

CHAPTER 20 MISCELLANEA

20.1 *Introduction*

This chapter is a miscellany of diverse and unrelated topics – namely surface tension, shear modulus and viscosity – discussed only for the purpose of presenting a few more examples of elementary problems in mechanics. It is not intended in any way to substitute for a comprehensive course in any of the vast and interesting fields of surface chemistry, elasticity or hydrodynamics. All of these subjects have a huge and specialized literature, each worthy of a full-length course, and I am not remotely competent to offer one. Nevertheless, the few simple problems chosen in this chapter are suitable for a bit more practice in classical mechanics.

20.2 *Surface Tension*

The cause of surface tension is often explained roughly as follows. Molecules within a liquid are subject to intermolecular forces whose exact nature and origin need not concern us other than to say that they are principally van der Waals forces and they hold the liquid together and prevent it from evaporating. A molecule deep within the liquid is surrounded in all directions by other molecules, and so the net force on it averages zero. But a molecule on the surface experiences forces from beneath the surface, and consequently it tends to get dragged beneath the surface. This results in as few molecules as possible remaining on the surface; i.e. it results in the surface contracting to as small an area as possible consistent with whatever other geometrical constraints may exist. That is, the surface appears to be in a state of tension causing it to contract to the least possible area.

This tension can be described qualitatively thus. In figure XX.1, the dashed line is an imaginary



FIGURE XX.1

line drawn in the surface of a liquid. The liquid to the left of the line is being pulled to the right as indicated by the red arrows; the liquid to the right of the line is being pulled equally to the left as indicated by the green arrows. The force per unit length perpendicular to a line drawn in the

surface of the liquid is the *surface tension*. Its SI unit is newtons per metre, and its CGS unit is dynes per centimetre. The dimensions are MT^{-2} .

I have seen various symbols, such as T , S and γ used for surface tension. The first two of these symbols are already heavily worked in thermodynamics, so I shall use the symbol γ (although, it must be admitted, γ is heavily worked in thermodynamics, too.)

Not everyone is comfortable with a definition involving forces perpendicular to an imaginary line drawn in the surface, and an alternative approach may be more palatable to some. The idea of a molecule beneath the surface being surrounded on all sides by other molecules and hence experiencing zero net average force, while a molecule on the surface is pulled asymmetrically by the molecules beneath it, remains. But instead of drawing an imaginary line on the surface, we reason that it requires *work* to move a molecule from within the liquid to the surface, and it requires a lot of work to move many molecules from beneath to the surface. That is, it requires work to create new surface. Thus we can define surface tension as the *work required to create unit area of new surface*. The conditions under which this work is done have to be carefully defined in any precise definition, and, from a thermodynamical point of view, the strict definition is the increase in the Gibbs free energy per unit area of new surface created under conditions of constant temperature and pressure. That is:

$$\gamma = \left(\frac{\partial G}{\partial A} \right)_{T,P} \quad 20.2.1$$

This is consistent with the definition of the Gibbs free energy as a quantity whose increase is equal to the work, other than PdV work, done on a system in a reversible, isothermal isobaric process.

Such a nicety will be of interest to those versed in thermodynamics (and I have added a bit about the thermodynamics of surface energy in Chapter 12 of Thermodynamics), but for those not so versed, you may, without any serious prejudice to understanding most of the matter in this section, think of surface tension either as the force per unit length perpendicular to an imaginary line in the surface, or as the work required to create unit area of new surface. You may express surface tension either in newtons per metre or in joules per square metre (or, if you are of CGS persuasion, dynes per centimetre or ergs per square centimetre). These are dimensionally equivalent.

20.2.1 *Excess Pressure Inside Drops and Bubbles*

The pressure inside a spherical drop is greater than the pressure outside. The way in which the excess pressure P depends on the radius a of the drop, and the surface tension γ and density ρ of the liquid is amenable to dimensional analysis. One can suppose that $P \propto a^\alpha \gamma^\beta \rho^\delta$, after which I leave it to the reader to show that $\alpha = -1$, $\beta = 1$, $\delta = 0$, and therefore $P \propto \gamma/a$.

However, it is also quite easy to calculate the excess pressure (other than as a mere proportionality) in terms of the surface tension and the radius of the drop. In figure XX.2 I have divided a spherical

drop of radius a into two hemispheres, and we are going to consider the equilibrium of the upper hemisphere.

The upper hemisphere is being pulled down by surface tension all round the base of the hemisphere, and this downward force is equal to the circumference of the base times the surface tension, or $2\pi\gamma a$.

If the excess pressure inside the drop is P , the upward component of the force due to this pressure is equal to P times the area of the base, πa^2 . In case this is not obvious, consider an elemental area dA

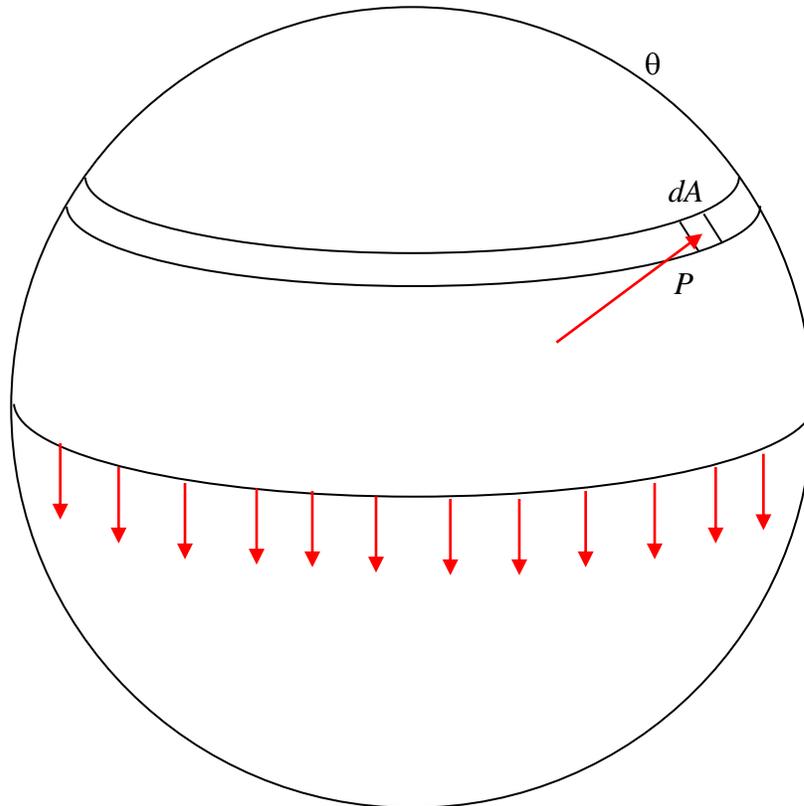


FIGURE XX.2

as shown, at a spherical angle θ from the top of the drop. The force on this element is equal to PdA . The upward component of this force is $P \cos \theta dA$, and this is equal to P times the horizontal projection of dA . Now you are welcome to do a nice double integration over the hemisphere, but since this (i.e. "this is equal to P times the horizontal projection of dA ") is true for every elemental area over the surface of the hemisphere, the total upward force must be equal to P times the area of the base. Thus $2\pi\gamma a = \pi a^2 P$, and so the excess pressure inside the drop is

$$P = \frac{2\gamma}{a} . \quad 20.2.2$$

The smaller the drop, the greater the excess pressure. You may regard this as an explanation as to why droplets cannot form from a vapour unless there is a dust nucleus of finite size for them to condense upon. Of course, two molecules colliding with each other cannot in any case coalesce unless there is something to remove or absorb the kinetic energy.

The case of a nonspherical drop might be mentioned in passing. It is a well known result in geometry (or at least it is well known to those who already know it) that if $z = z(x, y)$ is a nonspherical surface, and you take two vertical planes at right angles to each other, and if a_1 and a_2 are the radii of curvature of the intersections of the two planes with the surface, then $\frac{1}{a_1} + \frac{1}{a_2}$ is independent of the orientations of the two planes, as long as they remain perpendicular to each other. In other words, a_1 and a_2 do not have to be the maximum and minimum radii of curvature. The excess pressure inside a nonspherical drop is

$$P = \gamma \left(\frac{1}{a_1} + \frac{1}{a_2} \right). \quad 20.2.3$$

What about the pressure inside a spherical bubble of air (or other gas) under water (or other liquid)? If we are hasty, we might suggest that, since this is the opposite situation to a liquid drop in air, maybe the pressure is *less* inside an underwater bubble. This would be a very hasty conclusion, and quite wrong. If you go through exactly the same argument as we did for a drop, considering the equilibrium of one hemisphere, you will see immediately that there is (as for the drop) an *excess* pressure inside the bubble given again by equation 20.2.2. And exactly the same would apply to a spherical drop of one liquid under the surface of a second liquid, if the two liquid are immiscible. But, rather than just repeat the identical derivation, let's try a different approach.

Let us imagine that we have a bubble of radius a in a liquid of surface tension γ , and suppose that we are able, by means of a fine syringe, to inject some more air inside so as to increase the radius of the bubble by da at constant pressure and temperature. The surface area of a sphere of radius a is $A = 4\pi a^2$, so, if we increase the radius by da we increase the surface area by $8\pi a da$, and we increase the volume by $4\pi a^2 da$. The work done against the surface tension is $8\pi\gamma a da$, and this must also be equal to $4\pi P a^2 da$, where P is the excess pressure inside the bubble. Equating these two expressions leads again to equation 20.2.2.

What about a hollow spherical soap bubble in air? Here the soap has *two* surfaces – inside and out. If you repeat either of the above derivations to this case, you will see that the excess pressure inside a hollow spherical soap bubble is

$$P = \frac{4\gamma}{a}. \quad 20.2.4$$

20.2.2 Angle of Contact

When a static quantity of liquid is in contact with an impermeable solid surface, it generally rests so that there is a characteristic angle (measured in the liquid) between the surface of the liquid and the surface of the solid. This angle is the *angle of contact*, and is shown as the angle θ in figure XX.3. Figure XX.3(a) shows an acute angle of contact, in which the liquid spreads out a little and "wets" the surface. Figure XX.3(b) shows an obtuse angle of contact, in which the liquid "bunches up", and does not wet the surface, rather like drops of mercury on most surfaces, or drops of water on the surface of a car that has been freshly waxed. In many cases the angle of contact is close to either 0° or 180° , although it will be appreciated that if θ were *exactly* zero, the liquid would spread



FIGURE XX.3

out in an infinitesimally thin layer to cover or "wet" the entire surface; and if it were *exactly* 180° , the liquid, in the absence of other forces (such as its weight!), would form a spherical globule in contact with the surface only at a single point.

The angle of contact is determined by the nature of *both* surfaces, and is very sensitive to any surface contamination. In order to wet a surface, water may need to be helped by a small amount of wetting agent or detergent; only a small amount is necessary, because only the surface, and not the bulk, of the liquid is involved. The *chemical* nature of wetting agents and detergents is beyond the scope of these notes (i.e. it is beyond *my* scope!), but how the angle of contact depends on the surface tension provides a useful example of the technique of *virtual work* (see Section 9.4 of Chapter 9).

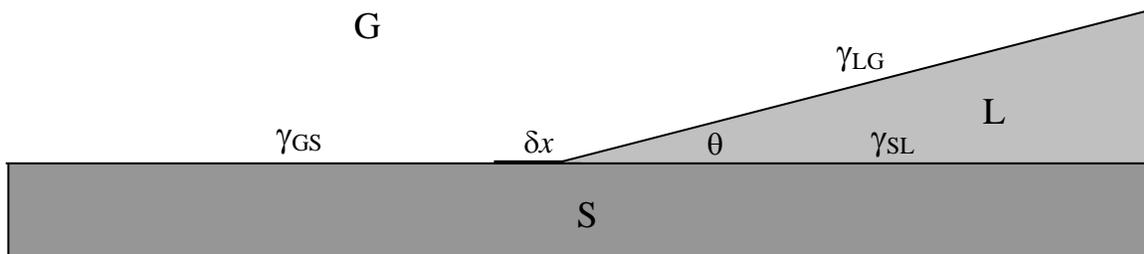


FIGURE XX.4

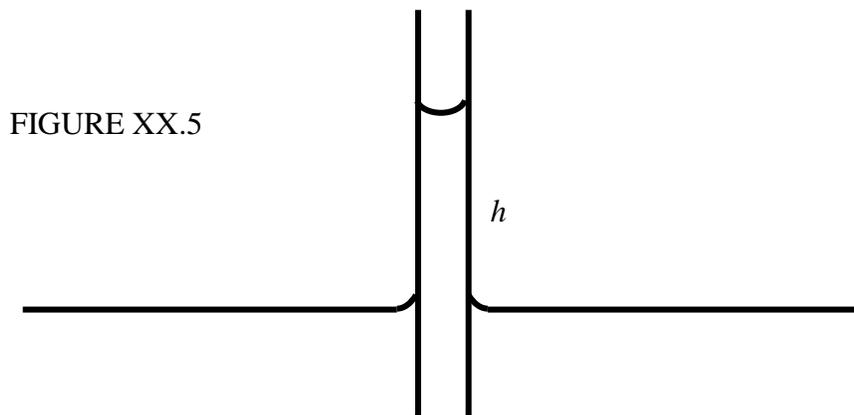
Figure XX.4 represents a liquid, L, (e.g. water) in contact with a solid, S, (e.g. glass) and a gas, G, (e.g. air). The angle of contact is θ , and the surface tensions (energy per unit area, or, for those who are versed in thermodynamics, the Gibbs free energy per unit area) of the three interfaces are as shown. We'll suppose that the three media extend for a distance l at right angles to the plane of the paper (or computer screen). The three phases are in equilibrium. Now, if we move the SLG boundary to the left by a distance δx , we create a new area $l\delta x$ of SL interface and a new area $l\cos\theta\delta x$ of LG interface, while we lose an area $l\delta x$ of GS interface. The work done on the system is therefore $\gamma_{SL}l\delta x + \gamma_{LG}l\cos\theta\delta x - \gamma_{GS}l\delta x$. By the principle of virtual work, this is zero, and therefore

$$\cos\theta = \frac{\gamma_{GS} - \gamma_{SL}}{\gamma_{LG}}. \quad 20.2.5$$

The angle of contact is acute or obtuse, according to whether γ_{GS} is greater than or less than γ_{SL} .

20.2.3 Capillary Rise

When the lower end of a narrow capillary tube is immersed in a liquid, the liquid inside the tube rises a little above the level of the liquid outside. It is then very simple to calculate how far the liquid rises in terms of the surface tension, the angle of contact and the inside radius of the tube. See figure XX.5.



The upward force due to surface tension is $2\pi a\gamma\cos\theta$, where a is the inside radius of the tube, and, if we neglect the very small mass of the liquid in the meniscus (the curved surface at the top of the liquid column), the weight of the liquid column is $\pi a^2 h \rho g$, and therefore

$$h = \frac{2\gamma\cos\theta}{\rho g a}. \quad 20.2.6$$

(Check the dimensions!)

Of course if θ is *obtuse* (as with mercury in contact with glass), h will be *negative*, and the level of the mercury in the tube will be *below* the outside level.

20.3 Shear Modulus and Torsion Constant

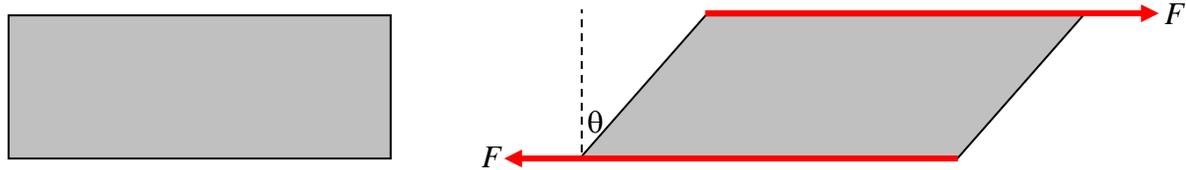


FIGURE XX.6

Imagine that we have a rectangular block of solid material, as shown on the left hand side of figure XX.6. We now apply a couple of tangential forces F as shown on the right hand side. (I have not decided to go all chatty and informal by saying “a couple” of forces; far from it – I am using the word “couple” in its formal sense in mechanics.) The material will undergo an angular deformation, and the ratio of the tangential force per unit area to the resulting angular deformation is called the *shear modulus* or the *rigidity modulus*. Its SI unit is $\text{N m}^{-2} \text{rad}^{-1}$ and its dimensions are $\text{ML}^{-1}\text{T}^{-2}\theta^{-1}$. (I’d advise against using “pascals” per radian. The unit “pascal” is best restricted to pressure, which is normal force per unit area, and is not quite the same thing as the tangential force per unit area that we are discussing here.) You should convince yourself that the definition must specify the force F , not the torque provided by the couple. If the block were twice as thick, and the forces were the same, you’d still get the same angular deformation.

If you hold one end of a wire or rod fixed and apply a torque to the other end, this end will twist through an angle, and the ratio of the applied torque to the angle through which the wire twists is the *torsion constant*, c , of the wire. You can see how the torsion constant depends on the shear modulus η of the metal and the radius a and length l of the wire by the method of dimensions. You can start by supposing that $c \propto \eta^\alpha a^\beta l^\gamma$, but you will soon find yourself in difficulty because a and l are each of dimension L . However, you will probably have no difficulty with making the assumption that $\gamma = -1$ (the longer the wire, the easier it is to twist), and dimensional analysis will soon show that $\alpha = 1$ and $\beta = 4$ - which, being interpreted, meaneth that it is much more difficult to twist a thick wire than a thin wire. But can we do better and get an expression other than a mere proportionality for the torsion constant? Can we find the proportionality constant? Let’s try some simpler problems first, and see how things go.

Let us consider a long, thin strip or ribbon of metal. By long and thin I mean that its length is much greater than its width, and its width is much greater than its thickness. I can use any symbol I like to represent any quantity I like, so I could, if I wished, use Ξ for the length, m_α for the width, and G_2 for the thickness. Instead, the symbols that I shall choose to represent the length, width and thickness of the strip are going to be, respectively, l , $2\pi r$ and δr . This seems silly at the moment,

but in the end you'll be glad that I made this choice. The strip is shown at the left hand side of figure XX.7.

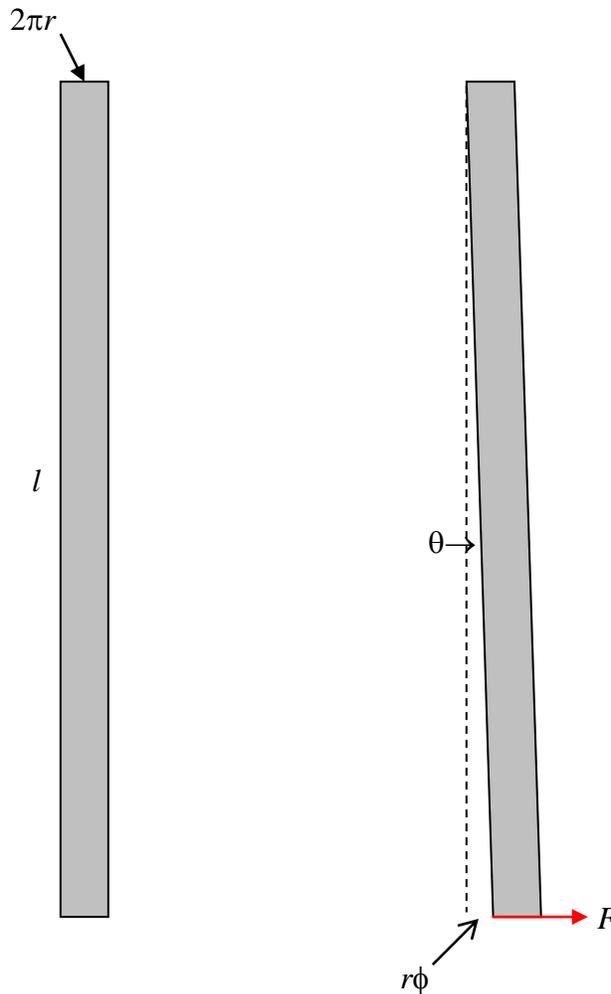


FIGURE XX.7

I am now going to fix the upper end of the strip and apply a force F to the lower end, as shown in the right hand side of figure XX.7, and I can use any symbol I like to represent the displacement of the lower end, and I choose the symbol $r\phi$. This means that the angular displacement θ is equal to $r\phi/l$. The tangential force per unit area is $F/(2\pi r \delta r)$, and therefore

$$\eta = \frac{F l}{2\pi\phi r^2 \delta r}, \quad 20.3.1$$

or

$$F = \frac{2\pi\eta\phi r^2 \delta r}{l}. \quad 20.3.2$$

Now I'm going to restore the strip to its original shape, and then I'm going to roll it into a hollow cylindrical tube, so that it now looks like a metal drinking straw. The circumference of the straw is $2\pi r$, its radius is r and its thickness is δr (figure XX.8). (Now my notation is beginning to make some sense!)

FIGURE XX.8



I shall hold the upper end of the tube fixed and I shall apply a torque $\tau = Fr$ to the lower end. The tube will evidently twist through an azimuthal angle ϕ given by

$$\tau = \frac{2\pi\eta r^3\delta r}{l}\phi. \quad 20.3.3$$

The torsion constant of the hollow tube is therefore

$$c = \frac{2\pi\eta r^3\delta r}{l}. \quad 20.3.4$$

The torsion constant of a long solid cylinder (a wire) of radius a is the integral of this from 0 to a , which is

$$c = \frac{\pi\eta a^4}{2l}. \quad 20.3.5$$

20.4 Viscosity

Consider a river flowing over a smooth bed, as in figure XX.9.

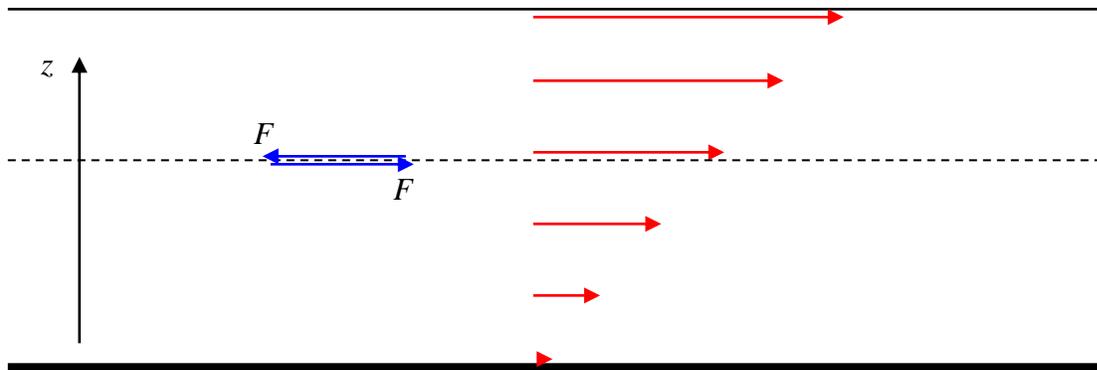


FIGURE XX.9

There will be a *transverse velocity gradient* dv/dz , with the liquid stationary at the river bottom, and the speed becoming faster as we ascend from the bottom. As a consequence of the transverse velocity gradient, the liquid below the dashed line will be dragged forward by the tangential force of the faster liquid above it, and the liquid above the dashed line will be dragged backward by the tangential force of the more sluggish liquid below it. The ratio of the tangential force per unit area to the transverse velocity gradient is called the coefficient of *dynamic viscosity*, for which the usual symbol is η . The dimensions of dynamic viscosity are $ML^{-1}T^{-1}$. The CGS unit of dynamic viscosity is the *poise*. The abbreviation for the unit is P – though it would be well to define it if you use it, since not everyone will recognize it. The unit is named after a nineteenth century French doctor, Jean Poiseuille, who was interested in blood pressure and hence in the rate of flow of liquids through tubes. That is to say, if, for a transverse velocity gradient of 1 cm s^{-1} per cm, the tangential force per unit area is 1 dyne cm^{-2} , the dynamic viscosity is one poise. The SI (MKS) unit is the decapoise (also spelled dekapoise), or equivalently and more often the pascal second (Pa s). If, for a transverse velocity gradient of 1 m s^{-1} per cm, the tangential force per unit area is 1 N m^{-2} , the dynamic viscosity is one decapoise or one pascal second,. The dynamic viscosity of water varies from about 1.8 centipoise at 0°C to about 0.3 centipoise at 100°C .

The ratio of the coefficient of dynamic viscosity to the density is the coefficient of *kinematic viscosity*, for which the usual symbol is the Greek letter ν . The dimensions of kinematic viscosity are L^2T^{-1} . The CGS unit of kinematic viscosity is the *stokes* (abbreviation St). It is named after nineteenth century British physicist, Sir George Stokes, who made major contributions to diverse areas of physics. The SI unit of kinematic viscosity is usually given simply as $m^2 s^{-1}$, and $1 m^2 s^{-1} = 10^4$ stokeses. The kinematic viscosity of water varies from about 1.8 centistokeses ($1.8 \times 10^{-6} m^2 s^{-1}$) at $0^\circ C$ to about 0.3 centistokeses ($3 \times 10^{-7} m^2 s^{-1}$) at $100^\circ C$.

Hydrodynamics is a huge and very difficult subject (at least I think it is), but there are a couple of simple problems that, if nothing else, make good homework problems. These are Poiseuille's law and the Couette viscometer.

20.4.1 Poiseuille's Law

Poiseuille's law tells you how the rate of nonturbulent flow of a liquid through a cylindrical pipe depends on the viscosity of the liquid, the radius of the pipe, and the pressure gradient. If all else fails, you can at least try dimensional analysis. Assume that the rate of flow of liquid (in cubic metres per second) is proportional to $\eta^\alpha a^\beta \left(\frac{dP}{dx}\right)^\gamma$, and show by dimensional analysis that $\alpha = -1$,

$\beta = -4$ and $\gamma = 1$, which shows that the rate of flow is very sensitive to the radius of the pipe. That $\beta = -4$ tells you that if your arteries are at all constricted, even by a little bit, you had better watch out. Gas flow is more complicated because gases are compressible, (so are liquids, but not by much), but $\beta = -4$ tells you that the rate at which you can pump out gas from a system depends a lot on the size of the smallest tube you have between the volume that you are trying to evacuate and the pump. Now let's try and analyse it further.

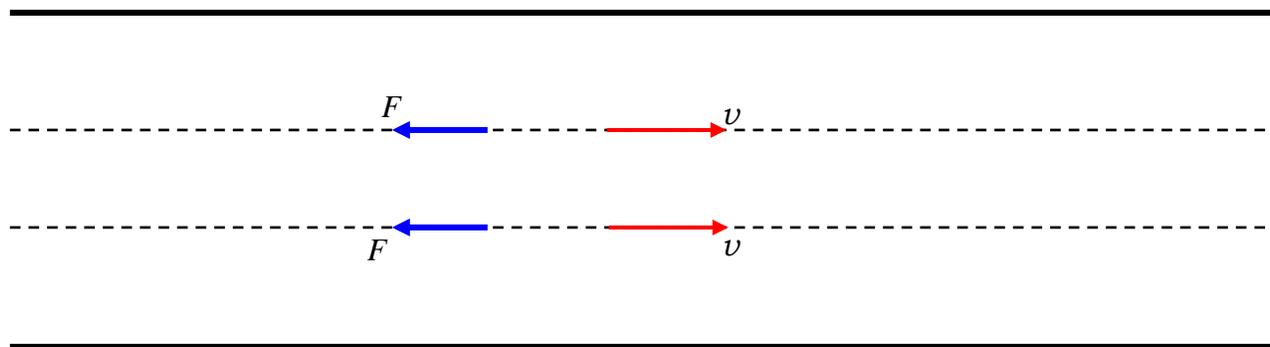


FIGURE XX.10

Figure XX.10 represents a pipe of radius a with liquid flowing to the right. At a distance r from the axis of the pipe the speed of the liquid is v . The length of the pipe is l , and there is a pressure gradient along the length of the pipe, the pressure at the left end being higher than the pressure at

the right by P . There is a velocity gradient in the pipe. The speed of the liquid along the axis of the pipe is v_0 , and the speed at the circumference of the pipe is zero. That is, the speed decreases from axis to circumference, so that the velocity gradient (dv/dr) is negative.

Now consider the equilibrium of the liquid inside radius r . (It is in equilibrium because it is moving at constant speed.) It is being pushed forward by the pressure gradient. This rightward force is $\pi r^2 P$. It is being dragged back by the viscous force acting on the area $2\pi r l$. This leftward force is $-2\pi\eta l r (dv/dr)$, this expression for the leftward force being positive.

$$\text{Therefore} \quad -2\eta l \frac{dv}{dr} = Pr. \quad 20.4.1$$

Integrate from the axis ($r = 0, v = v_0$) to r :

$$v = v_0 - \frac{Pr^2}{4\eta l}. \quad 20.4.2$$

Thus the speed decreases quadratically (parabolically) as you move away from the axis. The speed is zero at the circumference, and hence the speed on the axis is

$$v_0 = \frac{Pr^2}{4\eta l}. \quad 20.4.3$$

Verify the dimensions.

Now the volume flow through a cylindrical shell of radii r and $r + dr$ is the speed times the area $2\pi r dr$, which is $\frac{\pi Pr^3 dr}{2\eta l}$, and if you integrate that through the whole pipe, from 0 to a , you find that the rate of flow of liquid through the pipe (cubic metres per second) is

$$\frac{\pi a^4 P}{8\eta l}. \quad 20.4.4$$

This is *Poiseuille's Law*.

20.4.2 The Couette Viscometer

A cylindrical vessel of radius b contains the liquid whose viscosity is to be measured. A smaller, solid cylinder of radius a and length l is suspended from a torsion wire, whose torsion constant c is known, and is immersed in the liquid in the vessel, the two cylinders being coaxial. The vessel containing the liquid is spun about its axis at an angular speed Ω , thus setting the liquid in motion. This causes a viscous torque on the inner cylinder, which is therefore pulled round through an angle

ϕ . When the restoring torque of the torsion wire $c\phi$ is equal to the viscous torque, the system will be in equilibrium, and one can then calculate the viscosity η of the liquid. We shall refer to figure XX.11. In the simple analysis given below, we suppose that the angular and linear speed and gradients are sufficiently small that the flow is nonturbulent. We also neglect the effects of viscous drag on the flat ends of the cylinder. Thus the diameter of the cylinder, in our analysis, must be much less than its length.

Incidentally, for a long time I thought that the word “couette” must be French for something. It is – it’s French for “feather bed” or for “pigtail”. But the Couette viscometer is actually named after a little-known nineteenth century French scientist, Maurice Couette.

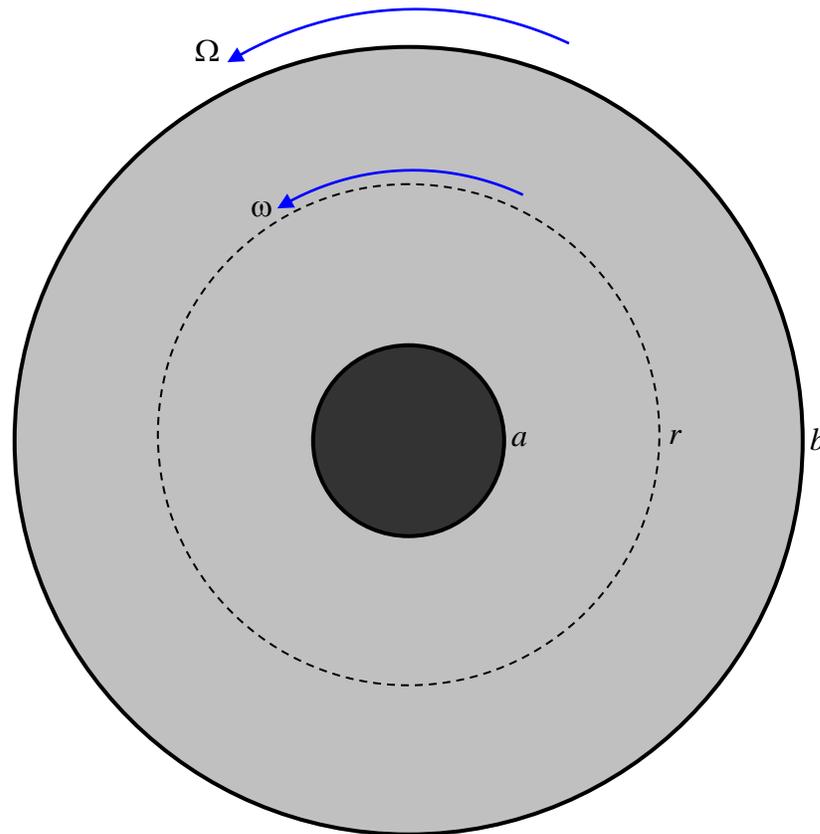


FIGURE XX.11

Let us calculate the viscous torque on the liquid within radius r . Notice that, since we have a steady-state situation, this torque is *independent of r* ; in particular the torque on the liquid within radius r is the same as the torque (which we can measure with the torsion wire) on the inner cylinder. The area of the curved surface of the liquid within radius r is $2\pi lr$. The viscous torque on this surface is r times η times the area times the transverse velocity gradient. But we have to be careful about this last term. If the whole body of the liquid were rotating as a solid body with angular speed ω , the speed at radius r would be $r\omega$ and hence there would be a transverse velocity gradient equal to ω – but no viscous drag! But the liquid is not, of course, rotating as a solid, and ω

(as well as v) is a function of r . Since $v = r\omega$, the velocity gradient is $\frac{dv}{dr} = r\frac{d\omega}{dr} + \omega$, and the only part of this that goes into the expression for the viscous torque is the part $r\frac{d\omega}{dr}$. Thus the expression for the torque on the liquid within radius r (and hence also on the inner cylinder) is

$$\tau = r \cdot \eta \cdot 2\pi r l \cdot r \frac{d\omega}{dr}. \quad 20.4.5$$

That is,
$$\frac{d\omega}{dr} = \frac{\tau}{2\pi\eta l r^3}. \quad 20.4.6$$

Integration from $r = a, \omega = 0$ to $r = b, \omega = \Omega$ gives

$$\tau = \frac{4\pi\eta l \Omega a^2 b^2}{b^2 - a^2}. \quad 20.4.7$$

In equilibrium, this is equal to $c\phi$, where c is the torsion constant of the suspension and ϕ is the angle through which the inner cylinder has turned, and hence the viscosity can be determined. You should, as usual, check the dimensions of equation 20.4.7.

20.4.3 Sphere falling through viscous liquid

If a sphere of radius a is moving at speed v in laminar (i.e. non-turbulent) flow through a viscous fluid it will experience a resistive force that depends on the speed v and radius a of the sphere and on the dynamic viscosity η of the fluid. Dimensional considerations show that the resistive force is proportional to ηav . Some Web sites that I have seen on the Internet say that the constant of proportionality is found “experimentally” to be 6π . To write that this constant is found “experimentally” is, of course, nonsense. The constant is indeed 6π , but this is determined theoretically from the theory of fluid dynamics. The theory of fluid dynamics is, unfortunately beyond my area of expertise, and consequently I do not know its derivation. The resistive force on a sphere moving without turbulence through a viscous fluid is $6\pi\eta av$. This is known as *Stokes’s Law*, and, if any viewer can supply a derivation of it, I’d be glad if s/he would contact me at tatumjb352 at gmail dot com. In Chapter 6, I used the symbol γ for the resistive force per unit mass (the “damping constant”); in other words, $\gamma = 6\pi\eta av/m$.

In Chapter 6, Motion in a Resisting Medium, Section 6.3b, I looked at a small massive body falling through a fluid. In this section the “body” will be a uniform sphere of radius a . The three forces acting on the body were its weight, the Archimedes hydrostatic upthrust, and the viscous force opposing the motion. The equation of motion (with y measured downwards) was given by equation 6.3.8, renumbered in this chapter as 20.4.8

$$\ddot{y} = g' - \gamma v, \quad 20.4.8$$

in which I referred to γ as the *damping constant*, and

$$g' = g \left(1 - \frac{\rho_0}{\rho} \right) \quad 20.4.9$$

in which ρ and ρ_0 are the densities of the fluid and the sphere respectively.

I want to describe the motion in terms of the radius a and density ρ of the sphere, and the dynamic viscosity η and density ρ_0 of the liquid. The term γv will be given by Stokes's law as

$$\frac{6\pi\eta a v}{m} = \frac{9\eta v}{2a^2\rho}$$

The equation of motion thus becomes

$$\frac{dv}{dt} = A - Bv \quad 20.4.10$$

where $A = (\rho_1 - \rho_0)g/\rho_1$ and
 $B = 9\eta/(2a^2\rho_1)$

Please verify, as you always do, that these are dimensionally correct.

The first time integral, with initial condition $t = 0, v = 0$, is

$$v = \frac{A[1 - \exp(-Bt)]}{B} \quad 20.4.11$$

By comparison with equation 6.3.12 of Chapter 6, reproduced here:

$$v = \hat{v}(1 - e^{-\gamma t}) \quad 6.3.12$$

we see that the damping constant is B and the terminal speed is A/B . That is to say:

$$\gamma = 9\eta/(2a^2\rho) \quad \text{and} \quad \hat{v} = 2a^2(\rho - \rho_0)g/(9\eta) \quad 20.4.12$$

(Check the dimensions, and also ask whether, qualitatively, it makes sense. For example, does it make sense that the damping constant is proportional to the viscosity of the liquid, and is inversely proportional to the density and the cross-sectional area of the sphere? Yes, it does!)

The second time integral was given in Chapter 6 as equation 6.3.14, reproduced here:

$$y = \hat{v}t - \frac{\hat{v}}{\gamma}(1 - e^{-\gamma t}) \quad 6.3.14$$

The problem of a sphere falling through a viscous fluid, described by equations 6.3.14 together with 20.4.11 lends itself to a variety of problems, such as: How long does it take to fall through a certain distance? What is the viscosity of the fluid? What is the density of the sphere? I suggest three such problems below, roughly in order of increasing difficulty. The second problem might describe a plausible experimental method for measuring the viscosity. The viewer might consider that there is an easier way of measuring the density of a sphere than by timing its fall through a viscous fluid, as in the third problem – for example if one had a caliper and a balance to measure its radius and mass – but that wouldn't give a very challenging computation.

Problem 1: Consider an iron sphere of diameter 1 cm ($a = 0.005$ m), density 7800 kg m^{-3} , falling through some oil of density 900 kg m^{-3} and dynamic viscosity 0.06 Pa s . The local gravitational acceleration is 9.81 m s^{-2} . In that case, the damping constant would be 1.38 s^{-1} and the terminal speed would be 6.27 m s^{-1} .

Starting from rest, how long would this sphere take to fall $y = 1$ m through the oil? All we have to do is to put the numbers for \hat{v}, γ, y in equation 6.3.14 (shown above) and solve for t .

“All we have to do...” It is probable that the readers of these notes will be divided into two populations. One population will know immediately how to solve equation 6.3.14 for t and will probably already have a program to do it. They will just have to type in the equation and its derivative, have a guess at the solution (it needn't be a sensible guess; a stupid one will do just as well) and charge ahead with a Newton-Raphson iteration.

Using a first guess of 1 s, it took me three iterations to converge to 0.540 s and 12 iterations to reach 0.539 684 037 s. The computer took no perceptible passage of time to do all twelve iterations. Click, and it was done. (Falling through 1 m *in vacuo* would take 0.452 s).

The second population may not have the slightest idea even how to begin. There will probably be few, if any, in an intermediate population. If you are in the second population, don't feel bad; if it is any comfort I can assure you that you are not alone. You are not completely lost, however. If you are good enough with a computer to compute a table of values of a function and to draw a graph of it, you could just plot a graph of the equation, and see for what value of t does it change sign and become zero. That is a perfectly acceptable way of solving any equation of the form $y = f(x)$. Sooner or later during a scientific career, you will have to learn a numerical way of arriving rapidly at a precise solution. Newton-Raphson iteration is often (usually, in my experience) one of the best ways of doing it. It is described, for example, in Chapter 1 Section 1.6 of <https://www.astro.uvic.ca/~tatum/celmechs.html>

Problem 2: A copper sphere of diameter 1 cm ($a = 0.005$ m), density 8900 kg m^{-3} , falls through some oil of density 900 kg m^{-3} . It takes 0.527 s to fall $y = 1$ m through the oil. What is the dynamic viscosity of the oil?

We use equation 6.3.14 again, in the form

$$f(\gamma) = \frac{g'}{\gamma} \left[t - \frac{1 - \exp(-\gamma t)}{\gamma} \right] - y = 0 \quad 20.4.13$$

This time, we solve for the damping constant, and the viscosity follows from the first of equations 20.4.12. The derivative is

$$f'(\gamma) = g'(2 - \gamma t - (2 + \gamma t) \exp(-\gamma t) / \gamma^3) \quad 20.4.14$$

and the Newton-Raphson procedure can forge ahead. I used a first guess of 1 s^{-1} for the damping constant. The Newton-Raphson procedure converged to 1.216 s^{-1} after two iterations and to $1.216976381 \text{ s}^{-1}$ after three iterations. Application of equation 20.4.12 resulted in a viscosity of 0.060 Pa s.

Problem 3: Suppose we use another sphere, of the same size, and we find that it takes 0.853 s to fall, from rest, through 1 m. What is the density of the sphere?

Again we have to solve equation 6.3.14, which I re-write here. In re-writing it, I have substituted the expressions 20.4.11 for the damping constant and the terminal speed, I have written the density of the sphere (the unknown quantity that we wish to find) as x rather than ρ , and I have slightly rearranged the terms. I have also defined

$$P = \frac{9\eta}{2a^2} \quad 20.4.12$$

The equation then becomes

$$f(x) = tx - t\rho_0 - \frac{(x^2 - \rho_0 x) \left[1 - \exp\left(-\frac{Pt}{x}\right) \right]}{P} - \frac{Py}{g} = 0 \quad 20.4.13$$

The numerical values in SI (MKS) units of all quantities other than x are known, and it just remains to solve this equation for x , the density of the sphere. The derivative may look long, complicated and awkward – but all you have to do is to type it into your computer, and it does not care how awkward it looks. It calculates it instantaneously, and the Newton-Raphson iteration converges just as rapidly as usual. The density of the sphere is evidently less than 7800 kg m^{-3} . For a first guess I chose 4000 kg m^{-3} , and I reached convergence to $2.70 \times 10^3 \text{ kg m}^{-3}$ in three iterations, and $2701.447746044115 \text{ kg m}^{-3}$ instantaneously in 12 iterations. The sphere is evidently made of aluminium.

