CHAPTER 19
THE CYCLOID

19.1 *Introduction*

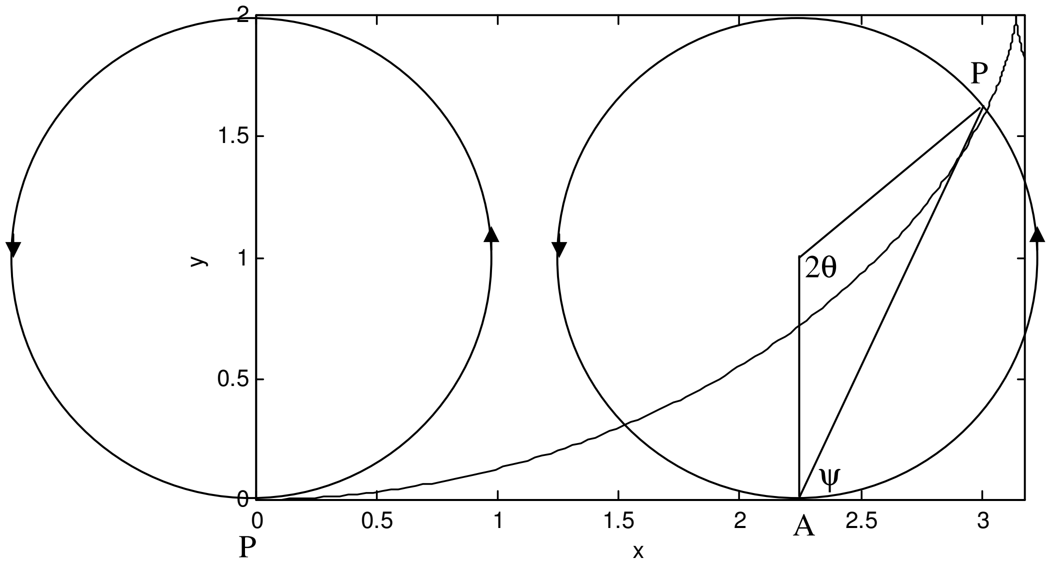

FIGURE XIX.1

Let us set up a coordinate system Oxy, and a horizontal straight line $y = 2a$. We imagine a circle of diameter $2a$ between the x-axis and the line $y = 2a$, and initially the lowest point on the circle, P, coincides with the origin of coordinates O. We now allow the circle to roll counterclockwise without slipping on the line $y = 2a$, so that the centre of the circle moves to the right. As the circle rolls on the line, the point P describes a curve, which is known as a *cycloid.*

When the circle has rolled through an angle 2θ, the centre of the circle has moved to the right by a horizontal distance $2a\theta$, while the horizontal distance of the point P from the centre of the circle is $a \sin 2\theta$, and the vertical distance of the point P below the centre of the circle is $a \cos 2\theta$. Thus the coordinates of the point P are

$$x = a(2\theta + \sin 2\theta), \qquad 19.1.1$$

and

$$y = a(1 - \cos 2\theta). \qquad 19.1.2$$

These are the parametric equations of the cycloid. Equation 19.1.2 can also be written

$$y = 2a \sin^2 \theta. \qquad 19.1.3$$

Exercise: When the x-coordinate of P is $2.500a$, what (to four significant figures) is its y-coordinate?

Solution: We have to find 2θ by solution of $2\theta + \sin 2\theta = 2.5$. By Newton-Raphson iteration (see Chapter 1 of the Stellar Atmospheres notes in this series) or otherwise, we find $2\theta = 0.931\,599\,201$ radians, and hence $y = 0.9316a$.

19.2 *Tangent to the Cycloid*

The slope of the tangent to the cycloid at P is dy/dx, which is equal to $(dy/d\theta)/(dx/d\theta)$, and these can be obtained from equations 19.1.1 and 19.1.2.

Exercise: Show that the slope of the tangent at P is $\tan\theta$. That is to say, the tangent at P makes an angle θ with the horizontal.

Having done that, now consider the following:

Let A be the lowest point of the circle. The angle ψ that AP makes with the horizontal is given by $\tan\psi = \dfrac{y}{x - 2a\theta}$.

Exercise: Show that $\psi = \theta$. Therefore the line AP is the tangent to the cycloid at P; or the tangent at P is the line AP.

19.3 *The Intrinsic Equation to the Cycloid*

An element ds of arc length, in terms of dx and dy, is given by the theorem of Pythagoras: $ds = \left((dx)^2 + (dy)^2\right)^{1/2}$, or, since x and y are given by the parametric equations 19.1.1 and 19.1.2, by

$ds = \left(\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2\right)^{1/2} d\theta$. And of course we have just shown that the intrinsic coordinate ψ (i.e. the angle that the tangent to the cycloid makes with the horizontal) is equal to θ.

Exercise: Integrate ds (with initial condition $s = 0$, $\theta = 0$) to show that the intrinsic equation to the cycloid is

$$s = 4a\sin\psi. \qquad 19.3.1$$

Also, eliminate ψ (or θ) from equations 19.3.1 and 19.1.2 to show that the following relation holds between arc length and height on the cycloid:

$$s^2 = 4ay. \qquad 19.3.2$$

19.4 *Variations*

In sections 19.1,2,3, we imagined that the cycloid was generated by a circle that was rolling counterclockwise along the line $y = 2a$. We can also imagine variations such as the circle rolling clockwise along $y = 0$, or we can start with P at the top of the circle rather than at the bottom. I summarise in this section four variations. The distinction between ψ and θ is as follows. The angle that the tangent to the cycloid makes with the positively-directed x-axis is ψ; that is to say, $dy/dx = \tan\psi$. The circle rolls through an angle 2θ. There is a simple relation between ψ and θ, which is different for each case.

In each figure, x and y are plotted in units of a. The vertical height between vertices and cusps is $2a$, the horizontal distance between a cusp and the next vertex is πa, and the arc length between a cusp and the next vertex is $4a$.

I. Circle rolls counterclockwise along $y = 2a$. P starts at the bottom. The cusps are up. A vertex is at the origin.
Figure XIX.2.

$$x = a(2\theta + \sin 2\theta) \qquad 19.4.1$$

$$y = 2a\sin^2\theta \qquad 19.4.2$$

$$s = 4a\sin\theta \qquad 19.4.3$$

$$s^2 = 8ay \qquad 19.4.4$$

$$\psi = \theta . \qquad 19.4.5$$

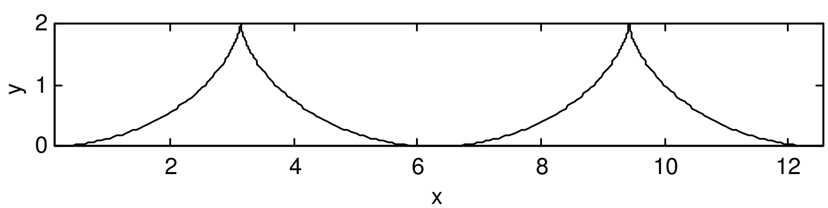

FIGURE XIX.2

II. Circle rolls clockwise along $y = 0$. P starts at the bottom. The cusps are down. A cusp is at the origin.
Figure XIX.3.

$$x = a(2\theta - \sin 2\theta) \qquad 19.4.6$$

$$y = 2a\sin^2\theta \qquad 19.4.7$$

$$s = 4a(1 - \cos\theta) \qquad 19.4.8$$

$$s^2 = 8a(y - s) \qquad 19.4.9$$

$$\psi = 90^\circ - \theta. \qquad 19.4.10$$

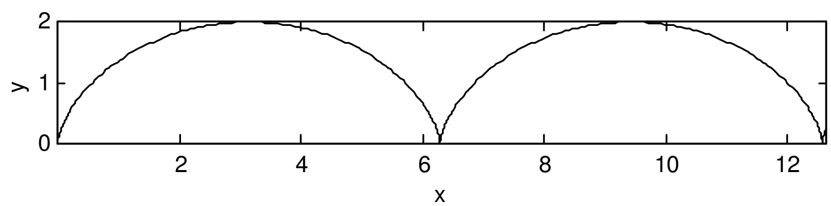

FIGURE XIX.3

III. Circle rolls clockwise along $y = 0$. P starts at the top. The cusps are down. A vertex is at $x = 0$. Figure XIX.4.

$$x = a(2\theta + \sin 2\theta) \tag{19.4.11}$$

$$y = 2a\cos^2\theta \tag{19.4.12}$$

$$s = 4a\sin\theta \tag{19.4.13}$$

$$s^2 = 8a(2a - y) \tag{19.4.14}$$

$$\psi = 180^\circ - \theta. \tag{19.4.15}$$

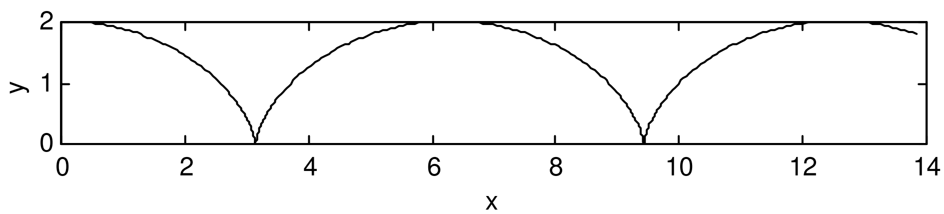

FIGURE XIX.4

IV. Circle rolls counterclockwise along $y = 2a$. P starts at the top. The cusps are up. A cusp is at $x = 0$.
Figure XIX.5.

$$x = a(2\theta - \sin 2\theta) \tag{19.4.16}$$

$$y = 2a\cos^2\theta \tag{19.4.17}$$

$$s = 4a(1 - \cos\theta) \tag{19.4.18}$$

$$s^2 - 8as + 8a(2a - y) = 0 \tag{19.4.19}$$

$$\psi = 90^\circ + \theta. \tag{19.4.20}$$

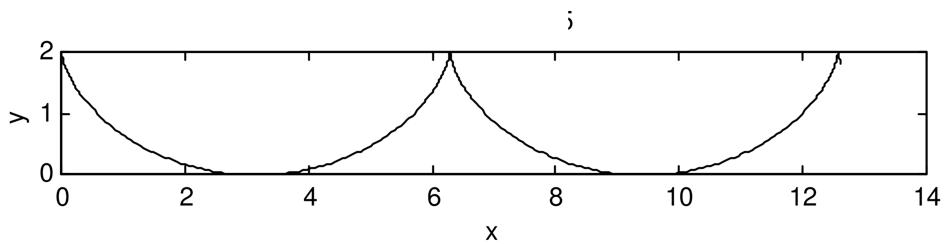

FIGURE XIX.5

19.5 *Motion on a Cycloid, Cusps Up*

We shall imagine either a particle sliding down the inside of a smooth cycloidal bowl, or a bead sliding down a smooth cycloidal wire, figure XIX.6.

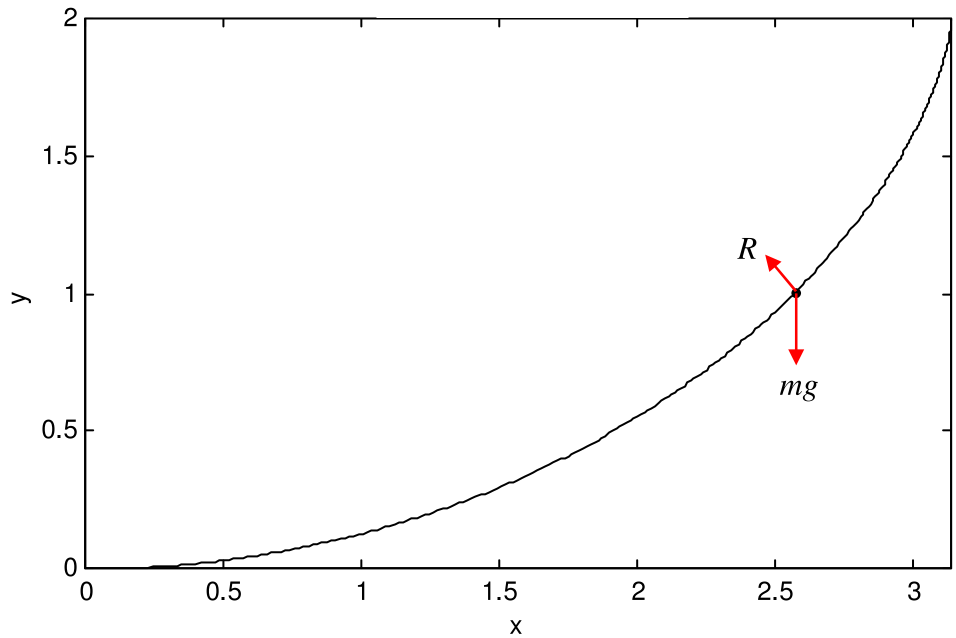

FIGURE XIX.6

We shall work in intrinsic coordinates to obtain the tangential and normal equations of motion. (For a reminder of the use of intrinsic coordinates, they were used briefly, for example, in Section 7.3) These equations are, respectively:

$$\ddot{s} = -g\sin\psi \qquad 19.5.1$$

and

$$\frac{mv^2}{\rho} = R - mg\cos\psi. \qquad 19.5.2$$

Here R is the normal (and only) reaction of the bowl or wire on the particle and ρ is the radius of curvature. The radius of curvature is $ds/d\psi$, which, from equation 19.3.1, (or equations 19.4.3 and 19.4.5) is

$$\rho = 4a\cos\psi. \qquad 19.5.3$$

From equations 19.3.1 and 19.5.1 we see that the tangential equation of motion can be written, without approximation:

$$\ddot{s} = -\frac{g}{4a}s. \qquad 19.5.4$$

This is simple harmonic motion of period $4\pi\sqrt{a/g}$, independent of the amplitude of the motion. This is the *isochronous* property of the cycloid. Likewise, if the particle is released from rest, it will reach the bottom of the cycloid in a time $\pi\sqrt{a/g}$, whatever the starting position.

Let us see if we can find the value of R where the generating angle is ψ. Let us suppose that the particle is released from rest at a height y_0 above the x-axis (generating angle = ψ_0); what is its speed υ when it has reached a height y (generating angle ψ)? Clearly this is given by

$$\tfrac{1}{2}m\upsilon^2 = mg(y_0 - y), \qquad 19.5.5$$

and, following equation 19.3.2, and recalling that $\theta = \psi$, this is

$$\upsilon^2 = 2ga(\cos 2\psi - \cos 2\psi_0). \qquad 19.5.6$$

On substituting this and equation 19.5.3 into equation 19.5.2, we find for R:

$$R = \frac{mg}{2\cos\psi}(1 + 2\cos 2\psi - \cos 2\psi_0). \qquad 19.5.7$$

19.6 *Motion on a Cycloid, Cusps Down*

We imagine a particle sliding down the outside of an inverted smooth cycloidal bowl, or a bead sliding down a smooth cycloidal wire. We shall suppose that, at time $t = 0$, the particle was at the top of the cycloid and was projected forward with a horizontal velocity υ_0. See figure XIX.7.

This time, the equations of motion are

$$\ddot{s} = g\sin\psi \qquad 19.6.1$$

and

$$\frac{m\upsilon^2}{\rho} = mg\cos\psi - R. \qquad 19.6.2$$

By arguments similar to those made in Section19.5, we find that

$$\ddot{s} = \frac{g}{4a}s. \qquad 19.6.3$$

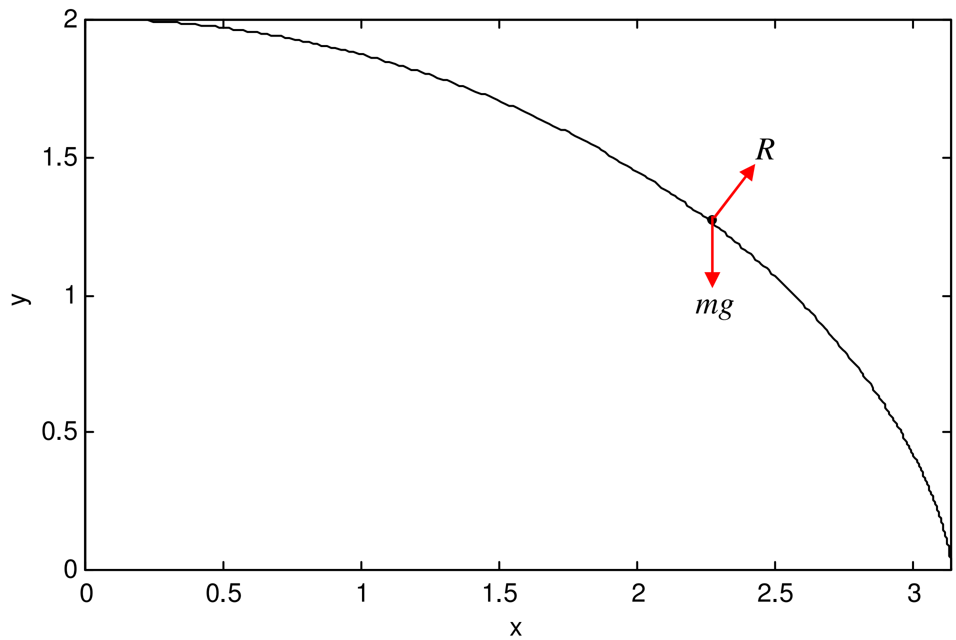

FIGURE XIX.7

The general solution to this is

$$s = Ae^{pt} + Be^{-pt}, \qquad 19.6.4$$

where

$$p = \sqrt{g/(2a)}. \qquad 19.6.5$$

With the initial condition given (at $t = 0$, s=0, $\dot{s} = v_0$), we can find A and B and hence:

$$s = v_0\sqrt{\frac{a}{g}}\left(e^{pt} - e^{-pt}\right). \qquad 19.6.6$$

Again proceeding as in Section 19.5, we find for R:

$$R = \frac{m}{4a\cos\psi}(4ga\cos 2\psi - v_0^2). \qquad 19.6.7$$

So – what happens?

If the constraint is *two-sided* (bead sliding on a wire) R becomes zero when $\cos 2\psi = \upsilon_0^2/(2ga)$, and thereafter R is in the opposite direction.

If the constraint is *one-sided* (particle sliding down the outside of a smooth cycloidal bowl):

1. If $\upsilon_0^2 > 4ga$, the particle loses contact at the moment of projection.

2. If $\upsilon_0^2 < 4ga$, the particle loses contact as soon as $\cos 2\psi = \upsilon_0^2/(2ga)$. If υ_0 is very small (i.e. very much smaller than $\sqrt{2ga}$), this will happen when $\psi = 45^\circ$; for faster initial speeds, contact is lost sooner.

Example. A particle is projected horizontally with speed $\upsilon_0 = 1$ m s^{-1} from the vertex of the smooth cycloidal hill

$$x = a(2\theta + \sin 2\theta)$$
$$y = 2a\cos^2\theta,$$

where $a = 2$ m. Assuming that $g = 9.8$ m s^{-2}, how long does it take to get halfway down the hill (i.e. to $y = a$)?

We have to use equation 19.6.6. With the numerical data given, this is

$$s = 0.451754\left(e^{1.565248t} - e^{-1.565248t}\right).$$

We can find s from equation 19.4.12, which gives us $s = 2.828427$ m. If we let $\xi = e^{1.565248t}$, we now have to solve $6.26099 = \xi - 1/\xi$, or $\xi^2 - 6.26099\xi - 1 = 0$. From this, $\xi = 6.41683$ and hence $t = 1.19$ s.

I leave it to the reader to calculate R at this time – and indeed to see whether the particle loses contact with the hill before then. Perhaps the fact that I got a positive real root for ξ means that we are all right and the particle is still in contact – but I wouldn't be sure of that. I leave it to the reader to investigate further.

19.7 *The Brachystochrone Property of the Cycloid*

A small point. The word is sometimes spelled brachistochrone, and I have no recommendation one way or the other. For what it's worth, the only dictionary within easy reach of my desk has brachiopod and brachycephalic. In any case, the word is derived from Greek, and means shortest time.

The famous brachystochrone problem problem is this: A smooth wire, which can be of any desired length, is to connect two points O and P; P is at a lower level than O, but is not vertically below O. The wire is to be bent to a shape, and cut to a length, such that the time taken for a bead to slide down the wire from O to P is least.

It is not easy to prove that the required curve is a cusps-up cycloid; but it is quite reasonable to *speculate* or to *guess* that this might be so. And, having speculated that it might be a cycloid, it is easy to verify that the required curve is indeed a cusps-up cycloid, the bead starting from rest at a cusp of the cycloid.

A speculation might go something like this. Generally one would expect that the further P is from O, the longer it will take for the bead to slide from O to P. But, if O and P are connected with a cycloidal wire, the time taken to go from O to P does *not* increase with distance. (See the isochronous property of the cycloid discussed in Section 19.5.) Thus, as you increase the distance between O and P, the time taken to travel by any route other than the cycloidal one must take longer than the cycloidal route. This argument may not sound like a rigorous proof, though it is enough to arouse our suspicions and to test whether it is correct.

Since I am going to deal with a bead sliding downwards under gravity, I am going to find it convenient to set up our coordinate axes such that x increases to the right, and y increases *downwards*. In that case, the parametric equations to a cusps-up cycloid, with the origin at a cusp, are

$$x = a(2\theta - \sin 2\theta), \qquad 19.7.1$$

and

$$y = 2a\sin^2\theta \qquad 19.7.2$$

– and these are the equations that we shall be testing.

The time taken for the bead to travel a distance ds along the wire, while it is moving at speed υ is ds/υ. In $(x\ ,\ y)$ coordinates, ds is $\sqrt{1+y'^2}\,dx$, where $y' = dy/dx$. Also, the speed reached is related (by equating the gain in kinetic energy to the loss of potential energy) to the vertical distance y dropped by $\upsilon = \sqrt{2gy}$. Thus the time taken to go from O to P is

$$\frac{1}{\sqrt{2g}}\int_{\mathrm{O}}^{\mathrm{P}}\frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx = \frac{1}{\sqrt{2g}}\int_{\mathrm{O}}^{\mathrm{P}} f(y,y')dx. \qquad 19.7.3$$

This is least (see Chapter 18 for a discussion of this theorem from the calculus of variations) for a function $y(x)$ that satisfies

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}. \qquad 19.7.4$$

We have:

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{y}}, \qquad 19.7.5$$

$$\frac{\partial f}{\partial y} = -\frac{(1 + y'^2)^{1/2}}{2y^{3/2}} , \qquad 19.7.6$$

and

$$\frac{\partial f}{\partial y'} = \frac{y'}{y^{1/2}(1 + y'^2)^{1/2}} . \qquad 19.7.7$$

It is left for the reader to see whether equations 19.7.1 and 2 satisfy equation 19.7.4. You should find that both sides of the equation are equal to $-1/\left(4\sqrt{2}a^{3/2}\sin^4\theta\right)$ Thus our speculation is confirmed, and a cusps-up cycloid is indeed the curve that offers passage from O to P in the shortest time.

19.8 *Contracted and Extended Cycloids*

As in Section 19.1, we consider a circle of radius a rolling to the right on the line $y = 2a$. The point P is initially below the centre of the circle, but, instead of being on the rim of the circle, its distance from the centre of the circle is r. If $r < a$, the path described by P will be a *contracted cycloid*; if $r > a$, the path is an *extended cycloid.* (I think there's a case for using this nomenclature the other way round, but most authors seem to use "contracted" for $r < a$ and "extended" for $r > a$.)

It should not take long to be convinced, by arguments similar to those in Section 19.1, that the parametric equations to a contracted or extended cycloid are

$$x = 2a\theta + r\sin 2\theta \qquad 19.8.1$$

and

$$y = a - r\cos 2\theta . \qquad 19.8.2$$

These are illustrated in figures XIX.8 and XIX.9 for a contracted cycloid with $r = 0.5a$ and an extended cycloid with $r = 1.5a$.

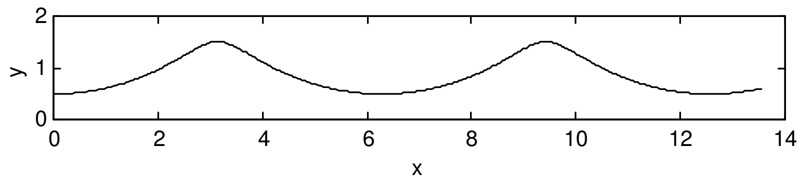

FIGURE XIX.8

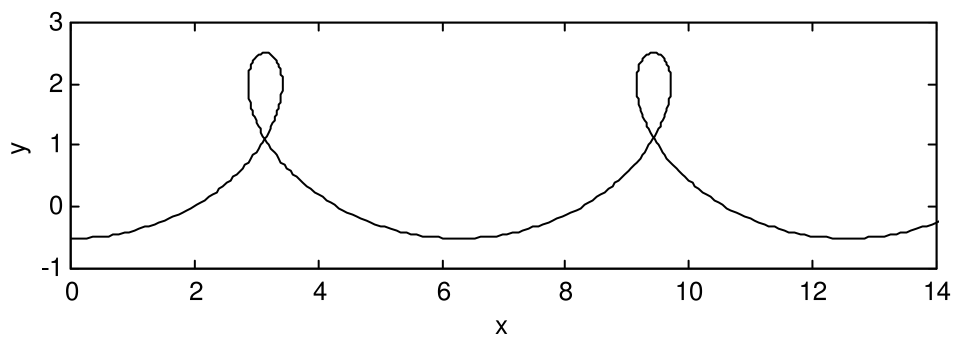

FIGURE XIX.9

19.9 *The Cycloidal Pendulum*

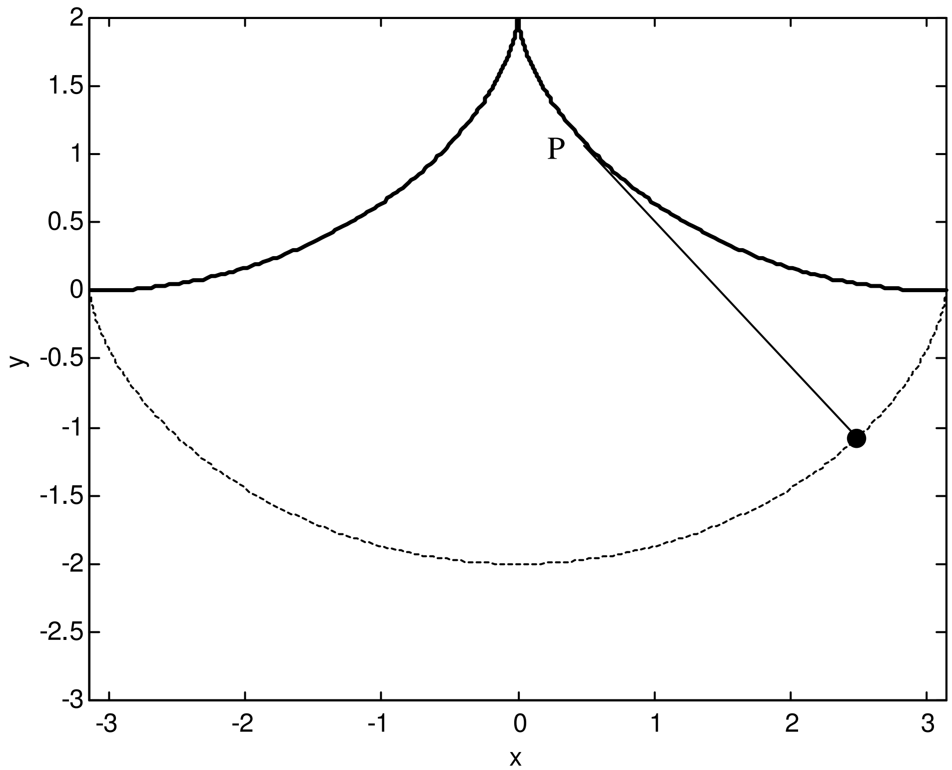

FIGURE XIX.10

Let us imagine building a wooden construction in the shape of the cycloid

$$x = a(2\theta - \sin 2\theta) \qquad 19.9.1$$

$$y = 2a\cos^2\theta \qquad 19.9.2$$

shown with the thick line in figure XIX.10. Now suspend a pendulum of length $4a$ from the cusp, and allow it to swing to and fro, partially wrapping itself against the wooden frame as it does so. If the arc length from the cusp to P is s, then the length of the "free" string is $4a - s$, and so the coordinates of the bob at the end of the pendulum are

$$x = a(2\theta - \sin 2\theta) + (4a - s)\cos(180^\circ - \psi) = a(2\theta - \sin 2\theta) + (4a - s)\sin\theta \qquad 19.9.3$$

and

$$y = 2a\cos^2\theta - (4a - s)\sin(180^\circ - \psi) = 2a\cos^2\theta - (4a - s)\cos\theta. \qquad 19.9.4$$

(You will need to remind yourself of the exact meaning of ψ and also make use of equation 19.4.20.) Now equation 19.4.18 tells us that $s = 4a(1 - \cos\theta)$, and, on substitution of this in equations 19.9.3 and 4, we find (after a very little algebra and trigonometry) for the parametric equations to the path described by the bob of the pendulum:

$$x = a(2\theta + \sin 2\theta) \qquad 19.9.5$$

and

$$y = -2a\cos^2\theta. \qquad 19.9.6$$

Thus the path of the pendulum bob (shown as a dashed line in figure XIX.10) is a cycloid, and hence its period is independent of its amplitude. (Recall Section 19.5.) Thus the pendulum is *isochronous* or *tautochronous*. It is astonishing to learn that Huygens constructed just such a pendulum as long ago as 1673.

19.10 *Examples of Cycloidal Motion in Physics*

Several examples of cycloidal motion in physics come to mind. One is the nutation of a top, which is described in Section 4.10 of Chapter 10. Earth's axis nutates in a similar fashion. Another well known example is the motion of an electron in crossed electric and magnetic fields. This is described in Chapter 8 of the Electricity and Magnetism section of these notes. In cosmology, if the mean density of the Universe is low, the Universe expands indefinitely, but, if the density is higher than a certain critical density, the (dimensionless) scale factor R of the Universe expands and contracts with time t according to the following parametric cycloidal equations:

$$R = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos 2\theta), \qquad 19.10.1$$

$$t = \frac{\Omega_0}{2(\Omega_0 - 1)^{3/2}}(2\theta - \sin 2\theta). \qquad 19.10.2$$

Here t is expressed in units of the reciprocal of the present Hubble constant, and Ω_0 is the ratio of the present density of the Universe to the density required to "close" the Universe.

A less well known example concerns the propagation of sound in the atmosphere. In the troposphere, which is the lower part of the atmosphere up to about 11 km, the temperature decreases roughly linearly with height. In that case sound travels through the troposphere in a cycloidal path. The speed of sound in a gas is proportional to the square root of the temperature. (If you are wondering how it depends on pressure P and density ρ, the answer is that it depends on the ratio P/ρ - and this ratio is proportional to the temperature.) In any case, if the temperature decreases linearly with height, the sound speed υ varies with height y as

$$\upsilon = \upsilon_0\sqrt{1 - cy}, \qquad 19.10.3$$

where c is a constant, equal to about 0.023 km^{-1}.Now to trace a sound ray through the atmosphere, we have to understand how the direction of propagation changes as the sound passes through layers of air of different temperature. This is governed , as with light, by Snell's law (see figure XIX.11):

$$\frac{d\upsilon}{\upsilon} = -\tan\psi\, d\psi. \qquad 19.10.4$$

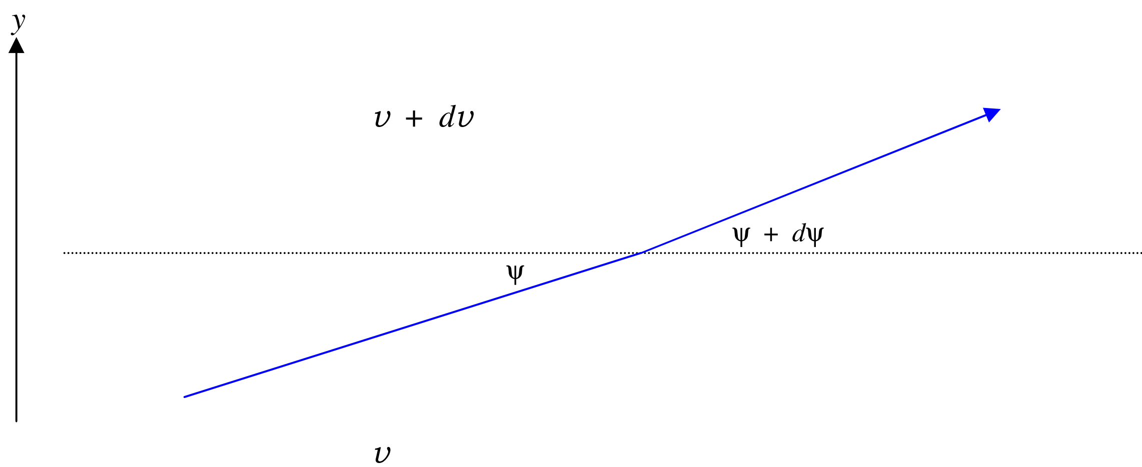

FIGURE XIX.11

Snell's law states that when sound (or light) enters a slower medium (i.e. one in which the speed of propagation is slower) it is bent towards the normal. I have drawn figure XIX.11 to represent the situation in the troposphere where the temperature (and hence the sound speed υ) decreases with height. That is, $d\upsilon/dy$ is negative. In other words $d\upsilon$ in figure XIX is negative, and equation 19.10.4 indicates that $d\psi$ is positive, as drawn. In case you do not recognize this differential form of Snell's law, try integrating it from υ_1 to υ_2 and from ψ_1 to ψ_2, and it should assume its more familiar integral form.

If you now eliminate υ between equations 19.10.3 and 4, you will get a differential relation between y and ψ, which, upon integration, becomes

$$cy = 1 - \frac{\cos^2\psi}{\cos^2\psi_0}, \qquad 19.10.5$$

where ψ_0 is the ground-level value of ψ. If we introduce

$$a = \frac{1}{2c\cos^2\psi_0}, \qquad 19.10.6$$

equation 19.10.5 can be conveniently re-written

$$y = 2a(\sin^2\psi - \sin^2\psi_0) = 2a(\cos^2\psi_0 - \cos^2\psi). \qquad 19.10.7$$

Now tan $\psi = dy/dx$, and elimination of y between this and equation 19.10.7 will give a differential relation between x and ψ, which, upon integration, becomes

$$x = a[2(\psi - \psi_0) + \sin 2\psi - \sin 2\psi_0]. \qquad 19.10.8$$

Equations 19.10.7 and 19.10.8 are the parametric equations of the sound path through the troposphere, and describe a cycloid.

Problem for a Rainy Day: If $x = 2.0$ and $y = 1.6$, what are ψ and ψ_0?

I make it $\psi = 69^\circ\ 17'$, $\psi_0 = 15^\circ\ 52'$.