Solutions to Problems

1. Notation:

V_1 = speed of comet immediately before collision.

V_2 = speed of combined object immediately after collision, $= \dfrac{V_1}{1+k}$, because linear momentum is conserved.

q = perihelion distance of original parabolic orbit, so that, at the end of the latus rectum (i.e. at the position of the collision) the heliocentric distance is $2q$.

Then, using the formula for the speed in a parabolic orbit (equation 9.5.32), we have

$$V_1^2 = \frac{GM}{q}, \tag{1}$$

where M is the mass of the Sun.

a = semi major axis the elliptic orbit of the combined object.

Then, using the formula for the speed in an elliptic orbit (equation 9.5.31), we have

$$V_2^2 = \frac{V_1^2}{(1+k)^2} = GM\left(\frac{1}{q} - \frac{1}{a}\right). \tag{2}$$

From equations (1) and (2) we obtain

$$\frac{1}{(1+k)^2 q} = \left(\frac{1}{q} - \frac{1}{a}\right),$$

or

$$\frac{q}{a} = \frac{k(2+k)}{(1+k)^2}. \tag{3}$$

Also, the collision results in no change in the angular momentum of the system, so that, using the formulas for the angular momentum *per unit mass* in parabolic and elliptic orbits (equations 9.5.27a and 9.5.28a), we obtain

$$2q = a(1+k)^2(1-e^2). \tag{4}$$

Equations (3) and (4) give us (by elimination of a)

$$e^2 = \frac{1 + 4k^2 + 4k^3 + k^4}{(1 + k)^4}, \tag{5}$$

or
$$1 - e^2 - 4e^2k + (4 - 6e^2)k^2 + 4(1 - e^2)k^3 + (1 - e^2)k^4 = 0. \tag{6}$$

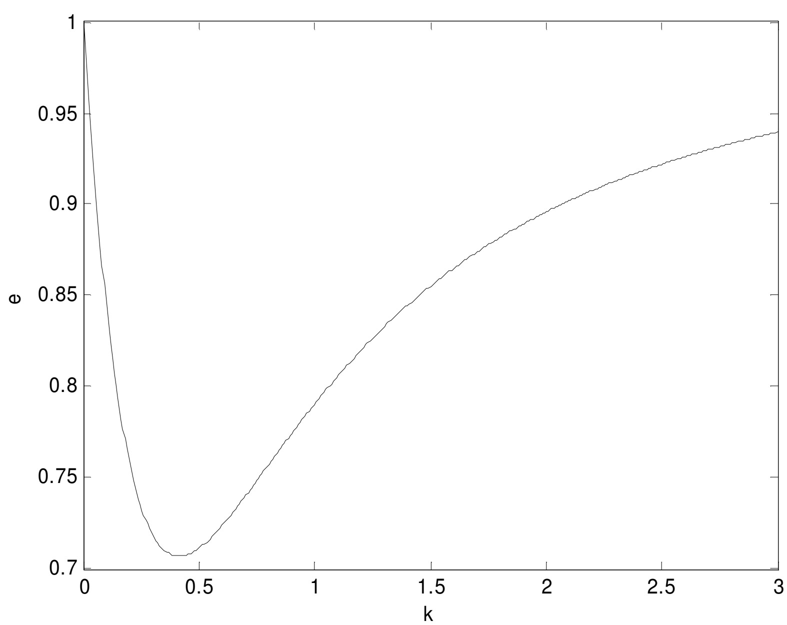
(a) We are told that $e = 0.8$, which results in $k = 1.06098$, so the mass of the object was <u>1.06098*m*.</u>

(b) Before embarking on a calculation, here are some preliminary qualitative thoughts. What happens if the mass of the object is very small, so that k is close to zero? In that case the orbit of the comet is barely changed, and the eccentricity of the orbit remains $e = 1$.

What happens if the object is very much more than that of the comet? In that case the object barely notices the impact of the comet, and it (and the comet, which by now is stuck to it) falls straight towards the Sun in a straight line, again with $e = 1$.

So, since e is 1 in the extreme cases of a very massive object and a very light object, it may be deduced that, for a body of mass comparable with that of the comet, the eccentricity of its orbit will be less than 1, and that, for some particular value of its mass, the eccentricity of the orbit will go through a minimum.

Indeed if we plot $e : k$ from equation (5), it looks like this:



Now for some calculation.

We have to use equation (5), to find for what value of k is e^2 least. Slightly tedious, but on setting the derivative to zero I find

$$1 + k - 4k^2 - 8k^3 - 5k^4 - k^5 = 0, \tag{7}$$

This has only one positive real root, which I found numerically, by Newton-Raphson iteration, to be 0.414214, which looked as though it might be $\sqrt{2}-1$. I checked by substituting $\sqrt{2}-1$ back into equation (7) and found that indeed $\sqrt{2}-1$ really is a solution. It was a bit tedious to do this.

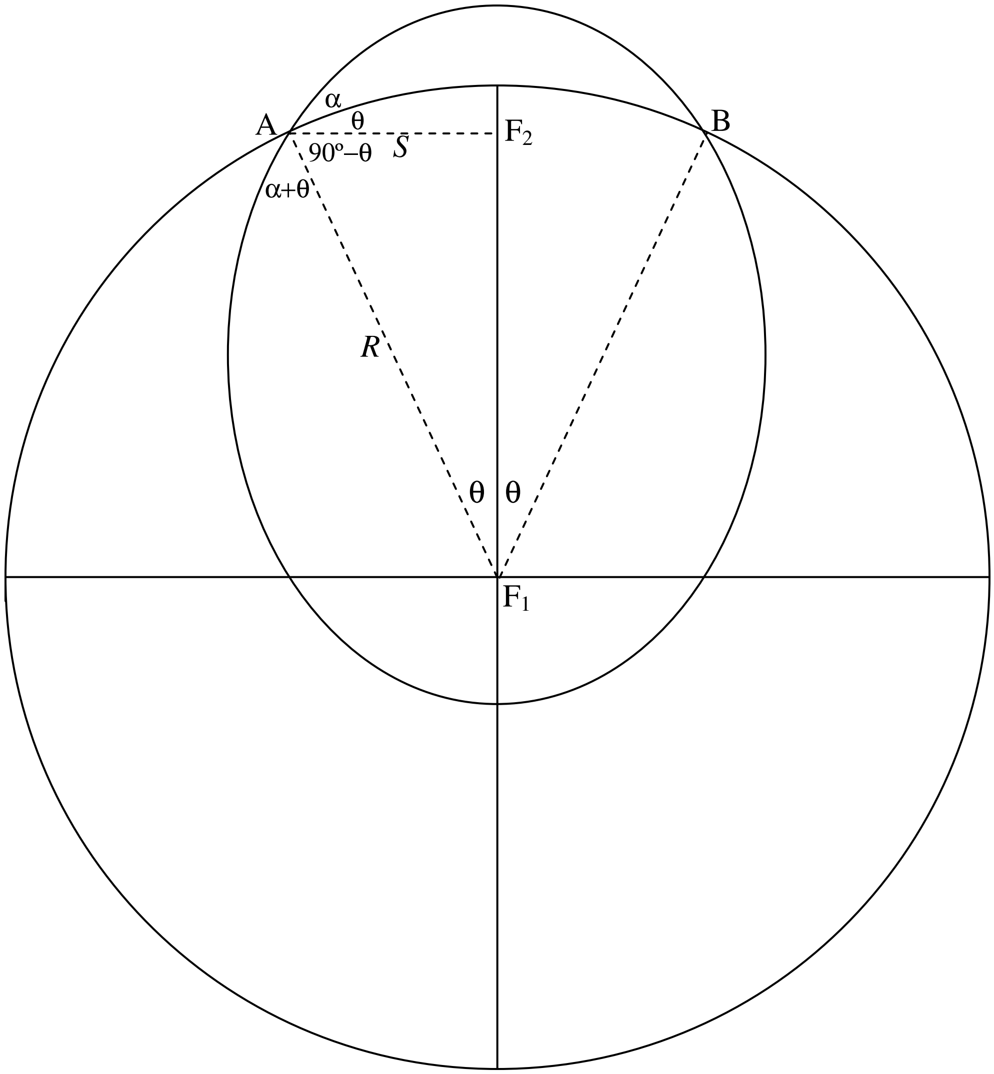
However, if you are better at algebra than I am, you might be able to factor equation (7) into

$$(1 + k)(1 - 2k - k^2)(1 + 2k + k^2) = 0, \tag{8}$$

which, indeed, has only one positive real root, namely $\underline{k = \sqrt{2}-1}$.

Then, on substitution of this value into equation (5), you obtain, after some tedium, $\underline{e = 1/\sqrt{2} = 0.7071}$.

2.



The trajectory is part of an ellipse, drawn in full above. The energy of an elliptic orbit depends on its semi major axis *a*, so the least-energy orbit is the one that passes through both cities and has the least semi major axis, so the determining the size and shape of the ellipse is a problem one of pure geometry.

Note that one focus, F_1 of the ellipse is at the centre of Earth. Let F_2 be the other (empty) focus. Then the semimajor axis of the ellipse is $a = \frac{1}{2}(R + S)$, and this is least when AF_2 is perpendicular to the major axis of the ellipse, as drawn. But $S = R\sin\theta$, so the semi major axis of the least-energy ellipse is $a = \frac{1}{2}R(1 + \sin\theta)$.

I have marked in some angles near to city A, from which I hope it will be agreed that $2(\alpha+\theta) + 90^\circ - \theta = 180^\circ$, and so we find that the launch angle for the least-energy orbit is $\underline{\underline{\alpha = 45^\circ - \frac{1}{2}\theta}}$.

The launch speed V_0 is the speed where $r = R$, so, using equation 9.5.31 for the speed in an elliptic orbit, we obtain

$$V^2 = GM\left(\frac{2}{R} - \frac{1}{a}\right) = GM\left(\frac{2}{R} - \frac{2}{R(1+\sin\theta)}\right) = \underline{\underline{\frac{2GM}{R}\left(\frac{1+\sin\theta}{\sin\theta}\right)}}.$$

Since we know the semi major axis, we can probably find the period.
The semi major axis is

$$a = \tfrac{1}{2}R(1 + \sin\theta) = 4531.75 \text{ km.}$$

Using equation 9.6.3, we see that the period is

$$P = \frac{2\pi}{\sqrt{GM}}a^{3/2} = 3.036168 \times 10^3 \text{ s.}$$

The angle $180^\circ - \theta$ is the true anomaly υ_1 at departure from A, and the angle $180^\circ + \theta$ is the true anomaly υ_1 at arrival at B. If we can find the mean anomaly at these two points, we should be able to find the time taken to get from A to B. We'll need to use Kepler's equation, so we'll need to know the eccentricity *e* of the ellipse.

The distance between the foci is $2ae = R\cos\theta$, from which we find that

$$e = 0.637\,070$$

To find the two eccentric anomalies, E_1 and E_2 we need one or more of equations 2.3.17d-g. Just make sure that you have the correct quadrant. I make them

$$E_1 = 129^\circ\!.5737 \qquad E_2 = 230^\circ\!.4263$$

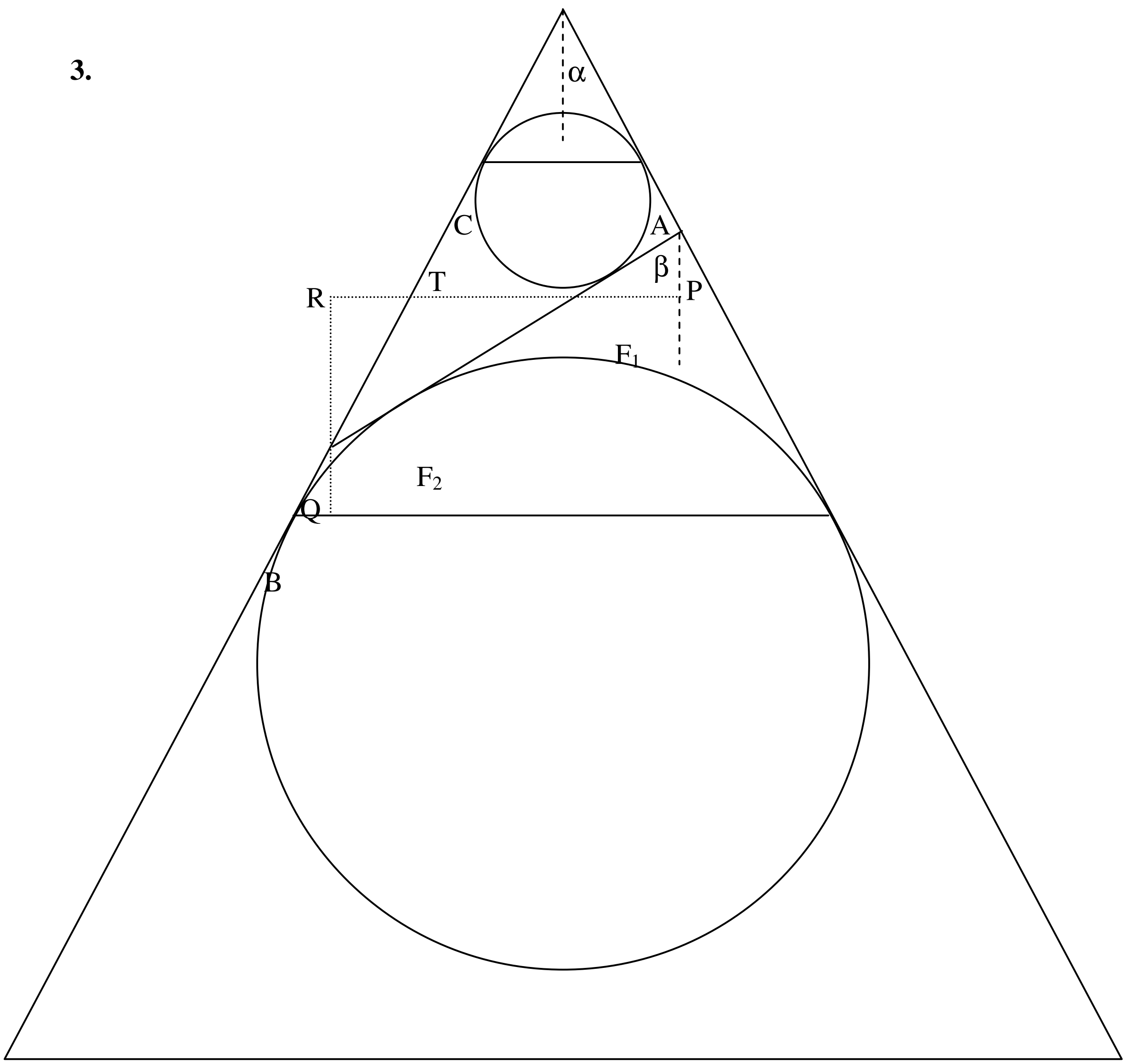
Now, Kepler's equation (equation 9.6.5), $\boldsymbol{M} = E - e\sin E$, gives us the mean anomalies. I make them

$$\boldsymbol{M}_1 = 101^\circ\!.4382 \qquad \boldsymbol{M}_2 = 258^\circ\!.5618$$

The difference is 157º.1236. The period of the entire orbit is 3.036168×10^3 s, so the duration of the journey is $\frac{157.1236}{360} \times 3.036168 \times 10^3 \text{ s} = 22.1$ minutes.

One last question: What is the maximum height above Earth reached by the vehicle? Since this is a relatively easy question, I'll leave it to the reader.

3.



PQ is the major axis of the ellipse, of length $2a$.
The distance BC is also equal to $2a$.
F_1F_2 is the distance between the foci, which is $2ae$.

I am going to try to find the vertical distance between P and Q. I.e., the height of P above Q. I.e. the distance QR.

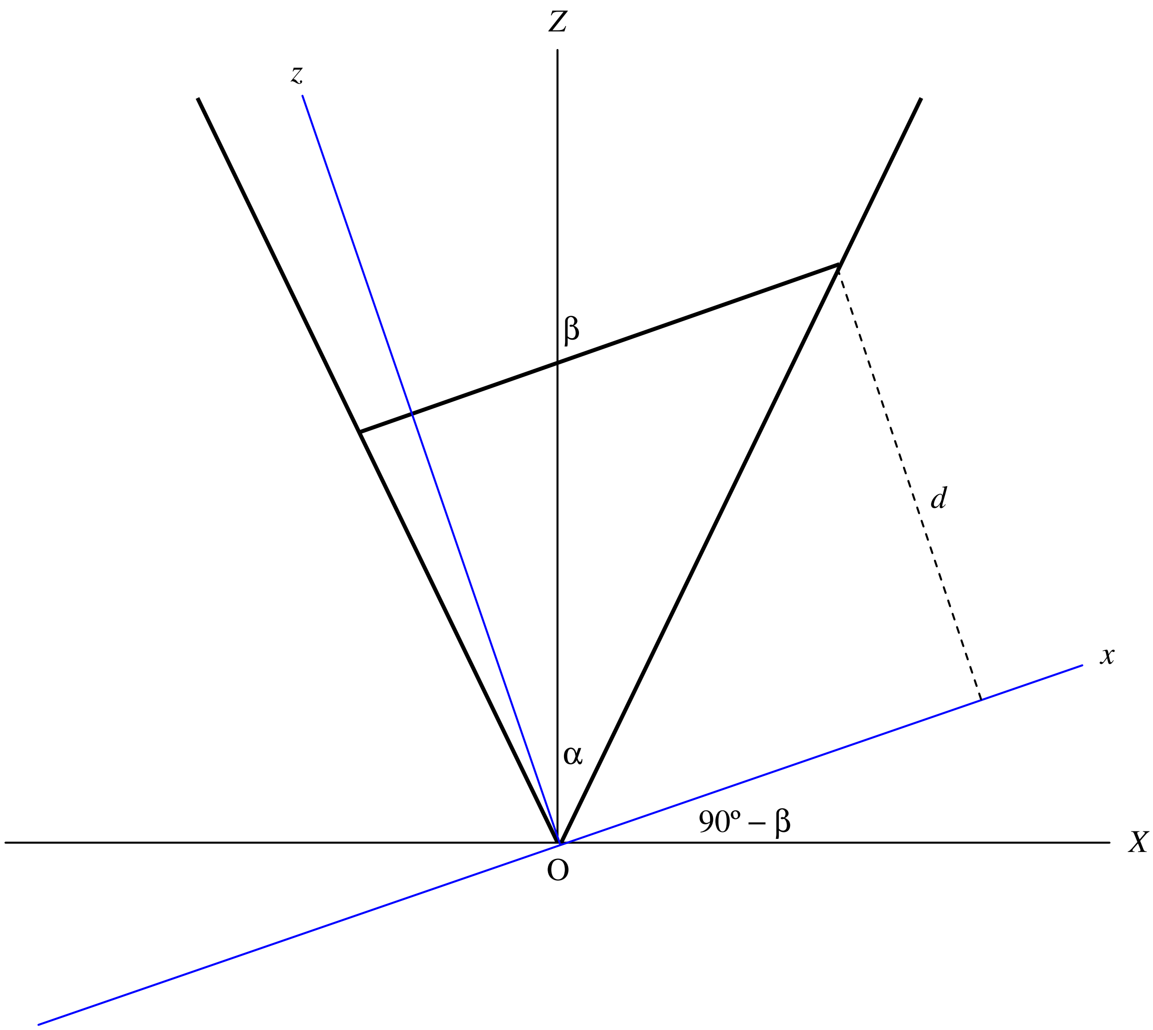
The lengths QB, QF_2, PA, PF_1, CT are all equal to $a(1-e)$.

The distance QT = $BC - BQ - CT = 2a - a(1-e) - a(1-e) = 2ae$.
Hence the distance QR is $2ae\cos\alpha$. But it is also equal to $2a\cos\beta$.

$$\text{Hence } \underline{\underline{e = \frac{\cos\beta}{\cos\alpha}}}.$$

I have drawn this for the case $\beta > \alpha$ (ellipse). You might also show that you get the same result for $\beta = \alpha$ (parabola) and for $\beta < \alpha$ (hyperbola). If the cone is a cylinder $(\alpha = 0)$, the eccentricity of the elliptical cross section is $\cos \beta$.

Here is an alternative solution, for which I am indebted to Pal Achintya of India:



The equation to the cone, referred to the axes OXYZ (the Y-axis directed away from the reader) is

$$X^2 + Y^2 - Z^2 \tan^2 \alpha = 0$$

We now refer to a second set of axes, Oxyz, as shown. Coordinates in the two systems are related by

$$\begin{aligned} X &= x\sin\beta - z\cos\beta \\ Y &= y \\ Z &= x\cos\beta + z\sin\beta \end{aligned}$$

The equation to the cone referred to this system is therefore

$$(x\sin\beta - z\cos\beta)^2 + y^2 - (x\cos\beta + z\sin\beta)^2\tan^2\alpha = 0$$

The equation to the plane is $\quad z = d$

Thus the equation to intersection between the plane and the cone is

$$(x\sin\beta - d\cos\beta)^2 + y^2 - (x\cos\beta + d\sin\beta)^2\tan^2\alpha = 0\,,$$

which, on making use of trigonometric identities, can be written

$$x^2(1-\cos^2\beta\sec^2\alpha) - x(d\sin 2\beta\sec^2\alpha) + y^2 + d^2(1-\sin^2\beta\sec^2\alpha) = 0$$

This can be cast in the form

$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which shows that the plane section of the cone (with $\beta > \alpha$) is an ellipse. It may require some effort to obtain the constants a, b and x_0 explicitly in terms of α, β and d. However, it is not necessary to do this, because it is almost immediately evident that the ratio $\dfrac{b^2}{a^2}$ must equal $1-\cos^2\beta\sec^2\alpha$. And since, for an ellipse, $b^2 = a^2(1-e^2)$, where e is the eccentricity, it follows that $e = \dfrac{\cos\beta}{\cos\alpha}$.

4. Notation:

Speed of asteroid immediately before the explosion = V.

Speed of part that moves in a circular orbit immediately after the explosion = V_1.

Speed of other part immediately after the explosion = V_2.

Semi major axis of the elliptic orbit of this part = a_2

Mass of Sun $= M$. Gravitational constant = G.

Some relevant equations:

$$V^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$V_1^2 = \frac{GM}{r}$$

$$V_2^2 = GM\left(\frac{2}{r} - \frac{1}{a_2}\right)$$

$$\tfrac{1}{2}mV_1^2 + \tfrac{1}{2}mV_2^2 = 2 \times \tfrac{1}{2}(2m)V^2$$

These should be enough to show that $\underline{\underline{a_2 = \frac{ar}{4r - 5a}}}$.