CHAPTER 2
BLACKBODY RADIATION

2.1 *Introduction.*

This chapter briefly summarizes some of the formulas and theorems associated with blackbody radiation. A small point of style is that when the word "blackbody" is used as an adjective, it is usually written as a single unhyphenated word, as in "blackbody radiation"; whereas when "body" is used as a noun and "black" as an adjective, two separate words are used. Thus a black body emits blackbody radiation. The Sun radiates energy only very approximately like a black body. The radiation from the Sun is only very approximately blackbody radiation.

2.2 *Absorptance, and the Definition of a Black Body.*

If a body is irradiated with radiation of wavelength λ, and a fraction $a(\lambda)$ of that radiation is absorbed, the remainder being either reflected or transmitted, $a(\lambda)$ is called the *absorptance* at wavelength λ. Note that λ is written in parentheses, to mean "at wavelength λ", not as a subscript, which would mean "per unit wavelength interval". The fractions of the radiation reflected and transmitted are, respectively, the *reflectance* and the *transmittance*. The sum of the absorptance, reflectance and transmittance is unity, unless you can think of anything else that might happen to the radiation.

A body for which $a(\lambda) = 1$ for all wavelengths is a black body.

A body for which a has the same value for all wavelengths, but less than unity, is a grey body.

(Caution: We may meet the word "absorbance" later. It is not the same as absorptance.)

2.3 *Radiation within a cavity enclosure.*

Consider two cavities at the same temperature. We'll suppose that the two cavities can be connected by a "door" that can be opened or closed to allow or to deny the passage of radiation between the cavities. We'll suppose that the walls of one cavity are bright and shiny with an absorptance close to zero, and the walls of the other cavity are dull and black with an absorptance close to unity. We'll also suppose that, because of the difference in nature of the walls of the two cavities, the radiation density in one is greater than in the other. Let us open the door for a moment. Radiation will flow in both directions, but there will be a net flow of radiation from the high-radiation-density cavity to the low-radiation-density cavity. As a consequence, the temperature of one cavity will rise and the temperature of the other will fall. The (now) hotter cavity can then be used as a source and the (now) colder cavity can be used as a sink in order to operate a heat engine which can then do external work, such work, for example, to be used for repeatedly opening and closing the door separating the two cavities. We have thus constructed a perpetual motion machine that can continue to do work without the expenditure of energy.

From this absurdity, we can conclude that, despite the difference in nature of the walls of the two cavities (which were initially at the same temperature), the radiation densities within the two cavities must be equal. We deduce the important principle that the radiation density inside an enclosure is determined solely by the temperature and is independent of the nature of the walls of the enclosure.

2.4 *Kirchhoff's Law*

Kirchhoff's law, as well as his studies with Bunsen (who invented the Bunsen burner for the purpose) showing that every element has its characteristic spectrum, represents one of the most important achievements of mid-nineteenth century physics and chemistry. The principal results were published in 1859, the same year as Darwin's *The Origin of Species*, and it has been claimed that the publication of Kirchhoff's law was at least as influential in the advance of science as the Darwinian theory of evolution. It is therefore distressing that so few people can achieve the triple task of spelling his name, pronouncing it correctly, and properly stating his law. Kirchhoff and Bunsen laid the foundations of quantitative and qualitative spectroscopy.

Imagine an enclosure filled with radiation at some temperature such that the energy density per unit wavelength interval at wavelength λ is $u_\lambda(\lambda)$. Here I have used a subscript and parentheses, according to the convention described in section 1.3, but, to avoid excessive pedantry, I shall henceforth omit the parentheses and write just u_λ. Imagine that there is some object, a football, perhaps, levitating in the middle of the enclosure and consequently being irradiated from all sides. The irradiance, in fact, per unit wavelength interval, is given by equation 1.17.1

$$E_\lambda = u_\lambda c/4 \qquad 2.4.1$$

If the absorptance at wavelength λ is $a(\lambda)$, the body will absorb energy per unit area per unit wavelength interval at a rate $a(\lambda)E_\lambda$.

The body will become warm, and it will radiate energy. Let the rate at which it radiates energy per unit area per unit wavelength interval (i.e. the exitance) be M_λ. When the body and the enclosure have reached an equilibrium state, the rates of absorption and emission of radiant energy will be equal:

$$M_l = a(\lambda)E_\lambda. \qquad 2.4.2$$

But E and u are related through equation 2.4.1, and u_λ is independent of the nature of the surface (of the walls of the enclosure or of any body within it), and so we see that the ratio of the exitance to the absorptance of any surface is independent of the nature of the surface. This is *Kirchhoff's Law*. (In popular parlance, "good emitters are good absorbers".) The ratio is a function only of temperature and wavelength. For a black body, the absorptance is unity, and the exitance is then the Planck function.

2.5 *An aperture as a black body.*

We consider an enclosure at some temperature and consequently filled with radiation of density u_l per unit wavelength interval. The inside walls of the enclosure are being irradiated at a rate given by equation 2.4.1. Now pierce a small hole in the side of the enclosure. Radiation will now pour out of the enclosure at a rate per unit area that is equal to the rate at which the walls are being radiated from within. In other words the exitance of the radiation emanating from the hole is the same as the irradiance within. Now irradiate the hole from outside. The radiation will enter the hole, and very little of it will get out again; the smaller the hole, the more nearly will all of the energy directed at the hole fail to get out again. The hole therefore absorbs like a black body, and therefore, by Kirchhoff's law, it also radiates like a black body. Put another way, a black body will radiate in the same way as will a small hole pierced in the side of an enclosure. Sometimes, indeed, a warm box with a small hole in it is used to emulate blackbody radiation and thus to calibrate the sensitivity of a radio telescope.

2.6 *Planck's equation*

The importance of Planck's equation in the early birth of quantum theory is well known. Its theoretical derivation is dealt with in courses on statistical mechanics. In this section I merely give the relevant equations for reference.

Planck's equation can be given in various ways, and here I present four. All will be given in terms of exitance. The radiance is the exitance divided by π .(Equation 1.15.2.). The four forms are as follows, in which I have made use of equations 1.3.1 and the expression $h\nu = hc/\lambda$ for the energy of a single photon.

The rate of emission of energy per unit area per unit time (i.e. the exitance) per unit wavelength interval:

$$M_\lambda = \frac{C_1}{\lambda^5\left(e^{K_1/\lambda T} - 1\right)} \qquad 2.6.1$$

The rate of emission of photons per unit area per unit time per unit wavelength interval:

$$N_\lambda = \frac{C_2}{\lambda^4\left(e^{K_1/\lambda T} - 1\right)} \qquad 2.6.2$$

The rate of emission of energy per unit area per unit time (i.e. the exitance) per unit frequency interval:

$$M_\nu = \frac{C_3\nu^3}{e^{K_2\nu/T} - 1} \qquad 2.6.3$$

The rate of emission of photons per unit area per unit time per unit frequency interval:

$$N_\nu = \frac{C_4 \nu^2}{e^{K_2 \nu / T} - 1} \qquad 2.6.4$$

The constants are:

$$C_1 = 2\pi hc^2 = 3.7418 \times 10^{-16} \text{ W m}^2 \qquad 2.6.5$$

$$C_2 = 2\pi c = 1.8837 \times 10^{9} \text{ m s}^{-1} \qquad 2.6.6$$

$$C_3 = 2\pi h/c^2 = 4.6323 \times 10^{-50} \text{ kg s} \qquad 2.6.7$$

$$C_4 = 2\pi/c^2 = 6.9910 \times 10^{-17} \text{ m}^{-2} \text{ s}^2 \qquad 2.6.8$$

$$K_1 = hc/k = 1.4388 \times 10^{-2} \text{ m K} \qquad 2.6.9$$

$$K_2 = h/k = 4.7992 \times 10^{-11} \text{ s K} \qquad 2.6.10$$

Symbols:
h = Planck's constant
k = Boltzmann's constant
c = speed of light
T = temperature
λ = wavelength
ν = frequency

2.7 *Wien's Law.*

The wavelengths or frequencies at which these functions reach a maximum, and what these maximum values are, can be found by differentiation of these functions. They do not all come to a maximum at the same wavelength. For the four functions (equations 2.6.1,2,3,4) the wavelengths or frequencies at which the maxima occur are given by:

For equation (2.6.1):

$$\lambda = W_1/T \qquad 2.7.1$$

For equation (2.6.2):

$$\lambda = W_2/T \qquad 2.7.2$$

For equation (2.6.3):

$$\nu = W_3 T \qquad 2.7.3$$

For equation (2.6.4):

$$\nu = W_4 T \qquad 2.7.4$$

Any of these equations (but more usually the first one) may be referred to as *Wien's law*.

The constants are

$$W_n = \frac{hc}{kx_n}, \qquad (n = 1,2) \qquad 2.7.5$$

$$W_n = \frac{kx_n}{h}, \qquad (n = 3,4) \qquad 2.7.6$$

where the x_n are the solutions of

$$x_n = (6 - n)\left(1 - e^{-x_n}\right) \qquad 2.7.7$$

and have the values

$$x_1 = 4.965114 \qquad 2.7.8$$

$$x_2 = 3.920690 \qquad 2.7.9$$

$$x_3 = 2.821439 \qquad 2.7.10$$

$$x_4 = 1.593624 \qquad 2.7.11$$

The Wien constants then have the values

$$W_1 = 2.8978 \times 10^{-3} \quad \text{m K} \qquad 2.7.12$$

$$W_2 = 3.6697 \times 10^{-3} \quad \text{m K} \qquad 2.7.13$$

$$W_3 = 5.8790 \times 10^{10} \quad \text{Hz K}^{-1} \qquad 2.7.14$$

$$W_4 = 3.3206 \times 10^{10} \quad \text{Hz K}^{-1} \qquad 2.7.15$$

The maximum ordinates of the functions are given by

$$M_\lambda\,(\max) = A_1 T^5 \qquad 2.7.16$$

$$N_\lambda\,(\max) = A_2 T^4 \qquad 2.7.17$$

$$M_\nu(\max) = A_3 T^3 \qquad 2.7.18$$

$$N_\nu(\max) = A_4 T^2 \qquad 2.7.19$$

The constants A_n are given by

$$A_n = \frac{2\pi k^{6-n} y_n}{h^4 c^3}, \qquad (n = 1,2) \qquad 2.7.20$$

$$A_n = \frac{2\pi k^{6-n} y_n}{h^2 c^2} \qquad (n = 3,4) \qquad 2.7.21$$

where the y_n are dimensionless numbers defined by

$$y_n = \frac{x_n^{6-n}}{e^{x_n} - 1} \qquad 2.7.22$$

That is, $y_1 = 21.20144$ 2.7.23

$$y_2 = 4.779841 \qquad 2.7.24$$

$$y_3 = 1.421435 \qquad 2.7.25$$

$$y_4 = 0.6476102 \qquad 2.7.26$$

The constants A_n therefore have the values

$$A_1 = 1.2867 \times 10^{-5} \quad \mathrm{W\ m^{-2}\ K^{-5}\ m^{-1}} \qquad 2.7.27$$

$$A_2 = 2.1011 \times 10^{17} \quad \mathrm{ph\ s^{-1}\ m^{-2}\ K^{-4}\ m^{-1}} \qquad 2.7.28$$

$$A_3 = 5.9568 \times 10^{-19} \quad \mathrm{W\ m^{-2}\ K^{-3}\ Hz^{-1}} \qquad 2.7.29$$

$$A_4 = 1.9657 \times 10^{4} \quad \mathrm{ph\ s^{-1}\ m^{-2}\ K^{-2}\ Hz^{-1}} \qquad 2.7.30$$

2.8 *Stefan's Law (The Stefan-Boltzmann Law).*

The total exitance integrated over all wavelengths or frequencies can be found by integrating equations 2.6.1,2,3,4. Integration of 2.6.1 over wavelengths or of 2.6.3 over frequencies each, of course, gives the same result:

$$M = \sigma T^4 \qquad 2.8.1$$

where $$\sigma = \frac{2\pi^5 k^4}{15h^3c^2} = 5.6705\times10^{-8} \quad \text{W m}^{-2}\ \text{K}^{-4} \qquad 2.8.2$$

This is Stefan's Law, or the Stefan-Boltzmann law, and σ is Stefan's constant.

Integration of 2.6.2 over wavelengths or of 2.6.4 over frequencies each, of course, gives the same result:

$$N = \rho T^3 \qquad 2.8.3$$

where $$\rho = \frac{4\pi\zeta(3)k^3}{h^3c^2} = 1.5205\times10^{-8} \qquad \text{ph s}^{-1}\ \text{m}^{-2}\ \text{K}^{-3} \qquad 2.8.4$$

Here $\zeta(3)$ is the *Riemann zeta-function*:

$$\zeta(3) = 1 + \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{3}\right)^3 + \left(\tfrac{1}{4}\right)^3 + \ldots = 1.202057 \qquad 2.8.5$$

2.9 *A Thermodynamical Argument*

I have pointed out that Wien's and Stefan's laws can be derived by differentiation and integration respectively of Planck's equation. Readers should try that for themselves if only to convince themselves that neither is particularly easy, nor indeed is the derivation of Planck's equation to begin with. Those who succeed may justifiably congratulate themselves. Those who fail may console themselves with the thought that Stefan's law was derived from a simple thermodynamical argument long before the derivation of Planck's equation, and it is not necessary to know Planck's equation, let alone how to differentiate it or integrate it, in order to arrive at Stefan's law. You do, however, have to know a little thermodynamics.

Among the plethora of thermodynamical relations is to be found one that reads:

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P \qquad 2.9.1$$

The derivation is usually started by writing entropy as a function of volume and temperature, and the derivation is commonly found as a preliminary to the derivation of the Joule effect for a nonideal gas. When equation 2.9.1 is applied to the equation of state of a non-ideal gas (for example a van der Waals gas), it can be used to calculate the drop in temperature during a Joule expansion. We wish to apply it, however, to radiation in an enclosure.

We assume that the radiation is isotropic and in a steady state. Under such conditions, photons will presumably bounce rather than stick at the walls, otherwise they would rapidly become depleted - or, if they are absorbed, others are being emitted at the same rate. Either way, the radiation pressure is given by equation 1.18.3, i.e. $P = u/3$. The energy density depends only on the temperature and not on the volume; therefore the term $(\partial U/\partial V)_T$ on the left hand side of equation 2.9.1 is just the energy density u. And since the pressure is $u/3$, the term $(\partial P/\partial T)_V$ is $\frac{1}{3}(du/dT)_V$.

Equation 2.9.1 therefore becomes

$$u = \frac{T}{3}\left(\frac{du}{dT}\right) - \frac{u}{3} \qquad 2.9.2$$

or

$$4u = T\frac{du}{dT}, \qquad 2.9.3$$

which yields Stefan's law upon integration.

2.10 *Dimensionless forms of Planck's equation*

The Planck functions (of wavelength or frequency and temperature) can be collapsed on to dimensionless functions of a single variable if we express the exitance in units of the maximum exitance, and the wavelength or frequency in units of the wavelength or frequency at which the maximum occurs. Equations 2.7.1-4 and 16-20 will be needed to achieve this, and the reader might enjoy doing it as a challenge. (I said "might".) The results are

$$M_\lambda = \frac{b_1}{\lambda^5\left(e^{x_1/\lambda} - 1\right)} \qquad 2.10.1$$

$$N_\lambda = \frac{b_2}{\lambda^4\left(e^{x_2/\lambda} - 1\right)} \qquad 2.10.2$$

$$M_\nu = \frac{b_3\nu^3}{e^{x_3\nu} - 1} \qquad 2.10.3$$

$$N_\nu = \frac{b_4\nu^2}{e^{x_4\nu} - 1} \qquad 2.10.4$$

where $$b_n = e^{x_n} - 1 \qquad (n = 1,2,3,4) \qquad 2.10.5$$

The numerical values of x_n are given in equations 2.7.8-11, and the values of b_n are

$$b_1 = 142.32492 \qquad 2.10.6$$

$$b_2 = 49.435253 \qquad 2.10.7$$

$$b_3 = 15.801016 \qquad 2.10.8$$

$$b_4 = 3.9215536 \qquad 2.10.9$$

The numbers x_n y_n, b_n are independent of the values of any physical constants such as h, c or k, and will not change as our knowledge of these values improves. These functions, which are independent of temperature, are drawn in figures II.1,2,3,4 shown at the end of this chapter.

Example

Here is an example to show the use of the dimensionless functions to calculate the blackbody radiance quickly. What is the radiance per unit wavelength of a 5000 K black body at 400 nm? You may prefer to calculate this directly from equation 2.6.1, but let's try it using the dimensionless form. From equation 2.7.1 we find that the wavelength at which maximum exitance per unit wavelength occurs is 579.56 nm, and therefore our dimensionless wavelength to be inserted into equation 2.10.1 is 0.6902. This gives a radiance per unit wavelength interval (in units of the maximum) of 0.6832. But equation 2.7.16 gives the maximum radiance per unit wavelength interval as 4.0178×10^{13} W m^{-2} m^{-1} and therefore the radiance at 400 nm is 2.745×10^{13} W m^{-2} m^{-1}.

I don't think there is much point in integrating the dimensionless functions 2.10.1-4 over all wavelengths and frequencies, but, for the record:

$$\int_0^\infty M_\lambda d\lambda = \frac{\pi^4 x_1}{15 y_1} = 1.52080 \qquad 2.10.10$$

$$\int_0^\infty N_\lambda d\lambda = \frac{2\zeta(3) x_2}{y_2} = 1.97199 \qquad 2.10.11$$

$$\int_0^\infty M_\nu d\nu = \frac{\pi^4}{15 x_3 y_3} = 1.61924 \qquad 2.10.12$$

$$\int_0^\infty N_\nu d\nu = \frac{2\zeta(3)}{x_4 y_4} = 2.32946 \qquad 2.10.13$$

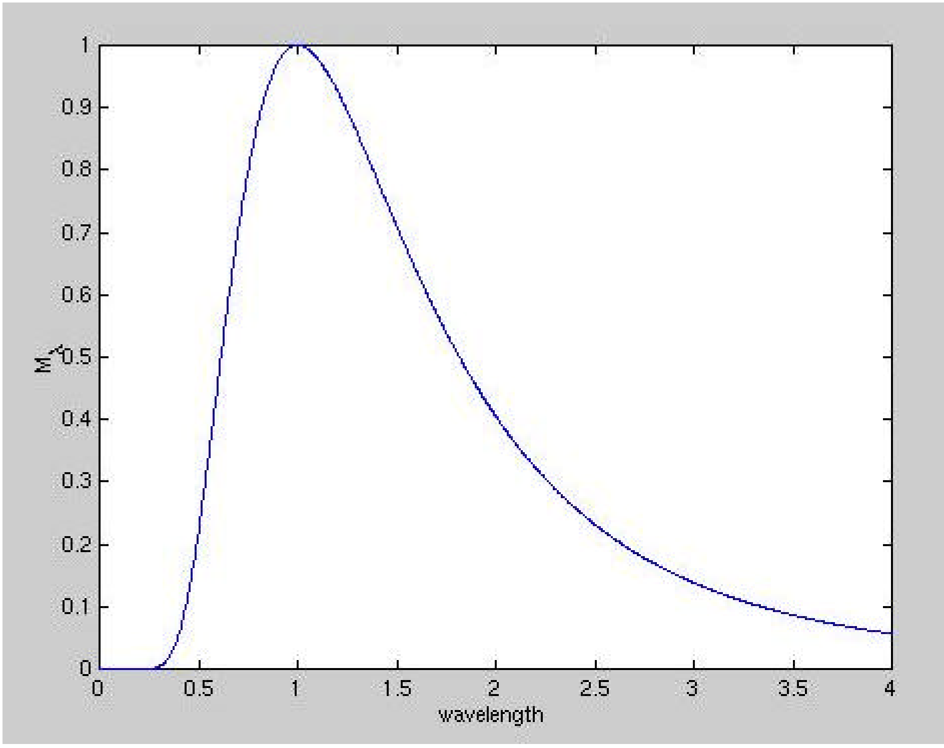

Figure II.1. Blackbody exitance per unit wavelength interval. The equation is equation 2.10.1, with constants given by equations 2.10.6 and 2.7.8. The maximum value is given by equations 2.7.16 and 2.7.27. It occurs at a wavelength given by equations 2.7.1 and 2.7.12.

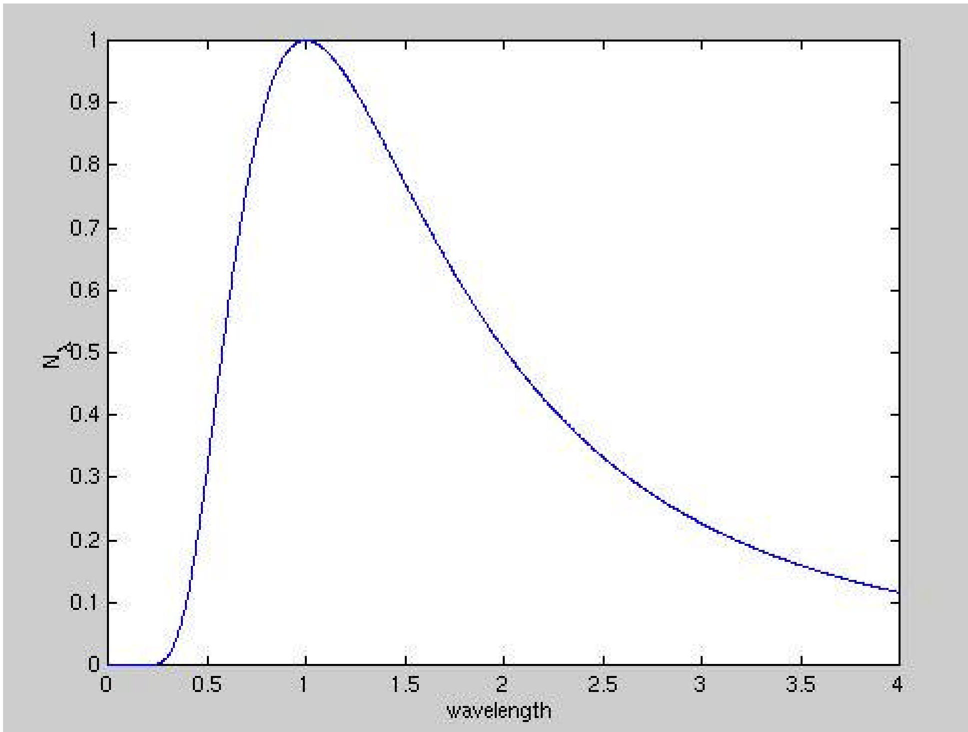

Figure II.2. Blackbody photon exitance per unit wavelength interval. The equation is equation 2.10.2 with constants given by equations 2.10.7 and 2.7.9. The maximum value is given by equations 2.7.17 and 2.7.28. It occurs at a wavelength given by equations 2.7.2 and 2.7.13.

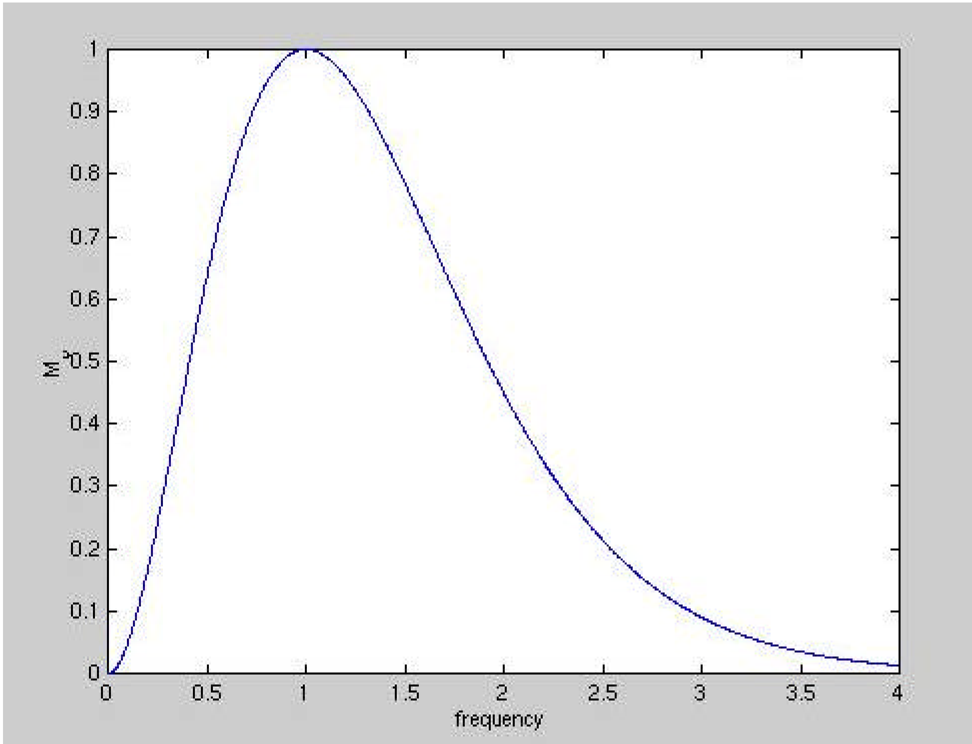

Figure II.3. Blackbody radiance per unit frequency interval. The equation is equation 2.10.3 with constants given by equations 2.10.8 and 2.7.10. The maximum value is given by equations 2.7.18 and 2.7.29. It occurs at a frequency given by equations 2.7.3 and 2.7.14.

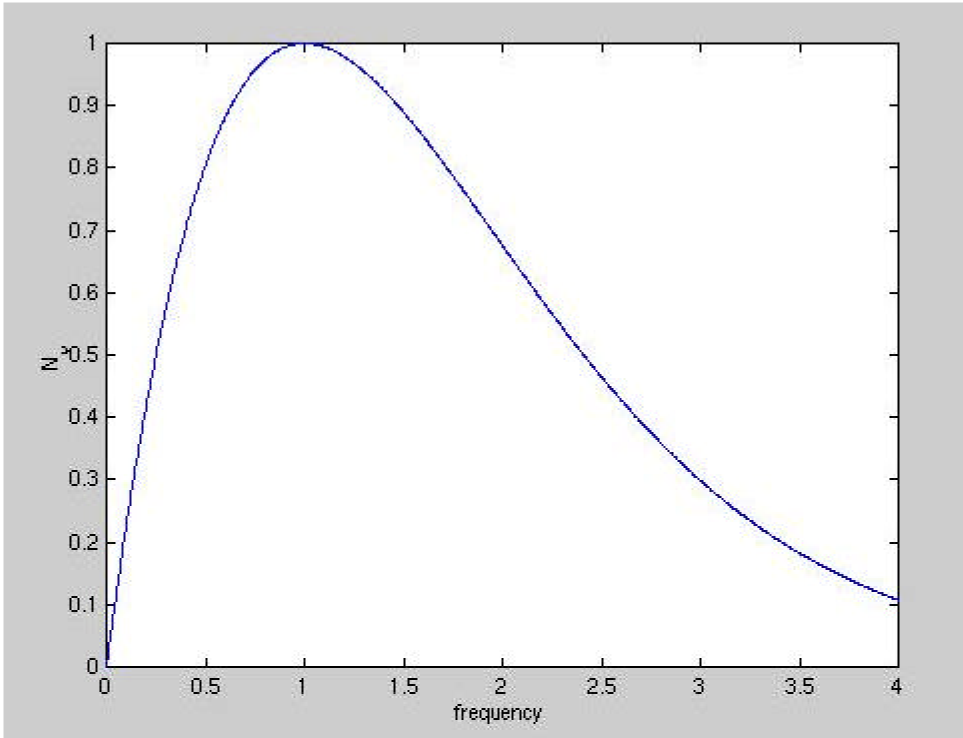

Figure II.4. Blackbody photon radiance per unit frequency interval. The equation is equation 2.10.4 with constants given by equations 2.10.9 and 2.7.11. The maximum value is given by equations 2.7.19 and 2.7 30. It occurs at a frequency given by equations 2.7.4 and 2.7.15.

2.11 *Derivation of Wien's and Stefan's Laws*

Wien's and Stefan's Laws are found, respectively, by differentiation and integration of Planck's equation. Neither of these is particularly easy, and they are not found in every textbook. Therefore, I derive them here.

Wien's Law

Planck's equation for the exitance per unit wavelength interval (equation 2.6.1) is

$$\frac{M}{C} = \frac{1}{\lambda^5 \left(e^{K/\lambda T} - 1 \right)}, \qquad 2.11.1$$

in which I have omitted some subscripts. Differentiation gives

$$\frac{1}{C}\frac{dM}{d\lambda} = -\frac{1}{\left(e^{K/\lambda T}-1\right)^2}\cdot\left[5\lambda^4.\left(e^{K/\lambda T}-1\right)+\lambda^5.\left(-\frac{K}{\lambda^2 T}\right)e^{K/\lambda T}\right]. \qquad 2.11.2$$

M is greatest when this is zero; that is, when

$$x = 5\left(1-e^{-x}\right), \qquad 2.11.3$$

where

$$x = \frac{K}{\lambda T}. \qquad 2.11.4$$

Hence, with equation 2.6.9, the wavelength at which M is a maximum, is given by

$$\lambda = \frac{hc}{kxT}. \qquad 2.11.5$$

The maximum value of M is found be substituting this vale of λ back into Planck's equation, to arrive at equation 2.7.16. The corresponding versions of Wien's Law appropriate to the other version's of Planck's equation are found similarly.

Stefan's Law.

Integration of Planck's equation to arrive at Stefan's law is a bit more tricky.

It should be clear that $\int_0^\infty M_\lambda d\lambda = \int_0^\infty M_\nu d\nu$, and therefore I choose to integrate the easier of the functions, namely M_ν. To integrate M_λ, the first thing we would do anyway would be to make the substitution $\nu = c/\lambda$.

Planck's equation for the blackbody exitance per unit frequency interval is

$$M_\nu = C_3\int_0^\infty \frac{\nu^3 d\nu}{e^{K_2\nu/T}-1}. \qquad 2.11.6$$

Let $x = K_2\nu/T$; then

$$M_\nu = \frac{2\pi k^4 T^4}{c^2 h^3}\int_0^\infty \frac{x^3 dx}{e^x-1}, \qquad 2.11.7$$

And, except for the numerical value of the integral, we already have Stefan's law. The integral can be evaluated numerically, but not without difficulty, and there is an analytical solution for it.

Consider the indefinite integral and integrate it by parts:

$$\int \frac{x^3 dx}{e^x - 1} = x^3 \ln\left(1 - e^{-x}\right) - 3\int x^2 \ln\left(1 - e^{-x}\right) dx + \text{const}.$$

Now put the limits in:

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = -3\int_0^\infty x^2 \ln\left(1 - e^{-x}\right) dx.$$

Write down the Maclaurin expansion of the integrand:

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = 3\int_0^\infty x^2 \left(e^{-x} + \tfrac{1}{2} e^{-2x} + \tfrac{1}{3} e^{-3x} + \ldots\right) dx$$

and integrate term by term to obtain

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right) \qquad 2.11.8$$

We must now evaluate $1 + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \ldots$

The series $\displaystyle\sum_1^\infty \frac{1}{n^m}$ is the *Riemann* ζ-*function*. For m = 1, it diverges. For m = 3, 5, 7, etc., it has to be evaluated numerically. For m = 2, 4, 6, etc., the sums can be written explicitly in terms of π. For example:

$$\zeta(2) = \frac{\pi^2}{6},$$

$$\zeta(4) = \frac{\pi^4}{90},$$

$$\zeta(6) = \frac{\pi^6}{945}.$$

One of the stages necessary in evaluating the ζ-function is to derive the infinite product

$$\frac{\sin \alpha\pi}{\alpha\pi} = \left[1 - \alpha^2\right]\left[1 - \left(\tfrac{1}{2}\alpha\right)^2\right]\left[1 - \left(\tfrac{1}{3}\alpha\right)^2\right]\ldots \qquad 2.11.9$$

If we can do that, we are more than halfway there.

Let's start by considering the Fourier expansion of $\cos\theta x$:

$$\cos\theta x = \sum_0^\infty a_n \cos nx \qquad 2.11.10$$

In equation 2.11.10 n is an integer, θ not necessarily so; we shall suppose that θ is some number between 0 and 1. There is no need to consider any sine terms, because $\cos\theta x$ is an even function of x. We work out what the Fourier coefficients are in the usual way, to get

$$a_n = (-1)^n \frac{2\theta \sin\theta\pi}{\theta^2 - n^2}, \qquad n = 1, 2, 3, \ldots \qquad 2.11.12$$

As usual, and for the usual reason, a_0 is an exception:

$$a_0 = \frac{\sin\theta\pi}{\theta\pi}. \qquad 2.11.13$$

We have therefore arrived at the Fourier expansion of $\cos\theta x$:

$$\cos\theta x = \frac{2\theta\sin\theta\pi}{\pi}\left(\frac{1}{2\theta^2} - \frac{\cos x}{\theta^2 - 1^2} + \frac{\cos 2x}{\theta^2 - 2^2} - \frac{\cos 3x}{\theta^2 - 3^2} + \ldots\right) \qquad 2.11.14$$

Put $x = \pi$ and rearrange slightly:

$$\pi\cot\theta\pi - \frac{1}{\theta} = 2\theta\left(\frac{1}{\theta^2 - 1^2} + \frac{1}{\theta^2 - 2^2} + \ldots\right). \qquad 2.11.15$$

Since we are assuming that θ is some number between 0 and 1, we shall re-write this so that the denominators are all positive:

$$\pi\cot\theta\pi - \frac{1}{\theta} = -\frac{2\theta}{1^2 - \theta^2} - \frac{2\theta}{2^2 - \theta^2} - \ldots \qquad 2.11.16$$

Now multiply both sides by $d\theta$ and integrate from $\theta = 0$ to $\theta = \alpha$. The integration must be done with care. The indefinite integral of the left hand side is $\ln\sin\theta\pi - \ln\theta +$ constant, i.e. $\ln\left(\frac{\sin\theta\pi}{\theta}\right) +$ constant. The definite integral between 0 and α is $\ln\left(\frac{\sin\alpha\pi}{\alpha}\right) - \lim_{\theta\to 0}\ln\left(\frac{\sin\theta\pi}{\theta}\right).$

The limit of the second term is $\ln\,\pi$, so the definite integral is $\ln\left(\frac{\sin\alpha\pi}{\alpha\pi}\right)$. Integrating the right hand side is a bit easier, so we arrive at

$$\ln\left(\frac{\sin\alpha\pi}{\alpha\pi}\right) = \ln\left(\frac{1^2-\alpha^2}{1^2}\right) + \ln\left(\frac{2^2-\alpha^2}{2^2}\right) + \dots \qquad 2.11.17$$

On taking the antilogarithm, we arrive at the required infinite product:

$$\frac{\sin\alpha\pi}{\alpha\pi} = \left[1-\alpha^2\right]\left[1-\left(\tfrac{1}{2}\alpha\right)^2\right]\left[1-\left(\tfrac{1}{3}\alpha\right)^2\right]\dots \qquad 2.11.18$$

Now expand this as a power series in α^2:

$$\frac{\sin\alpha\pi}{\alpha\pi} = 1 + (\;)\alpha^2 + (\;)\alpha^4 + (\;)\alpha^6 + \dots \qquad 2.11.19$$

The first one is easy, but subsequent ones rapidly get more difficult, but you do have to get at least as far as α^4.

Now compare this expansion with the ordinary Maclaurin expansion:

$$\frac{\sin\alpha\pi}{\alpha\pi} = 1 - \frac{\pi^2}{3!}\alpha^2 + \frac{\pi^4}{5!}\alpha^4 - \dots \qquad 2.11.20$$

and we arrive at the correct expressions for the Riemann ζ-functions. We then get for Stefan's law:

$$M = \frac{2\pi^5 k^4}{15h^3c^2}T^4 = \sigma T^4, \qquad 2.11.21$$

where $\sigma = 5.6705 \times 10^{-8}$ W m^{-2} K^{-4}.

Finally, now that you have struggled through Riemann's zeta-function, let's just make sure that you have understood the really simple stuff, so here are a couple of easy questions – and you won't have to bother with zeta-functions.

1. By what factor should the temperature of a black body be increased so that

   a) The integrated radiance (over all frequencies) is doubled?

   b) The frequency at which its radiance is greatest is doubled?

   c) The spectral radiance per unit wavelength interval at its wavelength of maximum spectral radiance is doubled?

2. A block of shiny silver (absorptance = 0.23) has a bubble inside it of radius 2.2 cm, and it is held at a temperature of 1200 K.

A block of dull black carbon (absorptance = 0.86) has a bubble inside it of radius 4.3 cm, and it is held at a temperature of 2300 K,

Calculate the ratio $\dfrac{\text{Integrated radiation energy density inside the carbon bubble}}{\text{Integrated radiation energy density inside the silver bubble}}$.

Answers. 1. a) 1.189 b) 2.000 c) 1.149

2. 13.5