CHAPTER 21
CENTRAL FORCES AND EQUIVALENT POTENTIAL

21.1 *Introduction*

When a particle is in orbit around a point under the influence of a central attractive force (i.e. a force $F(r)$ which is directed towards a central point, with no transverse component) it experiences, *when referred to an inertial reference frame*, a centripetal acceleration. If, however, the system is described *with respect to a co-rotating reference frame*, there is no centripetal acceleration; rather, it appears as though an additional force, the centrifugal force, is pushing it away from the centre of attraction. In the co-rotating frame, this force depends only on the distance of the particle from the centre of attraction, and it is therefore a conservative force – and, like any conservative force, it can be described by the negative of the derivative of a potential energy function.   When describing the motion with respect to the co-rotating frame, we must add this potential to any additional "real" potentials (such as originate from the gravitational fields of other bodies), to form an equivalent potential which constrains the motion of the particle. An excellent example of this method is the analysis of the restricted three-body problem given in some detail in Chapter 16 of my notes on Celestial Mechanics (http://orca.phys.uvic.ca/~tatum/celmechs.html). But I deal first, by way of example, with some simpler problems involving central forces, in which we shall be able, by simple arguments, to deduce some basic characteristics of the motion.

21.2 *Motion Under a Central Force*

I consider the two-dimensional motion of a particle of mass *m* under the influence of a conservative central force *F(r)*, which can be either attractive or repulsive, but depends only on the radial coordinate *r*. Recalling the formula $\ddot{r} - r\dot{\theta}^2$ for acceleration in polar coordinates (the second term being the centripetal acceleration), we see that the equation of motion is

$$m\ddot{r} - mr\dot{\theta}^2 \; = \; F(r). \qquad\qquad 21.2.1$$

This describes, in polar coordinates, two-dimensional motion in a plane. But since there are no transverse forces, the angular momentum $mr^2\dot{\theta}$ is constant and equal to *L*, say. Thus we can write equation 21.2.1 as

$$m\ddot{r} \; = \; F(r) \; + \; \frac{L^2}{mr^3}. \qquad\qquad 21.2.2$$

This has reduced it to a one-dimensional equation; that is, we are describing, relative to a co-rotating frame, how the distance of the particle from the centre of attraction (or repulsion) varies with time. In this co-rotating frame it is as if the particle were subject

not only to the force $F(r)$, but also to an additional force $\dfrac{L^2}{mr^3}$. In other words the total force on the particle (referred to the co-rotating frame) is

$$F'(r) = F(r) + \frac{L^2}{mr^3}.$$ 21.2.3

Now $F(r)$, being a conservative force, can be written as minus the derivative of a potential energy function, $F = -\dfrac{dV}{dr}$. Likewise, $\dfrac{L^2}{mr^3}$ is minus the derivative of $\dfrac{L^2}{2mr^2}$. Thus, in the co-rotating frame, the motion of the particle can be described as constrained by the potential energy function V', where

$$V' = V + \frac{L^2}{2mr^2}.$$ 21.2.4

This is the *equivalent potential energy*. If we divide both sides by the mass m of the orbiting particle, this becomes

$$\Phi' = \Phi + \frac{h^2}{2r^2}.$$ 21.2.5

Here h is the angular momentum per unit mass of the orbiting particle, Φ is the potential in the inertial frame, and Φ' is the *equivalent potential* in the corotating frame.

21.3 *Inverse Square Attractive Force*

This is dealt with in detail in Chapter 9 of my notes on Celestial Mechanics (http://orca.phys.uvic.ca/~tatum/celmechs.html). Here we investigate some general properties of the motion.

If $F = -\dfrac{GMm}{r^2}$ then $V = -\dfrac{GMm}{r}$, and hence

$$V' = -\frac{GMm}{r} + \frac{L^2}{2mr^2}.$$ 21.3.1

I sketch this in figure XXI.1. The total energy (potential + kinetic) is constant (independent of r) and is greater than (or equal to) the potential energy. If the total energy is less than zero, you can see from the graph that r has a lower (perihelion) and upper (aphelion) limit; this corresponds to an elliptic orbit. But if the total energy is

positive, r has a lower limit, but no upper limit; this corresponds to a hyperbolic orbit. If the total energy is equal to the minimum of V', only one value of r is possible, and the orbit is a circle.
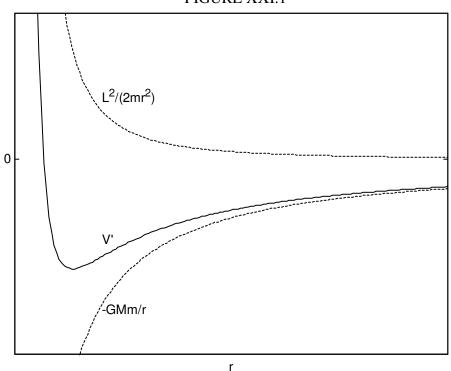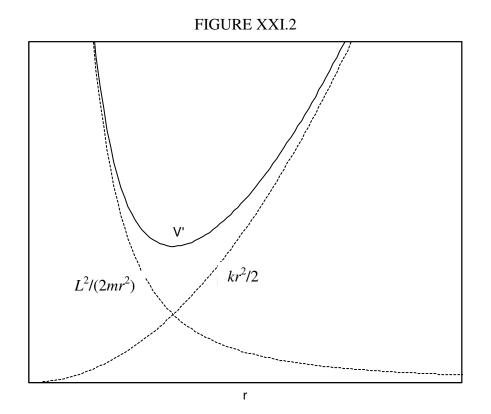
FIGURE XXI.1

21.4 *Hooke's Law*

We imagine a particle whirling around on the end of a spring, oscillating in and out as it does so. The force constant of the spring is k, the force on the particle is $-kr$ and the potential (elastic) energy is $V = \frac{1}{2}kr^2$. The effective potential energy is therefore

$$V' = \tfrac{1}{2}kr^2 + \frac{L^2}{2mr^2}.$$ 21.4.1

I sketch this in figure XVI.2.   The total energy (potential + kinetic) is constant (independent of r) and is greater than (or equal to) the potential energy. The distance of the particle from the centre of attraction is bounded above and below. The motion is a Lissajous ellipse, with the centre of attraction at the centre (not the focus) of the ellipse. The lower bound is the semi minor axis and the upper bound is the semi major axis.
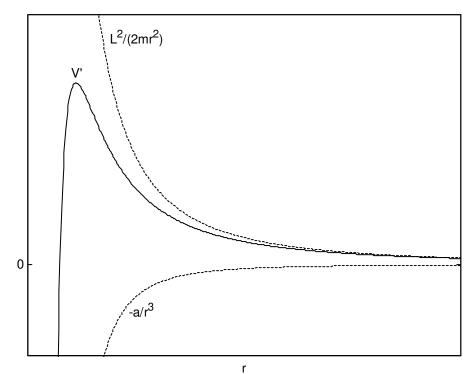
FIGURE XXI.2

An inverse square force (e.g. a gravitational force, or a Coulomb's law electrostatic force) and a Hooke's law force (*kx*) are obvious examples of real forces in nature. In what follows we shall investigate the behaviour of a particle under the influence of other force laws, such as inverse fourth power and inverse cube forces. It is difficult to imagine whether such forces actually exist in nature (the field of an electric dipole falls off as the cube of the distance - but the field is not radial, and the force is not a central force), and to that extent much of what follows is an exercise in mathematics more than in physics. But inverse square and Hooke's law forces are certainly not the only forces to operate in nature. What is the force law, for example, for the residual strong interactions between nucleons in an atomic nucleus, or the force law between the quarks within a nucleon? It will be worthwhile investigating the simpler hypothetical forces to be discussed here in order to understand the principles and methods that may be applicable to a more difficult problem.

21.5 *Inverse fourth power attractive force*

If $F = -\dfrac{3a}{r^4}$ then $V = -\dfrac{a}{r^3}$, and hence

$$V' = -\frac{a}{r^3} + \frac{L^2}{2mr^2}.$$ \hfill 21.5.1

I sketch this in figure XXI.3.   The total energy (potential + kinetic) is constant (independent of r) and is greater than (or equal to) the potential energy. If the total energy is negative, the distance r has an upper limit, but the only lower limit is the origin, or the centre of attraction, and particle will eventually end there. If the total energy is greater than the maximum of V', the motion is completely unbounded. If the total energy is positive but less than V'_{max}, the motion depends on the initial value of r. For small r the motion is bounded above, and the particle will eventually end at the origin. For large r, there is a minimum distance to which the particle can approach the origin, and the particle will eventually wander off to infinity. For total energy in this range, there is a range of r that is not possible.

FIGURE XXI.3

21.6 *A general central force*

Let us suppose that we have a particle that is moving under the influence of a central force $F(r)$. The equations of motion are

Radial: $\qquad\qquad\qquad\qquad m(\ddot{r} - r\dot{\theta}^2) = F(r)$ $\qquad\qquad\qquad\qquad$ 21.6.2

Transverse: $\qquad\qquad\qquad\qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$ $\qquad\qquad\qquad\qquad$ 21.6.3

These can also be written

$$\ddot{r} - r\dot{\theta}^2 = a(r) \qquad\qquad\qquad\qquad 21.6.4$$

$$r^2\dot{\theta} = h. \qquad\qquad\qquad\qquad 21.6.5$$

Here a is the radial force per unit mass (i.e. the radial acceleration) and h is the (constant) angular momentum per unit mass. [If you are unsure of why equations 21.6.3 and 21.6.5 are the same, differentiate equation 21.6.5 with respect to time.]

These are two simultaneous equations in r, θ, t. In principle, if we could eliminate t between them, we would obtain a relation between r and θ, which would tell us the shape of the path pursued by the particle. In Chapter 9 of my Celestial Mechanics notes we do this for the gravitational case, and we find that the path is an ellipse of the form $r = \dfrac{l}{1 + e\cos\theta}$. Or perhaps we could eliminate r and hence find out how the angle θ changes with time. Or again we might be able to eliminate θ and hence get a relation telling us how r varies with the time. Yet again we might be told the shape of the path $r(\theta)$, and asked to find the force law $F(r)$. Or again, rather than the force, we might be given the form of the potential energy $V(r)$, which is related to the force by $F = -dV/dr$. The *potential* Φ is the potential energy per unit mass, and $-d\Phi/dr$ is the radial force per unit mass - i.e. it is the radial acceleration $a(r)$ of the orbiting particle. The angular momentum of the particle, which is constant, is $L = mr^2\dot{\theta}$, and the angular momentum per unit mass is $h = r^2\dot{\theta}$, which is twice the rate at which the radius vector sweeps out area.

We might also remember that, if we are given the potential energy V or the potential Φ in an inertial frame, we might also want to work in a co-rotating frame, making use of the equivalent potential energy $V' = V + \dfrac{L^2}{2mr^2}$ or the equivalent potential

$$\Phi' = \Phi + \frac{h^2}{2r^2}.$$

One last thing to bear in mind before starting any problems of this class. It turns out that, very often, a change of variable $u = 1/r$ turns out to be useful. Conservation of angular momentum then takes the form $\dot{\theta}/u^2 = h$. Also

$$\dot{r} = \frac{dr}{du}\frac{du}{dt} = \frac{dr}{du}\frac{d\theta}{dt}\frac{du}{d\theta} = -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta} = -h\frac{du}{d\theta} \qquad 21.6.6$$

and $\qquad \ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = -h\frac{d}{dt}\frac{du}{d\theta} = -h\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta} = -h.hu^2.\frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2}.$

$$21.6.7$$

Equations 21.5.4 and 21.6.5 now become

$$h^2u^2\frac{d^2u}{dt^2} + \frac{1}{u}\dot{\theta}^2 = -a(r) \qquad 21.6.8$$

and $\qquad\qquad\qquad\qquad \dot{\theta} = hu^2. \qquad\qquad\qquad\qquad 21.6.9$

We can now easily eliminate the time which was one of our aims:

$$h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = -a(r). \qquad 21.6.10$$

[As ever, check the dimensions.] This equation, which does not contain the time, when integrated will give us the (r,θ) equation to the path.

With these remarks in mind, let us try a few problems. For example:

21.7 *Inverse cube attractive force*

A particle moves in a field such that the attractive force on it varies inversely as the cube of the distance from a centre of attraction.   What is the shape of the path? How does the angle θ vary with time?

Let's suppose that the radial acceleration is $a(r) = -k^2/r^3 = -k^2u^3$.   (I want the coefficient of $1/r^3$ to be negative, so that the force is attractive, which is why I have written the coefficient as $-k^2$.   Besides, the dimensions of k are then L^2T^{-1}, which are the same as those of h, the angular momentum per unit mass, which helps to make the algebra simple.) The differential equation to the path (equation 21.6.10) is then

$$h^2 u^2 \frac{d^2 u}{d\theta^2} + h^2 u^3 = k^2 u^3 \quad \text{or}$$

$$h^2 \frac{d^2 u}{d\theta^2} + h^2 u = k^3 u. \qquad \text{21.7.11}$$

That is,
$$\frac{d^2 u}{d\theta^2} = \frac{k^2 - h^2}{h^2} u. \qquad \text{21.7.12}$$

The form of the motion evidently depends on whether $k^2 > h^2$ (a strongly attractive force, or a small angular momentum), or if $k^2 < h^2$ (a weak force, or a large angular momentum.) If we start the particle rolling with just the right amount of angular momentum ($k^2 = h^2$), there will evidently be zero radial acceleration, and the particle will move in a circle.

Before integrating equation 21.7.12, let us look at the *equivalent potential*. For $a(r) = -k^2/r^3$, the potential in the inertial frame is $\Phi = -\frac{1}{2}k^2/r^2$ provided we take the potential at infinity to be zero. The equivalent potential is then (see equation 21.2.5)

$$\Phi' = -\frac{k^2}{2r^2} + \frac{h^2}{2r^2}. \qquad \text{21.7.13}$$

We see that, if $k^2 = h^2$, the potential is zero and independent of distance. If $h^2 < k^2$, the equivalent potential is negative, increasing to zero as $r \to \infty$, and the particle accelerates towards the centre of attraction. If $h^2 > k^2$, the potential is positive, decreasing to zero as $r \to \infty$, and the particle accelerates away from the centre of attraction. This sounds like a contradiction, but what is happening is that $h^2 > k^2$ means that the particle has initially been given a large angular momentum, and, in the corotating frame, the centrifugal force is larger than the attractive force.

If $h^2 < k^2$, the equation of motion (equation 21.7.12) is

$$\frac{d^2 u}{d\theta^2} = c^2 u, \qquad \text{21.7.14}$$

where
$$c^2 = \frac{k^2 - h^2}{h^2}. \qquad \text{21.7.15}$$

The general solution is

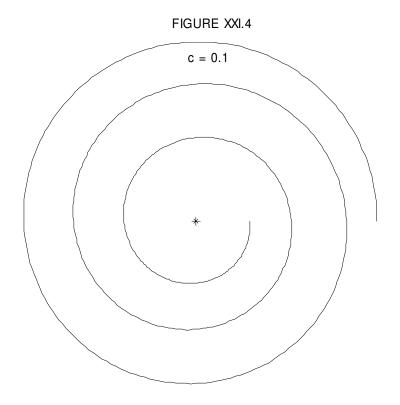$$u = Ae^{c\theta} + Be^{-c\theta} \qquad\qquad 21.7.16$$

If the initial conditions are that at $t = 0$, $r = r_0$, $u = u_0$, $\dfrac{du}{d\theta} = 0$ (this last condition means that the particle was launched in a direction at right angles to the radius vector, this solution becomes
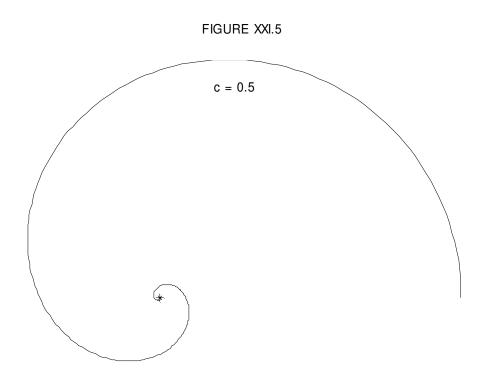
$$u = u_0 \cosh c\theta. \qquad\qquad 21.7.17$$

That is, $\qquad\qquad\qquad\qquad r = r_0 \operatorname{sech} c\theta. \qquad\qquad 21.7.18$

I have drawn this below for $c = 0.1$; that is, for $k \approx 1.05h$. And for $c = 0.5$; that is, for $k \approx 1.22h$, a smaller angular momentum.

FIGURE XXI.4

c = 0.1

FIGURE XXI.5

c = 0.5

We also need to consider the case $h^2 > k^2$, in which case the general solutuion is of the form $u = A\cos c\theta + B\sin c\theta$. Alas, I haven't had the energy to do this yet. Perhaps some view can beat me to it, and let me know at jtatum at uvic.ca