CHAPTER 11
PHOTOGRAPHIC ASTROMETRY

11.1 *Introduction*

Astrometry is the art and science of measuring positions of celestial objects, and indeed the first step in determining the orbit of a new asteroid or comet is to obtain a set of good astrometric positions. For much of the twentieth century, most astrometric positions were determined photographically, although transit circle measurements were (and still are in some applications) important. A photographic plate or film would be baked for several hours in an oven in an atmosphere of dry hydrogen and nitrogen. This "hypersensitization" was known to increase the sensitivity of the emulsion in long exposures. The film would then be exposed through a telescope to an area of the sky containing the asteroid. An hour or so later, a second photograph would be exposed, the asteroid presumably having moved slightly between the exposures. Exposure times would be from several minutes to an hour or even more, and the telescope had to be carefully guided throughout the long exposure. After exposure, the film had to be developed in a chemical solution in a dark-room, then "fixed" in another solution, washed under running water, and hung up to dry. After these procedures, which took some hours, preparation for measurement could start. The first thing to do would be to identify the asteroid. (In Mrs Beecham's words, "First catch your hare".) To do this, the two photographs would be viewed rapidly one after the other with a blink comparator (in which case the asteroid would move to and fro) or viewed simultaneously with a stereocomparator (in which case the asteroid would appear to be suspended in air above the film). Next, a number of comparison stars would have to be identified. This would be done by consulting a star catalogue and laboriously plotting the positions of the stars on a sheet of paper and comparing the pattern with what was seen on the photographs.

Each photograph would then be placed in a "measuring engine", or two-coordinate measuring microscope, and the x- and y-coordinates of the stars and the asteroid would be measured. Tedious calculations would be performed to convert the measurements to right ascension and declination. The results of this process, which would typically take several hours, would then be sent by mail to the Minor Planet Center of the International Astronomical Union in Cambridge, Massachusetts.

Starting in the early 1990s, photographic astrometry started to be superseded by CCD (charge coupled device) astrometry, and today almost no astrometry is done photographically, the CCD having taken over more or less completely. Everyone knows that the quantum efficiency of a CCD is far superior to that of a photographic emulsion, so that one can now image much fainter asteroids and with much shorter exposures. But that is only the beginning of the story – the CCD and other modern technologies have completely changed the way in which astrometry is carried out. For example, vast catalogues containing the positions of hundreds of millions of faint stars are stored in computer files, and the computer can automatically compare the positions of the stars in its catalogue with the star images on the CCD; thus the hitherto laborious process of identifying the comparison stars is carried out automatically and almost instantaneously. Further, there is no measurement to be done – each stellar image is already sitting on a particular pixel (or group of pixels), and all that has to be done is to read which pixels contain the stellar images. The positional measurements are all inherently completed as soon as the CCD is exposed. The positional measurements (of dozens of stars rather than a mere half-dozen) can then be automatically transferred into a computer program that carries out the necessary trigonometrical calculations to convert them to right ascension and declination, and the results can then be automatically sent by electronic mail to the Minor Planet Center. The entire process, which formerly took many hours, can now be done in less than a minute, to much higher precision than formerly, and for much fainter objects.

Why, then, would you ever want to read a chapter on photographic astrometry? Well, perhaps you won't. After all, to convert your observations to right ascension and declination today, a single key on your computer keyboard will do it all. But this is because *someone*, somewhere, and usually a very anonymous person, has written for you a highly efficient computer program that carries out all the necessary calculations, so that you can do useful astrometry even if you don't know the difference between a sine and a cosine. Thus you can probably safely bypass this chapter.

However, for those who wish to plod through it, this chapter describes how to convert the positional measurements on a photographic film (or on a CCD) to right ascension and declination – a process that is carried out by modern computer software, even if you are unaware of it. Much of this chapter is based on an article by the author published in the *Journal of the Royal Astronomical Society of Canada* **76**, 97 (1982), and you may want to consult that in the hope that I might have made it clear in either one place or the other.

11.2 *Standard Coordinates and Plate Constants*

We shall suppose that the optic axis of the telescope, whose effective focal length is F, is pointing to a point C on the celestial sphere, whose right ascension and declination are (A, D). The stars, as every astrophysicist knows, are scattered around on the surface of the celestial sphere, which is of arbitrary radius, and I shall take the radius to be equal to F, the focal length of the telescope. In figure XI.1, I have drawn the tangent plane to the sky at C, which is what will be recorded on the photograph. In the tangent plane (which is similar to the plane of the photographic plate or film) I have drawn two orthogonal axes: Cξ to the east and Cη to the north. I have drawn a star, Q, whose coordinates are (α, δ), on the surface of the celestial sphere, and its projection, Q', on the tangent plane, where its coordinates are (ξ, η). Every star is similarly mapped on to the tangent plane by a similar projection. The coordinates (ξ, η) are called the *standard coordinates* of the star, and our first task is to find a relation between the equatorial coordinates (α, δ) on the surface of the celestial sphere and the standard coordinates (ξ, η) on the tangent plane or the photograph.

In figure XI.2, I have re-drawn figure XI.1, and, in addition to the star Q and its projection Q', I have also drawn the north Celestial Pole P and its projection P'. The point P' is on the η axis. The spherical triangle PQC maps onto the plane triangle P' Q'C. On the spherical triangle PQC, the side PQ = $90^\circ - \delta$ and the side PC = $90^\circ - D$.

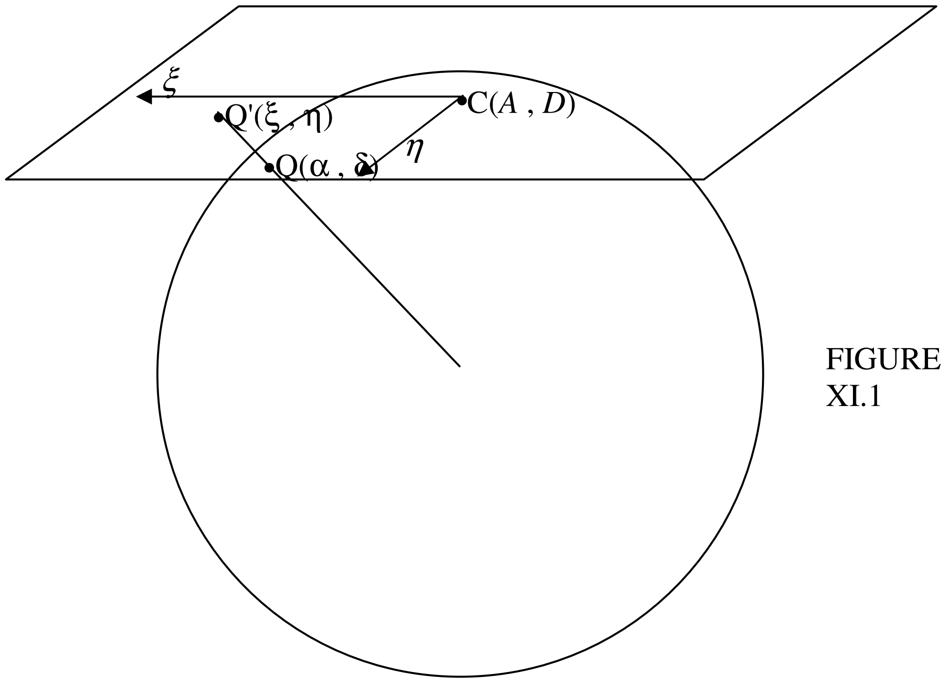

FIGURE XI.1

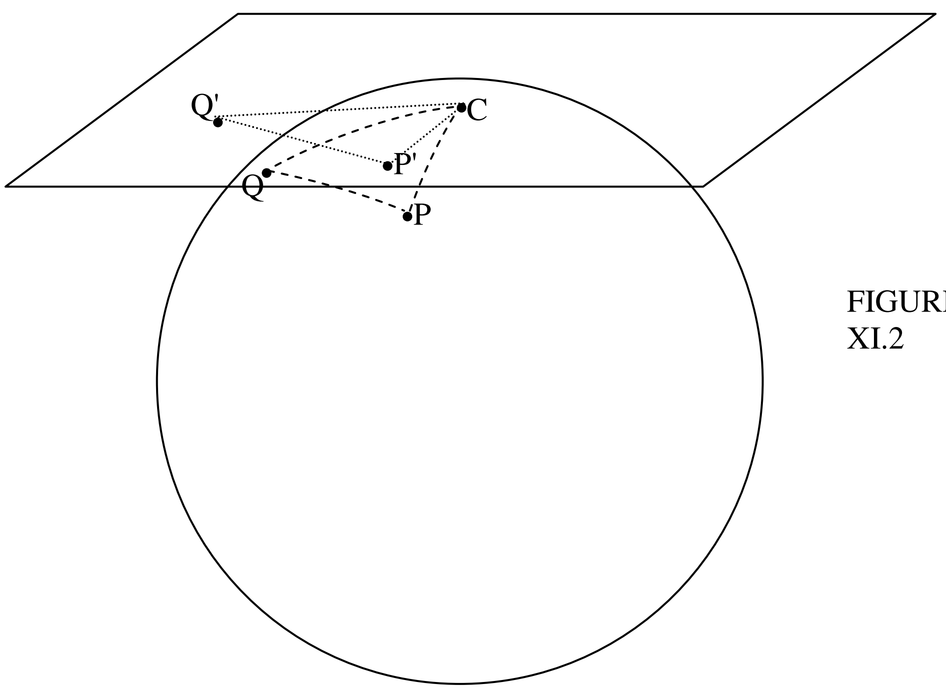

FIGURE XI.2

The angle PCQ in the spherical triangle PCQ is equal to the angle P'CQ' in the plane triangle P'CQ', and I shall call that angle γ. I shall call the arc CQ in the spherical triangle ε. In figure XI.3 I draw the tangent plane, showing the ξ- and η-axes and the projections, P' and Q' of the pole P and the star Q, as well as the plane triangle P'CQ'.

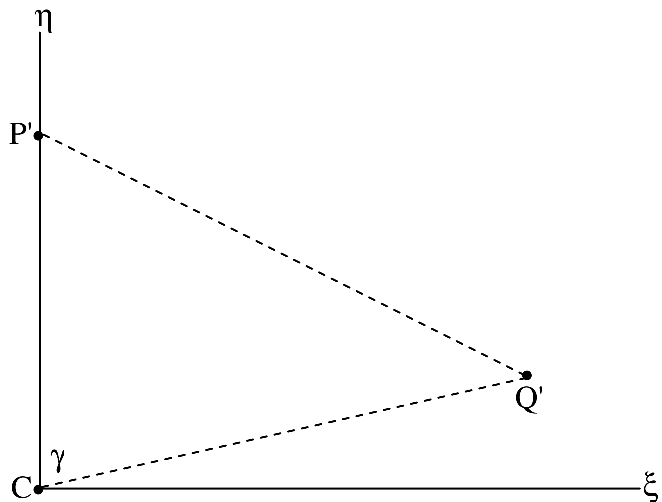

FIGURE XI.3

The ξ and η coordinates of Q' are (CQ'sin γ , CQ'cos γ). And by staring at figures XI.1 and XI.2 for a while, you can see that CQ' = F tan ε. Thus the standard coordinates of the image Q' of the star on the photograph, in units of the focal length of the telescope, are $(\tan\varepsilon\,\sin\gamma\,,\,\tan\varepsilon\cos\gamma)$. It remains now to find expressions for tan ε sin γ and tan ε cos γ in terms of the right ascensions and declinations of Q and of C. I draw now, in figure XI.4, the spherical triangle PCQ.

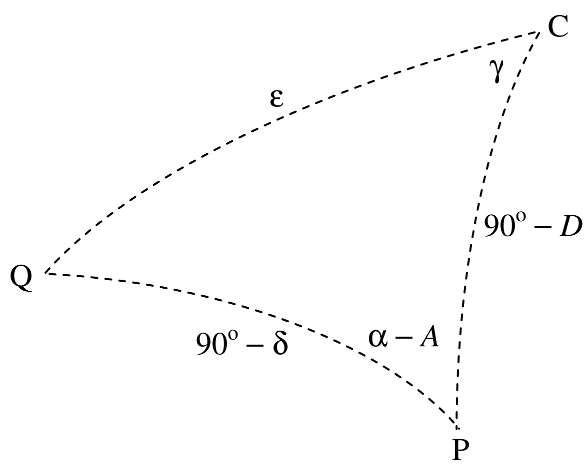

FIGURE XI.4

It is easy, from the usual formulas for spherical triangles, to obtain expressions for $\cos\varepsilon$ and for $\tan\gamma$:

$$\cos\varepsilon = \sin\delta\sin D + \cos\delta\cos D\cos(\alpha - A) \qquad 11.2.1$$

and

$$\tan\gamma = \frac{\sin(\alpha - A)}{\cos D\tan\delta - \sin D\cos(\alpha - A)}\,, \qquad 11.2.2$$

from which one can (eventually) calculate the standard coordinates $(\xi\,,\,\eta)$ of the star. It is also possible to calculate explicit expressions for $\tan\varepsilon\sin\gamma$ and for $\tan\varepsilon\cos\gamma$. Thus, by further applications of the spherical triangle formulas, we have

$$\tan\varepsilon = \frac{\cos D}{\sin D\cos\gamma + \cot(\alpha - A)\sin\gamma}\,. \qquad 11.2.3$$

Multiplication of equation 11.2.3 by $\sin\gamma$ gives $\tan\varepsilon\sin\gamma$ except that $\tan\gamma$ appears on the right hand side. This, however, can be eliminated by use of equation 11.2.2, and one obtains, after some algebra:

$$\xi = \tan\varepsilon\sin\gamma = \frac{\sin(\alpha - A)}{\sin D\tan\delta + \cos D\cos(\alpha - A)}\,. \qquad 11.2.4$$

In a similar way, you can multiply equation 11.2.3 by $\cos\gamma$, and again eliminate $\tan\gamma$ and eventually arrive at

$$\eta = \tan\varepsilon\cos\gamma = \frac{\tan\delta - \tan D\cos(\alpha - A)}{\tan D\tan\delta + \cos(\alpha - A)}\,. \qquad 11.2.5$$

These give the standard coordinates of a star or asteroid at $(\alpha\,,\,\delta)$ in units of the focal length F.

Now it would seem that all we have to do is to measure the standard coordinates $(\xi\,,\,\eta)$ of an object, and we can immediately determine its right ascension and declination by inverting equations 11.2.4 and 11.2.5:

$$\tan(\alpha - A) = \frac{\xi}{\cos D - \eta\sin D} \qquad 11.2.6$$

and

$$\tan\delta = \frac{(\eta\cos D + \sin D)\sin(\alpha - A)}{\xi}\,. \qquad 11.2.7$$

Indeed in principle that is what we have to do – but in practice we are still some way from achieving our aim.

One small difficulty is that we do not know the effective focal length F (which depends on the temperature) precisely. A more serious problem is that we do not know the exact position of the plate centre, nor do we know that the directions of travel of our two-coordinate measuring engine are parallel to the directions of right ascension and declination.

The best we can do is to start our measurements from some point near the plate centre and measure (in mm rather than in units of F) the horizontal and vertical distances (x , y) of the comparison stars and the asteroid from our arbitrary origin. These (x , y) coordinates are called, naturally, the *measured coordinates*.

The measured coordinates will usually be expressed in millimetres (or perhaps in pixels if a CCD is being used), and the linear distance s between any two comparison star images is found by the theorem of Pythagoras. The angular distance ω between any two stars is given by solution of a spherical triangle as

$$\cos\omega = \sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(\alpha_1 - \alpha_2) \qquad 11.2.8$$

The focal length F is then s/ω, and this can be calculated for several pairs of stars and averaged. From that point the standard coordinates can then be expressed in units of F.

The *measured coordinates* (x , y) are displaced from the *standard coordinates* (ξ , η) by an unknown translation and an unknown rotation (figure XI.5) , but the relation between them, if unknown, is at least linear (but see subsection 11.3.5) and thus of the form:

$$\xi - x = ax + by + c, \qquad 11.2.9$$

$$\eta - y = dx + ey + f. \qquad 11.2.10$$

The constants $a - f$ are the *plate constants.* They are determined by measuring the standard coordinates for a minimum of three comparison stars whose right ascensions and declinations are known and for which the standard coordinates can therefore be calculated. Three sets of equations 11.2.9 and 10 can then be set up and solved for the plate constants. In practice more than three comparison stars should be chosen, and a least squares solution determined. For how to do this, see either section 8 of chapter 1, or the article cited in section 1 of this chapter. In the photographic days, just a few (perhaps half a dozen) comparison stars were used. Today, when there are catalogues containing hundreds of millions of stars, and CCD measurement and automatic computation are so much faster, several dozen comparison stars may be used, and any poor measurements (or poor catalogue positions) can quickly be identified and rejected.

Having determined the plate constants, equations 11.2.9 and 10 can be used to calculate the standard coordinates of the asteroid, and hence its right ascension and declination can be calculated from equations 11.2.6 and 7.

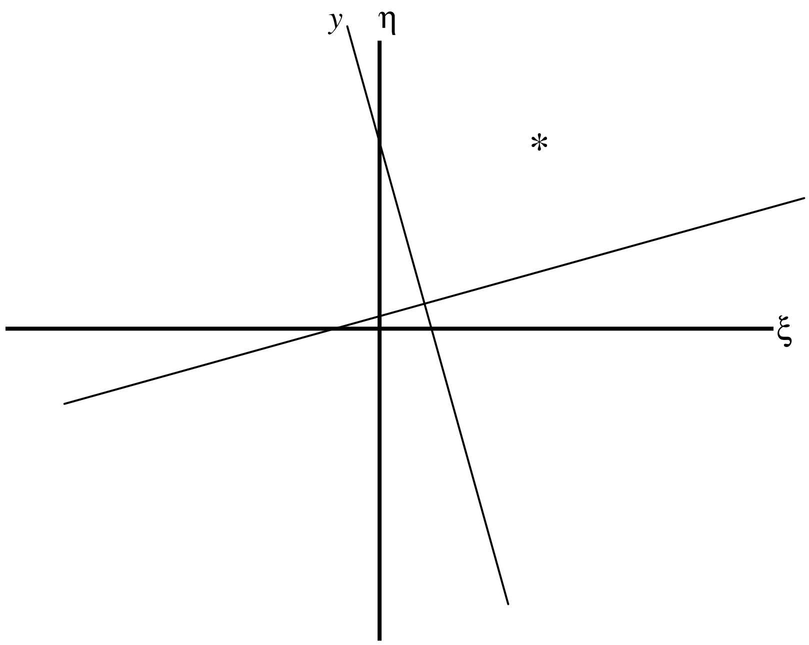

FIGURE XI.5

It should be noted that the position of the asteroid that you have measured – and should report to the Minor Planet Center – is the *topocentric* position (i.e. as measured from your position on the surface of Earth) rather than the *geocentric* position (as seen from the centre of Earth). The Minor Planet Center expects to receive from the observer the topocentric position; the MPC will know how to make the correction to the centre of Earth.

11.3 *Refinements and Corrections*

For precise work there are a number of refinements that should be considered, some of which should be implemented, and some which probably need not be. Things that come to mind include parallax and proper motion of the comparison stars, refraction, aberration, optical distortion, mistakes – which include such things as poor measurements, blends, poor or erroneous catalogue positions or any of a number of mistakes caused by human or instrumental frailty. If you write your own reduction programs, you will know which of these refinements you have included and which you have left out. If you use a "pre-packaged" program, you may not always know whether a given correction has been included.

Let us now look at some of these refinements.

11.3.1 *Parallaxes of the Comparison Stars*

Unless you are unlucky enough to choose as one of your comparison stars Proxima Centauri (whose parallax is much less than an arcsecond), the parallaxes of the comparison stars are not normally something that the asteroid astrometrist has to worry about.

11.3.2 *Proper Motions of the Comparison Stars*

Corrections for the proper motions of the comparison stars should certainly be made if possible.

Until a quarter of a century or so ago, a typical stellar catalogue used by asteroid observers was the *Smithsonian Astrophysical Stellar Catalog* containing the positions *and proper motions* of about a quarter of a million stars down to about magnitude 9. This catalogue gives the position (right ascension and declination referred to the equinox and equator of B1950.0) of each star at the time of the original epoch when the photograph on which the catalogue was based, and also the position of each star corrected for proper motion to the epoch 1950.0, as well as the proper motion of each star. Thus for the star SAO013800 the position (referred to the equinox and equator of B1950.0) *at the original epoch* is given as

$$\alpha_{1950.0} = 08^{\mathrm{h}}\ 14^{\mathrm{m}}\ 40^{\mathrm{s}}.390 \qquad \delta_{1950.0} = +65^{\circ}\ 09'\ 18''.87$$

and the proper motion is given as

$$\mu_\alpha = -0^{\mathrm{s}}.0058 \qquad \mu_\delta = -0''.085 \text{ per year}$$

The epoch of the original source is not immediately readable from the catalogue, but can be deduced from information therein. In any case the catalogue gives the position (referred to the equinox and equator of B1950.0) corrected for proper motion to the epoch 1950.0:

$$\alpha_{1950.0} = 08^{\mathrm{h}}\ 14^{\mathrm{m}}\ 40^{\mathrm{s}}.274 \qquad \delta_{1950.0} = +65^{\circ}\ 09'\ 17''.16$$

Now, suppose that you had taken a photograph in 1980. At that time we were still referring positions to the equinox and equator of B1950.0 (today we use J2000.0), but you would have to correct the position for proper motion to 1980; that is, you need to apply the proper motion for the 30 years since 1950. The position, then, in 1980, referred to the equinox and equator of B1950.0) was

$$\alpha_{1950.0} = 08^{\mathrm{h}}\ 14^{\mathrm{m}}\ 40^{\mathrm{s}}.100 \qquad \delta_{1950.0} = +65^{\circ}\ 09'\ 14''.61$$

and this is the position of the star that should be used in determining the plate constants.

One problem with this was that the proper motions were not equally reliable for all the stars (although the catalogue does list the formal standard errors in the proper motions), and there are a few stars in which the proper motion is even given with the wrong sign! In such cases, correcting for proper motion obviously does more harm than good. However, the stars with the "worst" proper motions are generally also those with the smallest proper motions; it can probably be assumed that the stars with significant proper motions also have proper motions that are well determined.

The situation changed in the 1990s with the widespread introduction of CCDs and the publication of the *Guide Star Catalog* containing positions of about half a billion stars down to about magnitude 21. With modern instrumentation one would never normally consider using comparison stars anything like as bright as magnitude 9 (the faint limit of the SAO Catalog). You now have the opportunity of choosing many more comparison stars, and faint ones, whose positions can be much more precisely measured than bright stars. Also, the *Guide Star Catalog* gives positions referred to the equinox and equator of J2000.0, which is the present-day norm for reporting astrometric positions. A difficulty is, however, that the GSC positions were obtained at only one epoch, so that proper motions are not available for the GSC stars, and hence proper motions cannot be applied. The standard response to this drawback is that, since faint stars (magnitude 16 and fainter) can be used, proper motions are negligible. Further, the epoch at which the GSC positions were obtained is recent, so again the proper motion correction is negligible. One always had certain qualms about accepting this assurance, since the apparent magnitude of a star depends not only on its distance but also on its absolute luminosity. Stars are known to have an enormous range in luminosity, and it is probable that stars of low luminosity stars are the commonest stars in the Galaxy, and consequently many of the apparently faint stars in the GSC may also be intrinsically faint stars that are nearby and may have appreciable proper motions. Furthermore, as time marches inexorably on, the epoch of the GSC becomes less and less "recent" and one cannot go on indefinitely declaring that proper motion corrections are negligible.

Today, however, the catalogue favoured for astrometric observations of asteroids is the USNU-B Catalog. (USNO = United States Naval Observatory.) This has positions *and proper motions* for more than a billion objects, so there is no longer any excuse for not applying proper motion corrections to the comparison stars.

11.3.3 *Refraction*

Refraction of starlight as it passes through Earth's atmosphere displaces the position of the star towards the zenith. The amount of the refraction is not large close to the zenith, but it amounts to about half a degree near the horizon. Earth's atmosphere is but a thin skin compared with the radius of Earth, and, provided that the star is not close to the horizon, we may treat the atmosphere as a plane-parallel atmosphere. The situation is illustrated in figure XI.6.

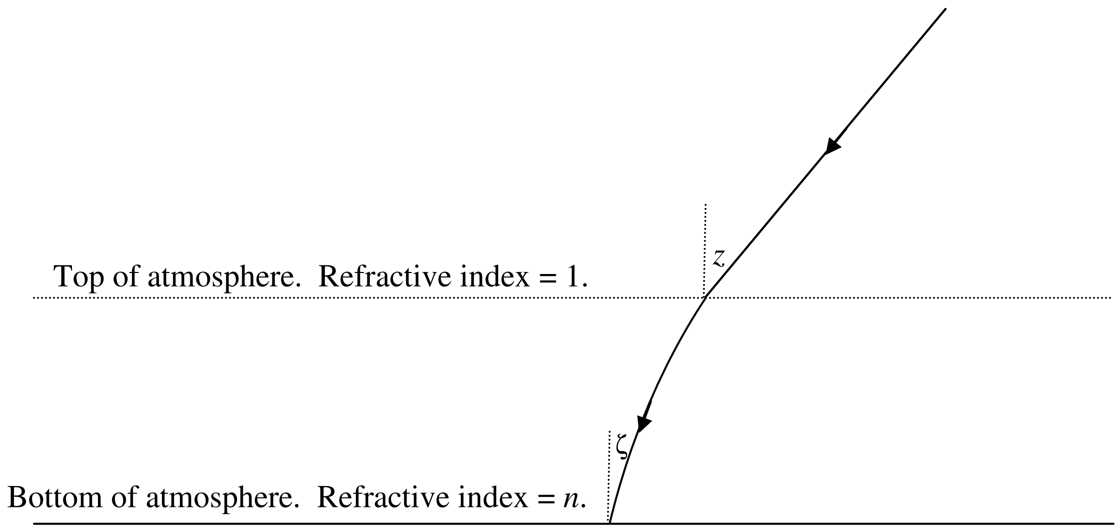

FIGURE XI.6

The angle z is the *true zenith distance* – i.e. the zenith distance it would have in the absence of an atmosphere. The angle ζ is the *apparent zenith distance*. By application of Snell's law, we have $\sin z = n \sin \zeta$, and if we let $\epsilon = z - \zeta$, this becomes

$$\sin\zeta\cos\epsilon + \cos\zeta\sin\epsilon = n\sin\zeta. \qquad 11.3.1$$

Divide both sides by $\sin\zeta$ and make the approximations (correct to first order in ϵ) $\sin\epsilon \approx \epsilon$, $\cos\epsilon \approx 1$, and we obtain

$$\epsilon = z - \zeta = (n-1)\tan\zeta. \qquad 11.3.2$$

The refractive index at ground level varies a little with temperature and pressure, but it averages about $n - 1 = 58''.2$. (You didn't know that refractive index was expressed in arcseconds, did you?)

We have made some approximations in deriving equation 11.3.2, but it must be borne in mind that, as far as astrometry is concerned, what is important is the *differential refraction* between the bottom and top of the detector (photographic film or CCD), and equation 11.3.2 should be more than adequate – unless one is observing very close to the horizon. The only time when one is likely to be observing close to the horizon would be for a bright comet, for which it is very difficult to make precise measurements anyway. The differential refraction between top and bottom obviously amounts to

$$\delta\epsilon = (n-1)\sec^2\zeta\,\delta\zeta, \qquad 11.3.3$$

where $\delta\zeta$ is the range of zenith distance covered by the detector. In the table below I show the differential refraction between top and bottom of a detector (such as a photographic film) with a 5-degree field, and for a detector (such as a CCD) with a 20-arcminute field, for four zenith distances. Obviously, the correction for differential refraction should be made for the 5-degree photographic field. It might be argued that, for the relatively small field of a 20-arcminute CCD, the correction for differential refraction is unimportant. However, the precision expected for modern CCD astrometry is rather higher than the precision that was expected during the photographic era, and certainly, for large zenith distances, if one hopes for sub-arcsecond astrometry, a correction for differential refraction is desirable. Bear in mind, too, that CCDs are becoming larger as technology advances, and that the larger the CCD, the more important will be the refraction correction.

Zenith distance in degrees	$\delta\varepsilon$ in arcseconds for 5° field	$\delta\varepsilon$ in arcseconds for 20' field
15	5.5	0.4
30	6.8	0.5
45	10.2	0.7
60	20.4	1.4

The most straightforward way of correcting for differential refraction is to calculate the true zenith distance z and azimuth A of each comparison star by the usual methods of spherical astronomy:

$$\cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H \qquad 11.3.4$$

and

$$\tan A = \frac{\sin H}{\cos\phi\tan\delta - \sin\phi\cos H}. \qquad 11.3.5$$

Here ϕ is the observer's latitude, and H is the hour angle of the star, to be found from its right ascension and the local sidereal time. Having found z, then calculate the apparent zenith distance ζ from equation11.3.2 (refraction does not, of course, change the azimuth), and then invert equations 11.3.4 and 11.3.5 to obtain the apparent hour angle H' (and hence apparent right ascension α') and apparent declination δ' of the star. Do this for all the comparison stars. (By hand, this might sound long and tedious, but of course when a computer is programmed to do it, it is all automatic and instantaneous.)

$$\sin\delta' = \sin\phi\cos\zeta + \cos\phi\sin\zeta\cos A \qquad 11.3.6$$

and

$$\tan H' = \frac{\sin A\tan\zeta}{\cos\phi - \sin\phi\cos A\tan\zeta}. \qquad 11.3.7$$

You can then carry out the measurements and from them calculate the apparent right ascension and declination of the asteroid. From these, calculate the apparent zenith distance. Correct this to obtain the true zenith distance, and finally calculate the true right ascension and declination of the asteroid – again all of this is done instantaneously once you have correctly programmed the computer.

Another aspect of refraction that might be considered is that blue (early-type) stars are refracted more than red (late-type) stars. In principle, therefore, one should use only comparison stars that are of the same colour as the asteroid. In practice, I imagine that few astrometrists always do this. If, by ill-fortune, one of the comparison stars is *very* red or *very* blue, this may result in a large residual for that star, and the star can be detected and rejected, as described in subsection 11.3.6. Yet another aspect is that, because of *dispersion*, the light from the star – especially if it is low down near the horizon – will be drawn out into a short spectrum, with the red end closer to the horizon than the blue end, and there is then a problem of how to measure the position of the star. The answer is probably to leave asteroids that are close to the horizon to observers who are at a more favourable latitude. As mentioned above, the only time you are likely to observe very low down would be for a long-period comet, on which you cannot set extremely precisely in any case.

11.3.4 *Aberration of light*

By "aberration" I am not referring to optical aberrations produced by lenses and mirrors, such as coma and astigmatism and similar optical aberrations, but rather to the *aberration of light* resulting from the vector difference between the velocity of light and the velocity of Earth. (In these notes, the word "velocity" is used to mean "velocity" and the word "speed" is used to mean "speed". The word "velocity" is not to be used merely as a longer and more impressive word for "speed".)

The effect of aberration is to displace a star towards the *Apex of the Earth's Way*, which is the point on the celestial sphere towards which Earth is moving. The apex is where the ecliptic intersects the observer's meridian at 6 hours local apparent solar time. The amount of the aberrational displacement varies with position on the sky, being greatest for stars 90° from the apex. It is then of magnitude v/c, where v and c are the speeds of Earth and light respectively. This amounts to 20.5 arcseconds. (You didn't know that the speed of Earth could be expressed in arcseconds, did you?) But what matters in astrometry is the *differential aberration* between one edge of the detector (photographic film or CCD) and the other. Evidently this is going to be a much smaller effect than differential refraction.

Let us examine the effect of aberration in figures XI.7a and b.

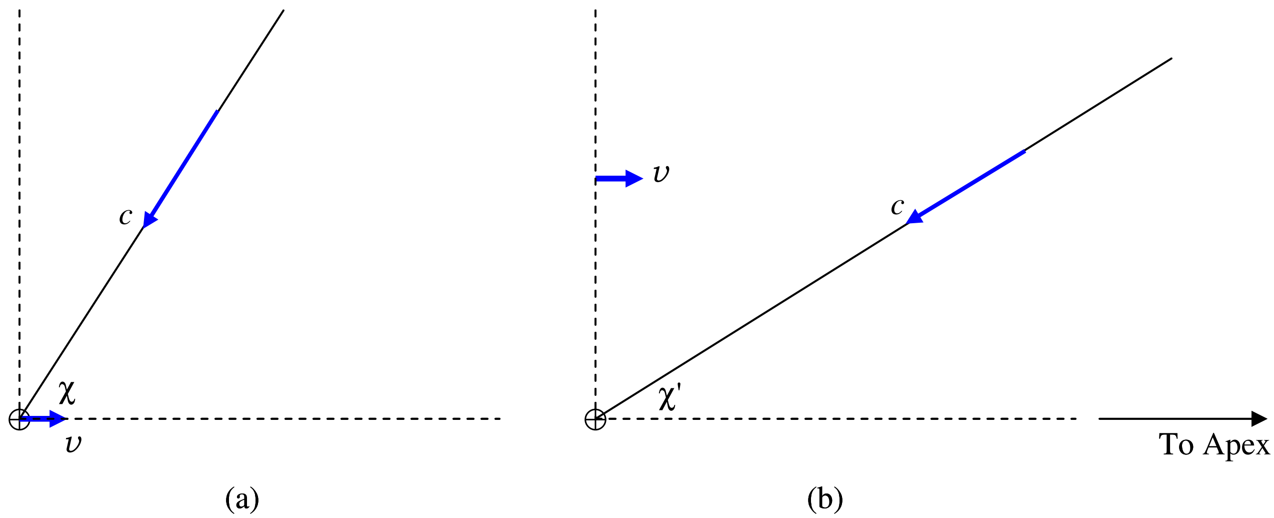

FIGURE XI.7

Part (a) of the figure shows a stationary reference frame. By "stationary" I mean a frame in which Earth, ⊕, is moving towards the apex at speed υ (29.8 km s^{-1}). Light from a star is approaching Earth at speed c from a direction that makes an angle χ, which I shall call the *true apical distance*, with the direction to the apex.

Part (b) shows the same situation referred to a frame in which Earth is stationary; that is the frame (b) is moving towards the apex with speed υ relative to the frame (a). Referred to this frame, the speed of light is c, and it is coming from a direction χ', which I shall call the *apparent apical distance.*

I refer to the difference $\varepsilon = \chi - \chi'$ as the *aberrational displacement.*

For brevity I shall refer to the direction to the apex as the "x-direction" and the upwards direction in the figures as the "y-direction".

Referred to frame (a), the x-component of the velocity of light is $-c\cos\chi$, and referred to frame (b), the x-component of the velocity of light is $-c\cos\chi'$. These are related by the Lorentz transformation between velocity components:

$$c\cos\chi' = \frac{c\cos\chi + \upsilon}{1 + (\upsilon/c)\cos\chi}. \qquad 11.3.8$$

Referred to frame (b), the y-component of the velocity of light is $-c\sin\chi$, and referred to frame (b), the y-component of the velocity of light is $-c\sin\chi'$. These are related by the Lorentz transformation between velocity components:

$$c\sin\chi' = \frac{c\sin\chi}{\gamma\left(1 + (\upsilon/c)\cos\chi\right)}, \qquad 11.3.9$$

in which, if need be, a c can be cancelled from each side of the equation. In equation 11.3.9, γ is the Lorentz factor $1/\sqrt{1-(\upsilon/c)^2}$.

Equations 11.3.8 and 9 are not independent; indeed one may be regarded as just another way of writing the other. One easy way to show this, for example, is to show that $\sin^2\chi' + \cos^2\chi' = 1$. In any case, either of them gives χ' as a function of χ and υ/c.

Figure XI.7 shows χ' as a function of χ for υ/c = 0.125, 0.250, 0.375, 0.500, 0.625, 0.750 and 0.875.

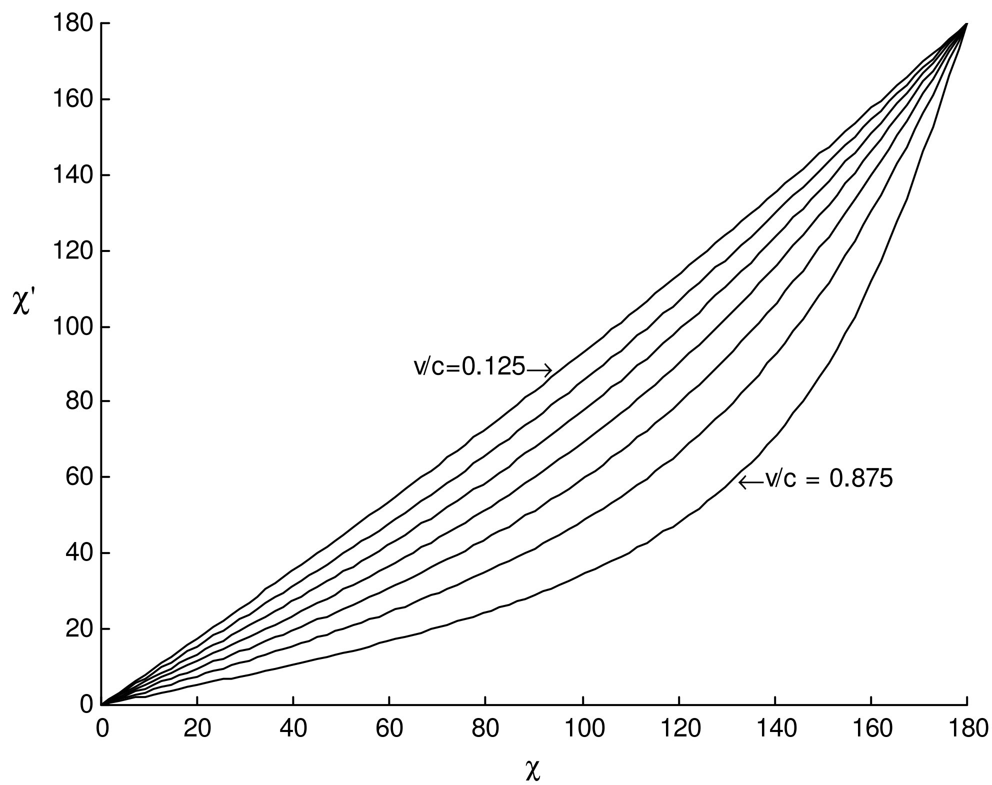

FIGURE XI.7

For Earth in orbit around the Sun, υ = 29.8 km s^{-1} and υ/c = 9.9 × 10^{-5}, which corresponds to an angle of 20".5. Thus the aberrational displacement is very small. If we write $\varepsilon = \chi - \chi'$, equation 11.3.8 takes the form to first order in ε:

$$\cos(\chi - \varepsilon) = \cos\chi + \varepsilon\sin\chi = \frac{\cos\chi + (\upsilon/c)}{1 + (\upsilon/c)\cos\chi}, \qquad 11.3.10$$

from which, after a very little algebra, we find

$$\varepsilon = \frac{(\upsilon/c)\sin\chi}{1 + (\upsilon/c)\chi}, \qquad 11.3.11$$

or, since $\upsilon/c \ll 1$,

$$\varepsilon \approx \frac{\upsilon\sin\chi}{c}. \qquad 11.3.12$$

Thus we see that the aberrational displacement is zero at the apex and at the antapex, and it reaches is greatest value, 20".5, ninety degrees from the apex.

As with refraction, however, it is the *differential* aberration that counts, and if the diameter of the detector field is δχ, the difference δε in the aberrational displacement across the field is

$$\delta\varepsilon = \frac{\upsilon\cos\chi\,\delta\chi}{c}. \qquad 11.3.13$$

Notice that the *differential* aberration is *greatest* at the apex and antapex, and is zero ninety degrees from the apex. It might be noted that the opposition point, where perhaps the majority of asteroid observations are made, is ninety degrees from the apex.

The following table, similar to the one shown for differential refraction, shows the differential aberration across five-degree and 20-arcminute fields for various apical distances.

Apical distance in degrees	δε in arcseconds for 5° field	δε in arcseconds for 20' field
0	1.8	0.12
15	1.7	0.12
30	1.5	0.10
45	1.3	0.09
60	0.9	0.06
90	0.0	0.00

It might be concluded that the effect of differential aberration is so small as to be scarcely worth worrying about in most circumstances. However, the expectations for the precision of asteroid astrometry are now rather stringent and are likely to become more exacting as time progresses, and for precise work the correction should be made. One of the problems with pre-packaged astrometry programs is that the user does not always know what corrections are included in the package. The surest way is to do it oneself.

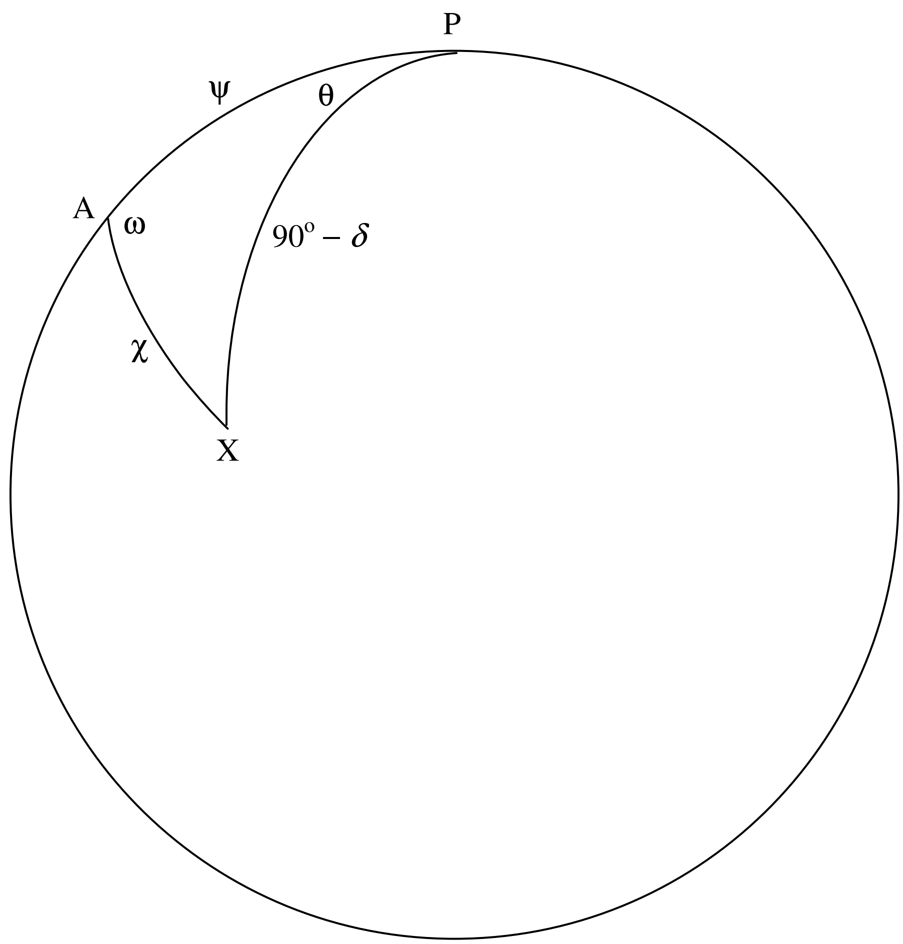

FIGURE XI.8

In figure XI.8, P is the north celestial pole, A is the apex of the Earth's way, and X is a star of true equatorial coordinates (α , δ). The apical distance AX is χ. The angle θ is $\alpha(X) - \alpha(A)$, the angle PAX is ω, and ψ is the distance from pole to apex. It is assumed that the observer knows how to calculate ω and ψ by the usual formulas of spherical astronomy, and hence that all angles in figure XI.8 are known.

From the cotangent formula, we have

$$\cos\psi\cos\omega = \sin\psi\cot\chi - \sin\omega\cos\theta. \qquad 11.3.14$$

If χ is increased by $\delta\chi$, the corresponding increase in θ is given by

$$\sin\omega\sin\theta\,\delta\theta = \sin\psi\csc^2\chi\,\delta\chi. \qquad 11.3.15$$

Here $\delta\theta = \alpha'(\mathrm{X}) - \alpha(\mathrm{X})$, where α and α' are, respectively, the true and apparent right ascensions of the star, and $\delta\chi$ is $\chi' - \chi$, which is $-\varepsilon$. It is easy to err in sign at this point, so I re-write equation 11.3.15 more explicitly:

$$(\alpha'(X) - \alpha(X)).\sin\omega.\sin(\alpha(X) - \alpha(P)) = -\varepsilon\sin\psi\csc^2\chi. \qquad 11.3.16$$

Here ε is the aberrational displacement of X towards A given by equation 11.3.12. On substitution of equation 11.3.12 into equation 11.3.16, this becomes, then,

$$(\alpha'(X) - \alpha(X)).\sin\omega.\sin(\alpha(X) - \alpha(P)) = -\frac{\upsilon}{c}\sin\psi\csc\chi. \qquad 11.3.17$$

This enables us to calculate the apparent right ascension of the star.

The declination is obtained from an application of the cosine formula:

$$\sin\delta = \cos\chi\cos\psi + \sin\chi\sin\psi\cos\omega, \qquad 11.3.18$$

from which $$\cos\delta\,\delta\delta = (-\cos\psi\sin\chi + \sin\psi\cos\omega\cos\chi)\delta\chi. \qquad 11.3.19$$

Here again, as in the usual convention of calculus, $\delta\chi$ represents an increase in χ and $\delta\delta$ is the corresponding increase in δ. But aberration results in a decrease of apical distance, so that $\delta\chi = -\varepsilon$.

Equation 11.3.19 enables us to calculate the apparent declination of the star.

From the measurements of the positions of the comparison stars and the asteroid, we can now calculate the apparent right ascension and declination of the asteroid, and, by inversion of equations 11.3.17 and 11.3.19, we can determine the true right ascension and declination of the asteroid.

11.3.5 *Optical Distortion*

I refer here to pincushion or barrel distortion introduced by the optical system. This results in a displacement of the stellar images towards or away from the plate centre. Unlike differential refraction or aberration of light, the stellar displacements are symmetric with respect to inversion through the plate centre. This is also true of the optical aberration known as coma. A comatic stellar image results in a displacement of the centre of the stellar image away from the plate centre. Thus we can deal with distortion and coma in a similar manner.

This can be best dealt with by assuming a quadratic relation for the difference between true and measured coordinates:

$$\xi - x = ax^2 + 2hxy + by^2 + 2gx + 2fy + c, \qquad 11.3.20$$

$$\eta - y = a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c'. \qquad 11.3.21$$

There are six plate constants in each coordinate, and therefore a minimum of six comparison stars are necessary to solve for them. If more that six are used (which is highly desirable) a least squares solution can be obtained for the plate constants. One then follows the same procedure as in the linear case.

11.3.6 *Errors, Mistakes and Blunders*

I was once told that the distinction between errors, mistakes and blunders was roughly as follows. *Errors* are the inevitable small variations caused by imprecision of measurement, or, in the case of computation, the small random errors produced by rounding off (which, incidentally, should not be done before the final "answer" is arrived at). *Mistakes* are things such as writing a 3 instead of 4, or 56 instead of 65, or writing 944 instead of 994 (this is a common one), or reading a poorly-handwritten 6 as a 0 or a 4, or writing a plus sign instead of a minus (this sort of mistake can be quite large!), or thinking that six times eight is 42. A *blunder* is a complete misconception of the entire problem!

Even with the greatest care, errors and mistakes can occur during measurement and reduction of an astrometric plate. The important thing is to find them and either correct or reject them. A stellar image can be contaminated by blending with another star or with a blemish on the plate. A star can be misidentified. There may be a mistake in the catalogued position, or the proper motion may be poor. A measurement can be poor simply because of fatigue or carelessness.

If only the minimum number of comparison stars are used (i.e. three for a linear plate solution, six for a quadratic plate solution), there is no way of detecting errors and mistakes other than carefully repeating the entire measurement and calculation. Error and mistake detection requires an overdetermination of the solution, by using more than the minimum number of comparison stars.

What has to be done is as follows. Once the plate constants have been determined, the right ascension and declination of each of the comparison stars must be calculated, and compared with the right ascension and declination given in the catalogue. The difference (O − C) is determined for each star, and the standard deviation of the residuals is calculated. Any star with a residual of more that two or three standard deviations should be rejected. The exact criterion for rejection will depend on how many stars we used. Statistical tests will determine the probability that a given residual is a random or gaussian deviation from zero. A full and proper statistical test is slightly laborious (although a computer can make short work of it), and many measurers may decide to reject any star whose residual is more than 2.5 standard deviations from zero, even if this is not strictly the correct statistical way of doing it.