magnetism, http://orca.phys.uvic.ca/~tatum/elmag.html

To determine the dimensions of a physical quantity, the easiest way is usually to look at the definition of that quantity. Most readers will have no difficulty in understanding that, since work is force times distance, the dimensions of work (and hence also of energy) are $\mathrm{ML}^2\mathrm{T}^{-2}$. A more challenging one would be to find [dynamic viscosity]. One would have to refer to its definition (see Chapter 20) as tangential force per unit area per unit transverse velocity gradient.

$$\text{Thus [dynamic viscosity]} \;=\; \left[\frac{\text{force}}{\text{area}}\,\frac{\text{distance}}{\text{velocity}}\right] = \frac{\mathrm{MLT}^{-2}}{\mathrm{L}^2}\,\frac{\mathrm{L}}{\mathrm{LT}^{-1}} \;=\; \mathrm{ML}^{-1}\mathrm{T}^{-1}.$$

22.2 *Table of Dimensions*

I supply here a table of dimensions and MKS units of some mechanical quantities. Some are obvious and trivial. Others might be less so, and readers to whom this topic is new are encouraged to derive some of them from the definitions of the quantities concerned. Let me know (jtatum at uvic.ca) if you detect any mistakes.

I don't know whether angle is a dimensionless or a dimensioned quantity. I can convince you that it is dimensionless by reminding you that it is defined as a ratio of two lengths. I can convince you that it is dimensioned by pointing out that it is necessary to state the units (e.g. radians or degrees) in which it is expressed. This might make for an interesting lunchtime conversation

Mass	M	kg	
Length	L	m	
Time	T	s	
Density	ML^{-3}	$kg\ m^{-3}$	
Speed	LT^{-1}	$m\ s^{-1}$	
Acceleration	LT^{-2}	$m\ s^{-2}$	
Force	MLT^{-2}	$kg\ m\ s^{-2}$	N
Work, Energy, Torque	ML^2T^{-2}	$kg\ m^2\ s^{-2}$	J, N m
Action	ML^2T^{-1}	$kg\ m^2\ s^{-1}$	J s
Rotational inertia	ML^2	$kg\ m^2$	
Angular speed	T^{-1}	s^{-1}	$rad\ s^{-1}$
Angular acceleration	T^{-2}	s^{-2}	$rad\ s^{-2}$
Angular momentum	$ML^2\ T^{-1}$	$kg\ m^2\ s^{-1}$	J s
Pressure, elastic modulus	$ML^{-1}T^{-2}$	$kg\ m^{-1}\ s^{-2}$	Pa
Gravitational constant	$M^{-1}L^3T^{-2}$	$kg^{-1}\ m^3\ s^{-2}$	$N\ m^2\ kg^{-2}$
Dynamic viscosity	$ML^{-1}T^{-1}$	$kg\ m^{-1}\ s^{-1}$	dekapoise
Kinematic viscosity	L^2T^{-1}	$m^2\ s^{-1}$	
Force constant	MT^{-2}	$kg\ s^{-2}$	$N\ m^{-1}$
Torsion constant	ML^2T^{-2}	$kg\ m^2\ s^{-2}$	$N\ m\ rad^{-1}$
Surface tension	MT^{-2}	$kg\ s^{-2}$	$N\ m^{-1}$
Schrödinger wavefunction Ψ	$L^{-3/2}T^{-1/2}$	$m^{-3/2}s^{-1/2}$	
Schrödinger wavefunction ψ	$L^{-3/2}$	$m^{-3/2}$	

22.3 *Checking Equations*

When you are doing a complicated calculation involving difficult equations connecting several physical quantities, you must, routinely, check the dimensions of every line in your calculation. If the equation does not balance dimensionally, you know immediately that you have made a mistake, and the dimensional imbalance may even give you a hint as to what the mistake is. If the equation does balance dimensionally, this, of course, does not guarantee that it is correct - you may, for example, have missed a dimensionless constant in the equation.

Suppose that you have deduced (or have read in a book) that the period of oscillations of a torsion pendulum is $P = 2\pi\sqrt{\frac{I}{c}}$, where I is the rotational inertia and c is the torsion constant. You have to check to see whether the dimensions of the right hand side are indeed that of time. We have $\left[\sqrt{\frac{I}{c}}\right] = \sqrt{\frac{\mathrm{ML^2}}{\mathrm{ML^2T^{-2}}}}$, which does indeed come to T, and so the equation balances dimensionally.

22.4 *Deducing Relationships*

i. We may suppose that the period P of a simple pendulum depends upon its mass m, its length l, and the gravitational acceleration g. In particular we suppose that the period is proportional to some power α of the mass, some power β of the length, and some power γ of the gravitational acceleration. That is

$$P \propto m^\alpha l^\beta g^\gamma .$$

Both sides must have the same dimension - namely T.

That is $$[m^\alpha l^\beta g^\gamma] = \mathrm{T}$$

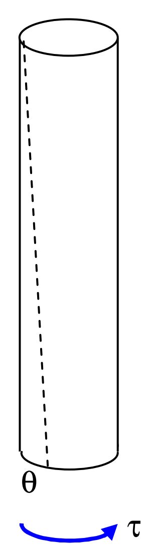
That is $$\mathrm{M^\alpha L^\beta (LT^{-2})^\gamma} = \mathrm{T}$$

We equate powers of M, L and T to get three equations in α, β, γ:

$$\alpha = 0, \ \beta + \gamma = 0, \ -2\gamma = 1,$$

with solutions $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = -\frac{1}{2}$, which shows that

$$P \propto m^0 l^{\frac{1}{2}} g^{-\frac{1}{2}}, \ \text{or} \ P \propto \sqrt{\frac{l}{g}}.$$



ii. Here's another:
The torque τ required to twist a solid metallic cylinder through an angle θ is proportional to θ : $\tau = c\theta$.

c is the torsion constant. How does c depend upon the length l and radius a of the cylinder, its density ρ and its shear modulus η?

There is an immediate difficulty, in that we have four quantities to consider – l, a, ρ and η, yet we have only three dimensions –

L, M , T to deal with. Hence we shall have three equations in four unknowns. Further, two of the quantities, l and a have similar dimensions, which adds to the difficulties.

In cases like this we may have to make a sensible assumption about one of the quantities. We may, for example, find it easy to accept that, the longer the cylinder, the easier it is to twist, and we may make the assumption that the torsion constant is inversely proportional to the first power of its length. Then we can suppose that

$$cl \propto a^{\alpha}\rho^{\beta}\eta^{\gamma}$$

in which case

$$[cl] \equiv [a^{\alpha}\rho^{\beta}\eta^{\gamma}]$$

That is $$\mathrm{ML^2T^{-2}L} \equiv \mathrm{L}^{\alpha}(\mathrm{ML^{-3}})^{\beta}(\mathrm{ML^{-1}T^{-2}})^{\gamma}$$

Equate the powers of M, L and T:

$$1 = \beta + \gamma; \quad 3 = \alpha - 3\beta - \gamma; \quad -2 = -2\gamma.$$

This gives $\alpha = 4$, $\beta = 0$, $\gamma = 1$, and hence $c \propto \dfrac{\eta a^4}{l}$.

iii. How does the orbital period P of a planet depend on the radius of its orbit, the mass M of the Sun, and the gravitational constant G?

Assume $$P \propto G^{\alpha}M^{\beta}a^{\gamma}$$

It is left to the reader to show that $P \propto \sqrt{\dfrac{a^3}{GM}}$.

iv. A sphere of radius a moves slowly at a speed υ through a fluid of density ρ and dynamic viscosity η. How does the viscous drag F depend upon these four variables?

Four variables, but only three dimensions, and hence three equations! What to do?
If you have better insight than I have, or if you already know the answer, you can assume that it doesn't depend upon the density. I haven't got such clear insight, but I'd be willing to suppose that the viscous drag is proportional to the first power of the dynamic viscosity. In which case I'd be happy to assume that

$$\frac{F}{\eta} \propto a^{\alpha}\rho^{\beta}\upsilon^{\gamma}$$

Then $$\frac{\mathrm{MLT^{-2}}}{\mathrm{ML^{-1}T^{-1}}} \equiv \mathrm{L}^{\alpha}(\mathrm{ML^{-3}})^{\beta}(\mathrm{LT^{-1}})^{\gamma}.$$

Equate the powers of M, L and T:

$$0 = \beta\,; \qquad 2 = \alpha - 3\beta + \gamma; \qquad -1 = -\gamma.$$

This gives $\alpha = 1$, $\beta = 0$, $\gamma = 1$, and hence $F \propto \eta a v$.

22.5 *Dimensionless Quantities*

are used extensively in fluid dynamics. For example, if a body of some difficult shape, such as an aircraft, is moving through a fluid at speed V, it will experience all sorts of forces, external and internal. The ratio of the internal forces to the external forces will depend upon its speed, and the viscosity of the fluid, and the size of the body. By "size" of a body of "difficult" shape we could take the distance between two defined points on the body, such as its top and bottom, or its front and back, or its greatest width, or whatever. Call that distance l. But the ratio of the internal to the viscous forces is dimensionless, so it must depend on some combination of the viscosity, speed V and linear size l that is dimensionless. Since V and l do not contain M in their dimensions, the viscosity concerned must be the *kinematic* viscosity ν, which is the ratio of dynamic viscosity to density and does not have M in its dimensions. So, what combination of ν, V and l is dimensionless?

It is easy to see that $\dfrac{Vl}{\nu}$ - or any power of it, positive, negative, zero, integral, nonintegral - is dimensionless. $\dfrac{Vl}{\nu}$ is called the Reynolds number, and is usually given the symbol Re. It is supposed that if you make a small model of the aircraft (or whatever the body is) and move it through some fluid and some speed, the ratio of internal to viscous forces in the model will be the same as in the real thing provided that the Reynolds numbers in the model and in the real thing are the same.

There are oodles of similar dimensionless numbers used in fluid dynamics, such as Froude's number and Mach number, but this example of Reynolds number should give the general idea.

22.6 *Different Fundamental Quantities*

We stated at the beginning of this chapter that any mechanical quantity could be expressed in terms of three fundamental quantities, *mass*, *length* and *time*. But there is nothing particularly magic about these quantities. For example, we might decide that we could express any mechanical quantity in terms of, say, *energy* E, *speed* V and *angular momentum* J. We might then say that the dimensions of area could be expressed as $E^{-2}V^2J^2$. (Verify this!)

While agreeing that such a system might be possible, you might feel that it would be totally absurd and there is no point in reading further.

But stop! Such a system is not only possible, but it is *normally and routinely used* in the field of *high-energy particle physics*. That, perhaps, is a surprise, but, if you are thinking of taking an interest in particle physics, read on.

The *units* generally used in particle physics to express the fundamental quantities energy, speed and angular momentum are GeV (or MeV, or TeV , etc) for energy, the speed of light c for speed, and the modified Planck constant $\hbar$ for angular momentum. There are often referred to as "natural" units, the speed of light being a "natural" unit of speed and $\hbar$ being a "natural" unit for angular momentum, whereas metre, kilogram and second are not so "natural" in this sense as they are "man-made". It is true that a GeV is not particularly "natural", but at least a system with GeV, c and $\hbar$ as fundamental quantities is certainly more "natural" than metre-kilogram-second.

In any case, the dimensions of *mass* in this system are EV^{-2}. (You can see this immediately, for example from Einstein's famous equation $E = mc^2$.) The units used in this system are GeV/c^2. Thus the rest mass of a proton is 0.9383 GeV/c^2, and the rest mass of an electron is 0.5110 MeV/c^2. One way to interpret this, if you like, is to say that the rest-mass energy of a proton (i.e. its m_0c^2) is 0.9383 GeV.

Likewise the dimensions of linear momentum are EV^{-1}, and units in which it is expressed are GeV/c. (You can see this, for example, if you look at the energy and momentum of a photon: $E = h\nu$, $p = h/\lambda$, from which $\frac{p}{E} = \frac{1}{\nu\lambda} = \frac{1}{c}$.)

Torque (which has the same dimensions as energy) is equal to rate of change of angular momentum, from which we see that time has dimensions $E^{-1}J$ and could be expressed in units of $\hbar$/GeV. Alternatively you can see that [time] = $\hbar$/GeV immediately from Planck's equation . $E = \hbar\omega$ And speed is distance over time, so that we see that distance, or length, has dimensions $E^{-1}VJ$, and hence units $\hbar c$/GeV.

Using data from the 2010 Particle Physics Booklet, I calculate as follows.

Mass: 1 GeV/c^2 = $1.782\,661\,76 \times 10^{-27}$ kg

Length: 1 $\hbar c$/GeV = $1.973\,269\,63 \times 10^{-16}$ m

Time: 1 $\hbar$/GeV = $6.582\,118\,99 \times 10^{-26}$ s

Energy: 1 GeV = $1.602\,176\,49 \times 10^{-10}$ J

Linear Momentum 1 GeV/c = $5.344\,285\,50 \times 10^{-19}$ kg m s^{-1}

I give here a table of the dimensions (in terms of EVJ) of the same quantities as in the table of page 2. I dare say some of them are never likely to be needed, but some certainly will be needed, and, rather than predict which will be useful and which not, I might as well give them all. The dynamic viscosity of water at room temperature is about 10^{-3} kg m^{-1} s^{-1}, or 10^{-3} dekapoise. I cannot imagine anyone needing to know that the dynamic viscosity of water at room temperature is about 7.3×10^{-18} (GeV)^3/($c^3\hbar^2$) , or that its surface tension is so many (GeV)^3/($c\hbar$)^2 – but you never know.

Mass	EV^{-2}	GeV/c^2
Length	$\mathrm{E}^{-1}\mathrm{VJ}$	$\hbar c$/GeV
Time	$\mathrm{E}^{-1}\mathrm{J}$	$\hbar$/GeV
Density	$\mathrm{E}^2\mathrm{V}^{-5}\mathrm{J}^{-3}$	(GeV)^2/($c^5\hbar^3$)
Speed	V	c
Acceleration	EVJ^{-1}	GeV $c/\hbar$
Force	$\mathrm{E}^2\mathrm{V}^{-1}\mathrm{J}^{-1}$	(GeV)^2/($c\hbar$)
Work, Energy, Torque	E	GeV
Action	J	$\hbar$
Rotational inertia	$\mathrm{E}^{-1}\mathrm{J}^2$	$\hbar^2$/GeV
Angular speed	EJ	GeV $\hbar$
Angular acceleration	$\mathrm{E}^2\mathrm{J}^2$	(GeV)$^2\hbar^2$
Angular momentum	J	$\hbar$
Pressure, elastic modulus	$\mathrm{E}^4\mathrm{V}^{-3}\mathrm{J}^{-3}$	(GeV)^4/($c\hbar$)^3
Gravitational constant	$\mathrm{E}^{-2}\mathrm{V}^5\mathrm{J}$	$c^5\hbar$/(GeV)^2
Dynamic viscosity	$\mathrm{E}^3\mathrm{V}^{-3}\mathrm{J}^{-2}$	(GeV)^3/($c^3\hbar^2$)
Kinematic viscosity	$\mathrm{E}^{-1}\mathrm{V}^2\mathrm{J}$	$c^2\hbar$/(GeV)
Force constant	$\mathrm{E}^3\mathrm{V}^{-2}\mathrm{J}^{-2}$	(GeV)^3/($c\hbar$)^2
Torsion constant	E	GeV
Surface tension	$\mathrm{E}^3\mathrm{V}^{-2}\mathrm{J}^{-2}$	(GeV)^3/($c\hbar$)^2
Schrödinger wavefunction Ψ	$\mathrm{E}^2\mathrm{V}^{-3/2}\mathrm{J}^{-2}$	(GeV)^2/($c^{3/2}\hbar^2$)
Schrödinger wavefunction ψ	$\mathrm{E}^{3/2}\mathrm{V}^{-3/2}\mathrm{J}^{-3/2}$	(GeV)$^{3/2}$/($c^{3/2}\hbar^{3/2}$)