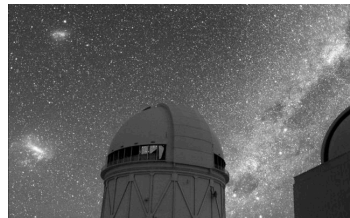


Astronomy 303

The Orbits of the Stars

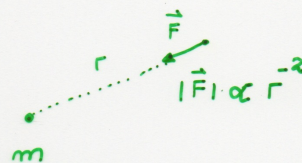
1. Rudiments of Gravitational Theory
2. Motion under gravity: weighing the Galaxy
3. Two-body relaxation
4. Epicycle Theory: the orbits of disk stars



POTENTIAL THEORY

NEWTON'S LAW OF GRAVITATIONAL ATTRACTION BETWEEN PARTICLES.

- CENTRAL
- INVERSE-SQUARE
- LINEAR



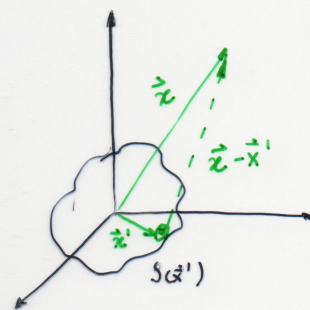
$$\vec{F}(\vec{x}) = G \int \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \rho(\vec{x}') d^3x'$$

$$= G \sum_{i=1}^N m_i \frac{(\vec{x}_i - \vec{x})}{|\vec{x}_i - \vec{x}|^3}$$

(for a collection of particles)

$$\rho(\vec{x}) = \sum_{i=1}^N m_i \delta(\vec{x} - \vec{x}_i)$$

Dirac's delta



- CONSERVATIVE

SINCE $\nabla_x \left(\frac{1}{|\vec{x}' - \vec{x}|} \right) = \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \Rightarrow \vec{F}(\vec{x}) = \nabla_x \int \frac{G \rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' \Rightarrow$

$\Rightarrow \vec{F}(\vec{x}) = -\nabla \Phi$ WHERE $\Phi(\vec{x}) = -G \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$

GRAVITATIONAL POTENTIAL $\equiv$ A SCALAR FUNCTION

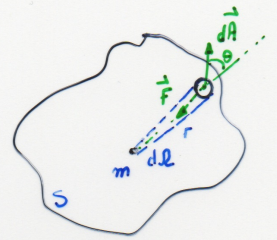
GAUSS' THEOREM

$$\vec{F} \cdot d\vec{A} = F dA \cos \theta =$$

$$= -G m \frac{\cos \theta dA}{r^2} =$$

$$= -G m d\Omega$$

$$\oint_S \vec{F} \cdot d\vec{A} = -4\pi G m$$



FOR A COLLECTION OF PARTICLES OR MASS DISTRIBUTION INSIDE S

$$\oint_S \vec{F} \cdot d\vec{A} = -4\pi G \sum_i m_i = -4\pi G \int_{V(S)} \rho(\vec{x}') d^3x'$$

USING DIVERGENCE THEOREM (ACTUALLY THIS DEFINES DIVERGENCE)

$$\oint_S \vec{F} \cdot d\vec{A} = \int_V \nabla \cdot \vec{F}(\vec{x}') d^3x' = -4\pi G \int_V \rho(\vec{x}') d^3x'$$

$$\boxed{\nabla \cdot \vec{F} = -4\pi G \rho(\vec{x})}$$

DIFF. FORM OF GAUSS THEOR.

$$\nabla \cdot \nabla \phi = \boxed{\nabla^2 \phi = 4\pi G \rho}$$

POISSON'S EQ.

IF $\rho = 0$

$$\boxed{\nabla^2 \phi = 0}$$

LAPLACE'S EQ.

POTENTIAL ENERGY

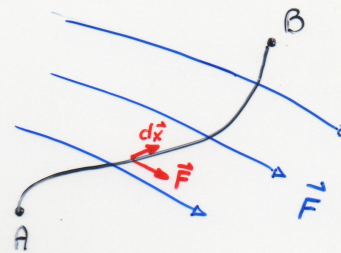
WORK DONE MOVING A TEST (UNIT)
MASS BETWEEN A and B IS

$$W_{AB} = - \int_A^B \vec{F} \cdot d\vec{x} =$$

$$= \int_A^B \vec{\nabla} \phi \cdot d\vec{x} = \int_A^B d\phi = \phi_B - \phi_A$$

NOTE THAT

$$\oint \vec{F} \cdot d\vec{x} = 0$$

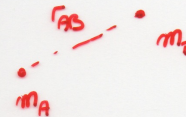


$\phi(x)$ IS THE POTENTIAL ENERGY PER UNIT MASS $\equiv$

$\equiv$ WORK DONE TO BRING UNIT MASS FROM ∞ (WHERE $\phi = 0$)

TO ASSEMBLE A SYSTEM OF 2 PARTICLES

$$W = -G \frac{m_A m_B}{r_{AB}}$$



FOR A SYSTEM OF PARTICLES

$$W = -G \sum_{i=1}^N \sum_{j=1}^{i-1} \frac{m_i m_j}{r_{ij}} = -\frac{G}{2} \sum_{i \neq j} \frac{m_i m_j}{r_{ij}} =$$

$$= -\frac{G}{2} \iint \frac{\rho(\vec{x}) \rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x d^3x' = +\frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3x =$$

$$= +\frac{1}{8\pi G} \int \phi(\vec{x}) \nabla^2 \phi(\vec{x}) d^3x = -\frac{1}{8\pi G} \int |\nabla \phi|^2 d^3x$$

INT. BY
PARTS OVER ALLSPACE

SO THAT $-\frac{|\nabla\phi|^2}{8\pi G}$ CAN BE THOUGHT OF AS THE

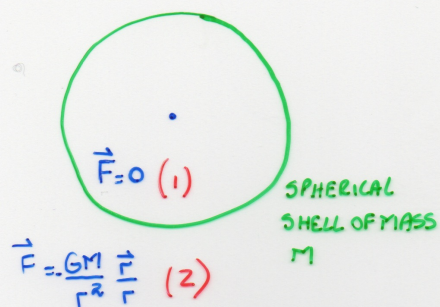
"ENERGY DENSITY" OF THE GRAVITATIONAL FIELD -

EXERCISE WHAT IS THE POT. ENERGY OF A SINGLE ISOLATED
POINT MASS? (SELF-ENERGY vs INTERACTION ENERGY)
OF 2 POINT MASSES?

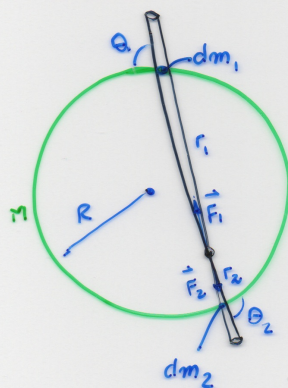
"ENERGY DENSITY" CONTAINS BOTH SELF AND INTERACTION
ENERGY TERMS -

SPHERICAL SYSTEMS

NEWTON'S THEOREMS -



PROOF OF (1):



$$\frac{dm_1}{r_1^2} = \frac{dm_2}{r_2^2}$$

$$\Rightarrow \vec{F}_1 = -\vec{F}_2$$

Integrating OVER ALL SOLID

ANGLES $\Rightarrow \vec{F} = 0$

$$\Rightarrow \nabla \phi = 0 \Rightarrow \phi = \text{CONSTANT}$$

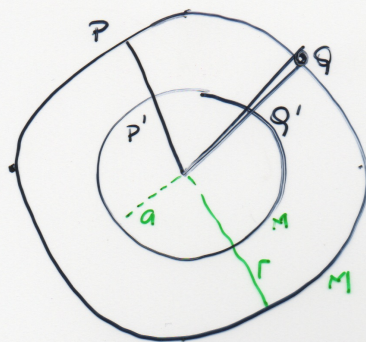
EVALUATING AT $r=0$

$$\phi = -\frac{GM}{R}$$

EXERCISE: IS $\vec{F}=0$ INSIDE A CIRCULAR RING OF UNIFORM SURFACE DENSITY?

PROOF OF (2)

$$P-Q' \equiv P'-Q$$



ϕ at P due to inner sphere is the same

as ϕ at P' due to outer sphere.

$$\text{But } \phi \text{ at P'} = -\frac{GM}{r}$$

So $\phi(P) = -\frac{GM}{r}$ = same as point mass at center of sphere

WE can use this to show that the potential of a spherical system can be written as

$$\phi(r) = -4\pi G \left[\underbrace{\frac{1}{r} \int_0^r \rho(r') r'^2 dr'}_{\text{inner shells}} + \underbrace{\int_r^\infty \rho(r') r' dr'}_{\text{outer shells}} \right]$$

FORCE AT r depends only on material inside r

$$\vec{F}(\vec{r}) = -\frac{d\phi}{dr} \hat{e}_r = -\frac{GM(r)}{r^2} \hat{e}_r$$

FOR A PARTICLE ON A CIRCULAR ORBIT THIS WOULD BE
EQUAL TO THE CENTRIPETAL ACCELERATION.

$$\frac{v_c^2}{r} = \frac{GM(r)}{r^2} \Rightarrow v_c^2(r) = \frac{GM(r)}{r} \quad \text{CIRCULAR VELOCITY}$$

A PARTICLE AT r CAN ESCAPE IF IT HAS ESCAPE SPEED

$$v_e(r) = \sqrt{2|\phi(r)|} \quad \frac{E}{m} = \frac{1}{2} v^2 + \phi$$

SIMPLE EXAMPLES.

- POINT MASS

$$\phi(r) = -\frac{GM}{r} ; v_c(r) = \sqrt{\frac{GM}{r}} ; v_e(r) = \sqrt{\frac{2GM}{r}}$$

- HOMOGENEOUS SPHERE

$$S(r) = \text{const.}$$

$$M(r) = \frac{4}{3} \pi S r^3$$

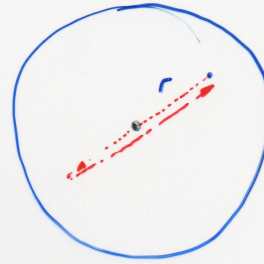
$$v_c(r) = \left(\frac{4\pi G S}{3} \right)^{1/2} r$$

$$T_c = \frac{2\pi r}{v_c} = \left(\frac{3\pi}{G S} \right)^{1/2}$$

indep. of radius

A BODY RELEASED FROM REST AT r
HAS AN EQ. OF MOTION

$$\ddot{r} = -\frac{4\pi G \rho}{3} r$$



HARMONIC OSCILLATOR OF FREQUENCY

$$T = \left(\frac{3\pi}{G\rho} \right)^{1/2}$$

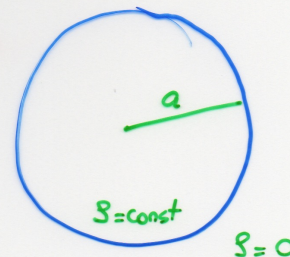
REGARDLESS OF INITIAL RADIUS, PARTICLES REACH $r=0$ in time

$$t_{dyn} = \frac{T}{4} = \left(\frac{3\pi}{16G\rho} \right)^{1/2}$$

This is by definition the **dynamical time** OF A SYSTEM OF
MEAN DENSITY ρ . THE **free-fall time** is $t_{ff} = t_{dyn} \sqrt{2}$

$\phi(r)$ is infinite unless mass is truncated somewhere,
say at a radius a .

$$\begin{aligned} \phi(r) &= -2\pi G \rho \left(a^2 - \frac{1}{3} r^2 \right) \quad r < a \\ &= -\frac{4\pi G \rho a^3}{3r} \quad r > a \end{aligned}$$



• THE ISOCHRONE POTENTIAL

$$S \rightarrow S_0 \quad r \rightarrow 0 \quad \phi \rightarrow r^2 - C$$

$$S \rightarrow 0 \quad r \rightarrow \infty \quad \phi \rightarrow \frac{1}{r}$$

$$\Phi(r) = -\frac{GM}{b + (b^2 + r^2)^{1/2}}$$

$$V_c^2(r) = \frac{GM r^2}{(b+a)^2 a} = r \frac{d\phi}{dr} \quad a \equiv (b^2 + r^2)^{1/2}$$

$$S(r) = \frac{\nabla^2 \Phi}{4\pi b} = \frac{1}{4\pi b} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = M \left[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi (b+a)^3 a^3} \right]$$

$$S \rightarrow \frac{3M}{16\pi b^3} \quad r \rightarrow 0$$

$$S \rightarrow \frac{bM}{2\pi r^4} \quad r \rightarrow \infty$$

• MODIFIED HUBBLE PROFILE

$$S(r) = \frac{S_0}{\left[1 + (r/a)^2\right]^{-3/2}}$$

$$M(r) = 4\pi a^3 S_0 \left\{ \ln \left[\frac{r}{a} + \sqrt{\frac{r^2}{a^2} + 1} \right] - \frac{r}{a} \left(\frac{r^2}{a^2} + 1 \right)^{-1/2} \right\} \quad \text{a mess!}$$

$$\phi(r) = - \frac{GM(r)}{r} - \frac{4\pi G S_0 a^2}{\left(1 + (r/a)^2\right)^{1/2}}$$

$M(r) \rightarrow \infty$
 $r \rightarrow \infty$ but $\phi(r)$ remains finite

This potential-density pair is interesting because the projected mass density is similar to Hubble's profile for E galaxies.

$$\Sigma(R) = \frac{2 S_0 a}{1 + \left(\frac{R}{a}\right)^2}$$

$$\Sigma_H(R) = \frac{\Sigma_0}{\left(1 + \frac{R}{R_H}\right)^2}$$

• POWER LAW DENSITY PROFILES

$$\rho(r) = \rho_0 \left(\frac{r}{r_0} \right)^{-\alpha}$$

$$\Sigma(R) = \frac{\rho_0 r_0^\alpha}{R^{\alpha-1}} \frac{\left(-\frac{1}{2}\right)! \left(\frac{\alpha-3}{2}\right)!}{\left(\frac{\alpha-2}{2}\right)!}$$

$$M(r) = \frac{4\pi \rho_0 r_0^\alpha}{3-\alpha} r^{3-\alpha}$$

$$\alpha < 3$$

MASS INFINITE FOR $\alpha \geq 3$

$$V_c^2(r) = \frac{4\pi G \rho_0 r_0^\alpha}{3-\alpha} r^{2-\alpha}$$

$V_c \approx \text{constant}$ (Spiral GALAXIES)

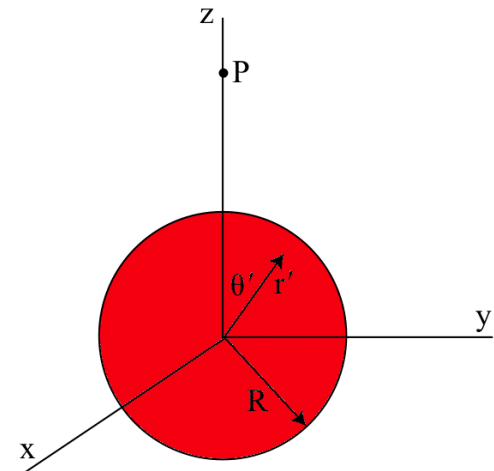
$\Rightarrow \alpha \approx 2 \Rightarrow \text{SINGULAR ISOTHERMAL SPHERE}$

$$V_e^2(r) = \frac{2 V_c^2(r)}{\alpha-2}$$

$3 > \alpha > 2 \Rightarrow \text{otherwise infinite mass or potential}$

Multipole Expansion

- Corollaries from Newton's theorems
- Far away from a finite distribution of mass $\Phi \sim -GM/r$. The *monopole* term dominates
- Dipole is zero in the center of mass reference frame and outside a mass distribution



Fundamental Principles and Relationships

$$\text{wavy } r : \eta = r - r'. \quad (1)$$

$$\text{Electric Potential : } V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r') d\tau'}{\eta} \quad (2)$$

$$\frac{1}{\eta} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n(\cos \theta') \quad (3)$$

Multipole :

$$V = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos \theta') \rho(r') d\tau' \quad (4)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int \rho(r') d\tau' \quad (\text{monopole}) \quad (5)$$

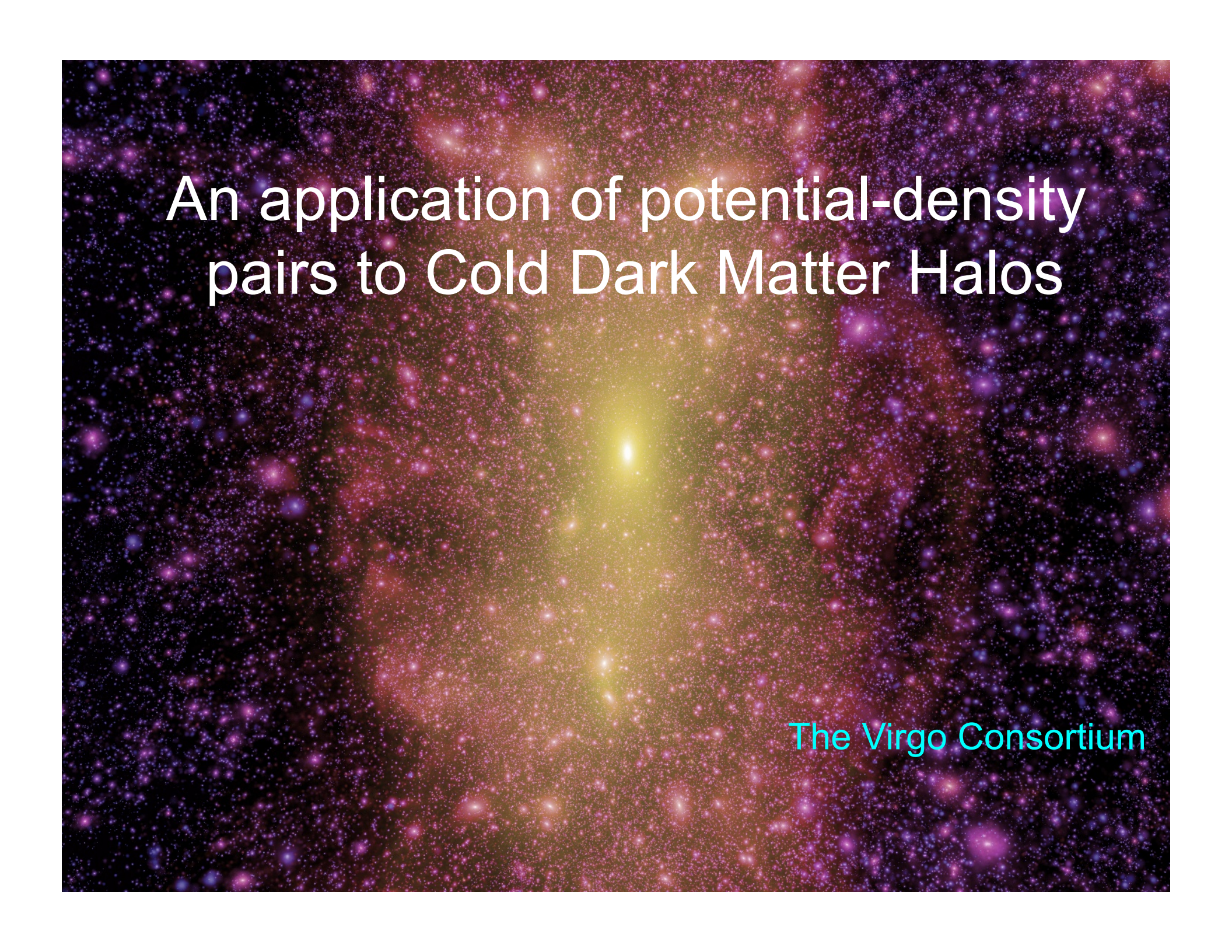
$$+ \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int r' \cos \theta' \rho(r') d\tau' \quad (\text{dipole}) \quad (6)$$

$$+ \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int r'^2 \left(\frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) \rho(r') d\tau' \quad (\text{quadrupole}) \quad (7)$$

+ ...

$$\rho(r', \theta') = k \frac{R}{r'^2} (R - 2r') \sin \theta' \quad (8)$$

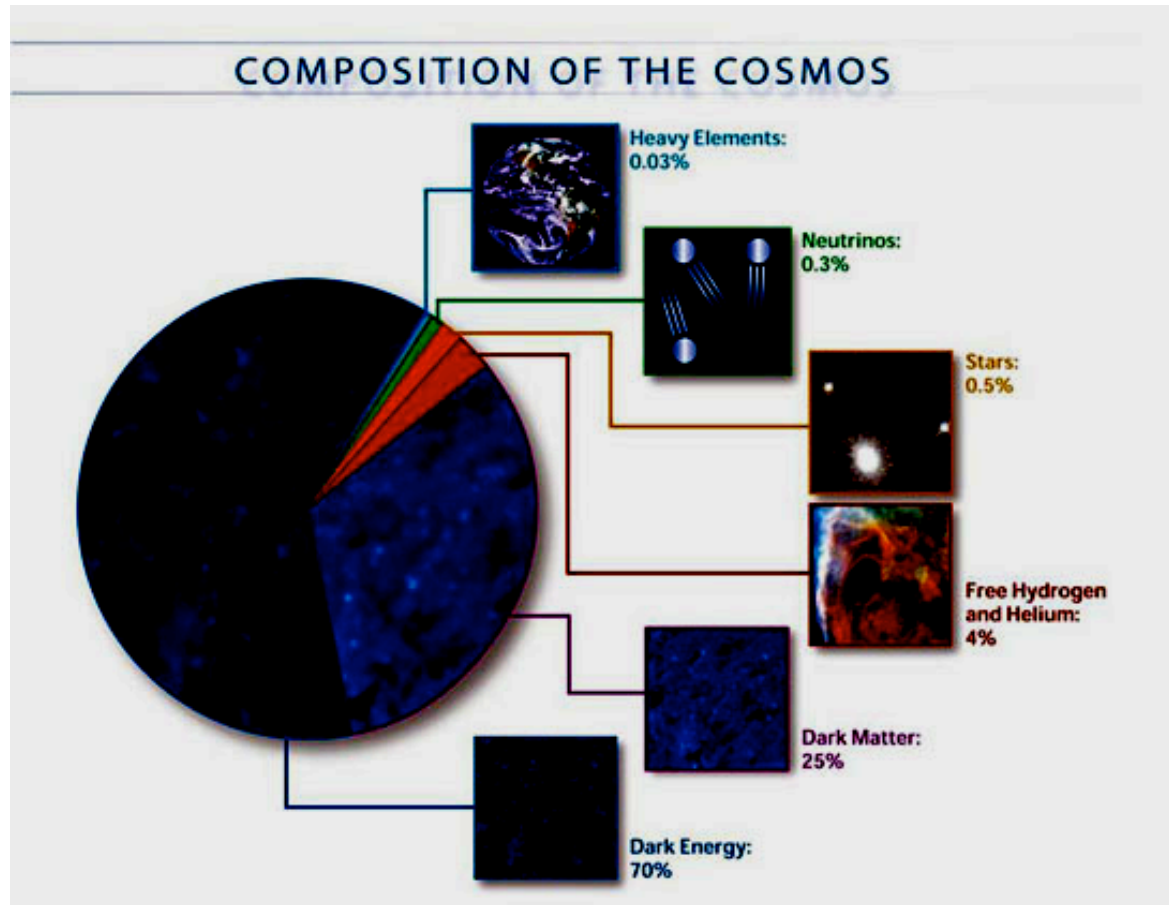
$$V(z) = \frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{24z^2} \quad (9)$$

The background of the slide is a simulated galaxy cluster. It features a dense field of galaxies, primarily in shades of purple and blue, scattered across a dark space. In the center, there is a bright, glowing yellow and orange region, representing the core of the cluster where the density of matter is highest. The overall effect is a vibrant, cosmic scene.

An application of potential-density pairs to Cold Dark Matter Halos

The Virgo Consortium

The Current Paradigm



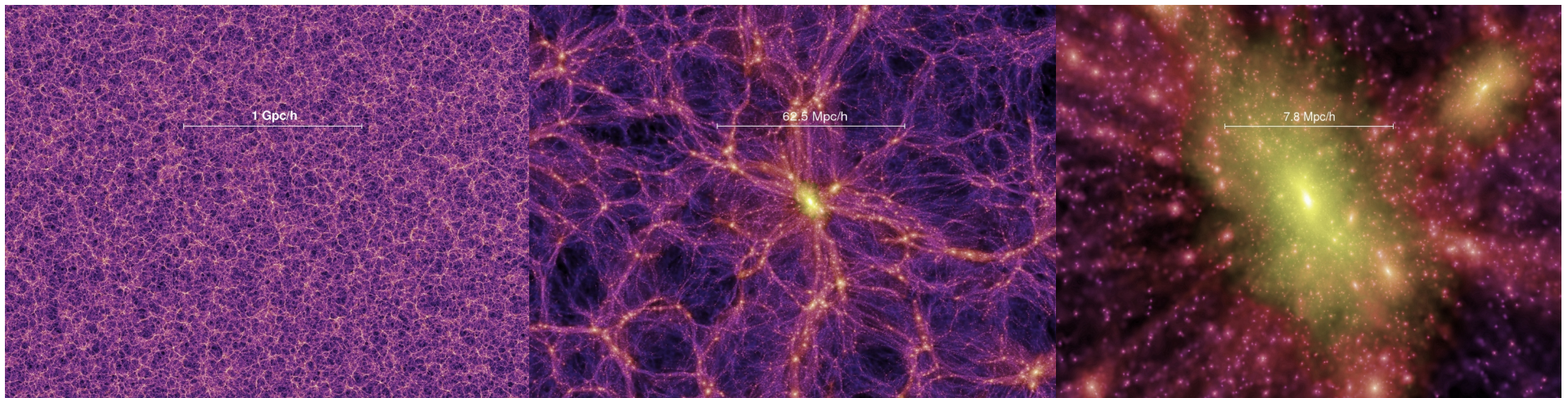
**“The Pie of Ignorance”
our current inventory of the matter-
energy content of the Universe**

Cosmological measurements have challenged the very foundations of our physical understanding of Nature.

- **What is the dark matter?**
- **What is the dark energy?**
- **What is the eventual fate of the Universe?**
- **How did galaxies like our own Milky Way form in such Universe?**

N-body simulations track the clustering evolution of the dark matter from the Big Bang to the present

Millennium Simulation



Springel et al '05



Simulation data, movies, pictures available at:

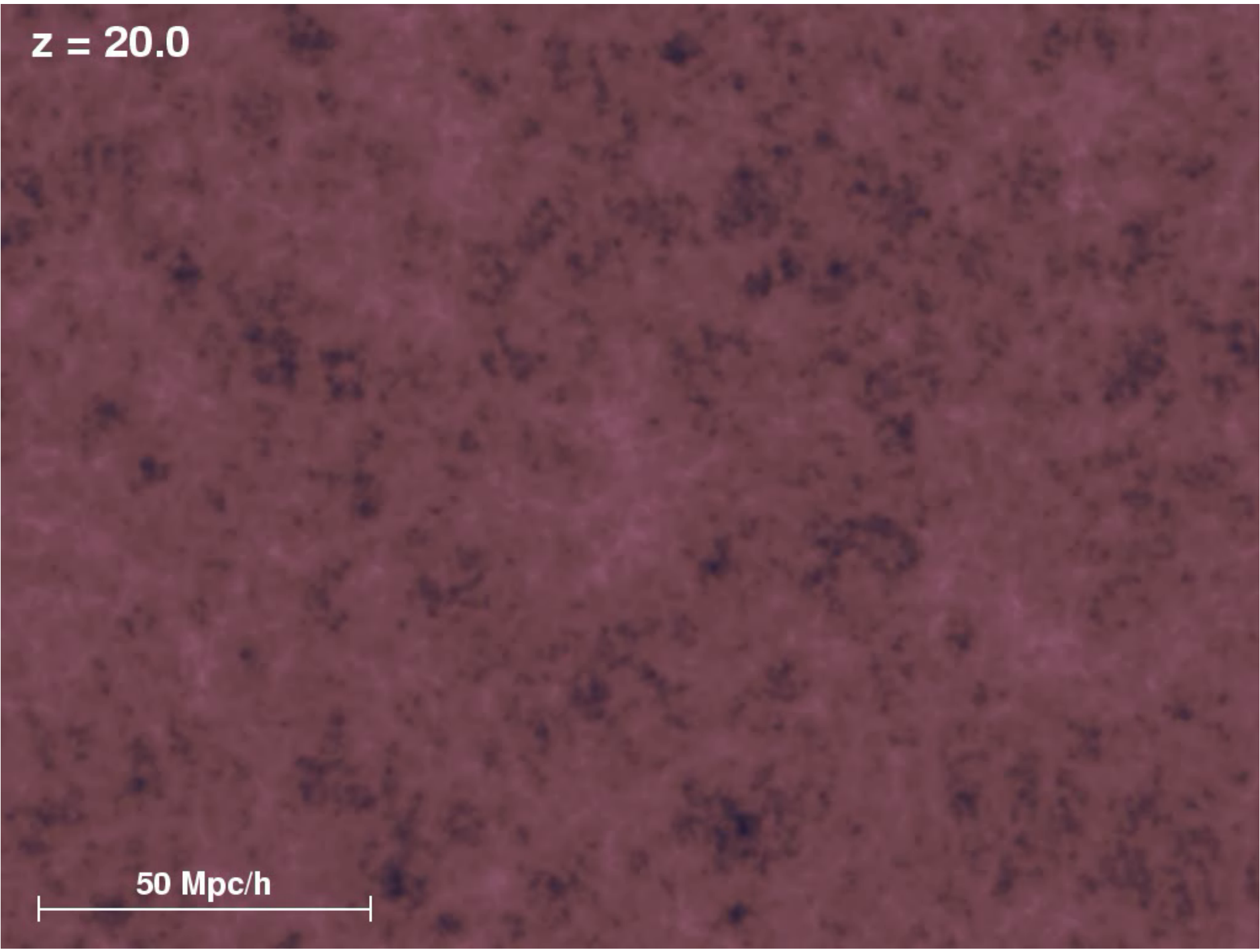
www.mpa-garching.mpg.de/Virgo

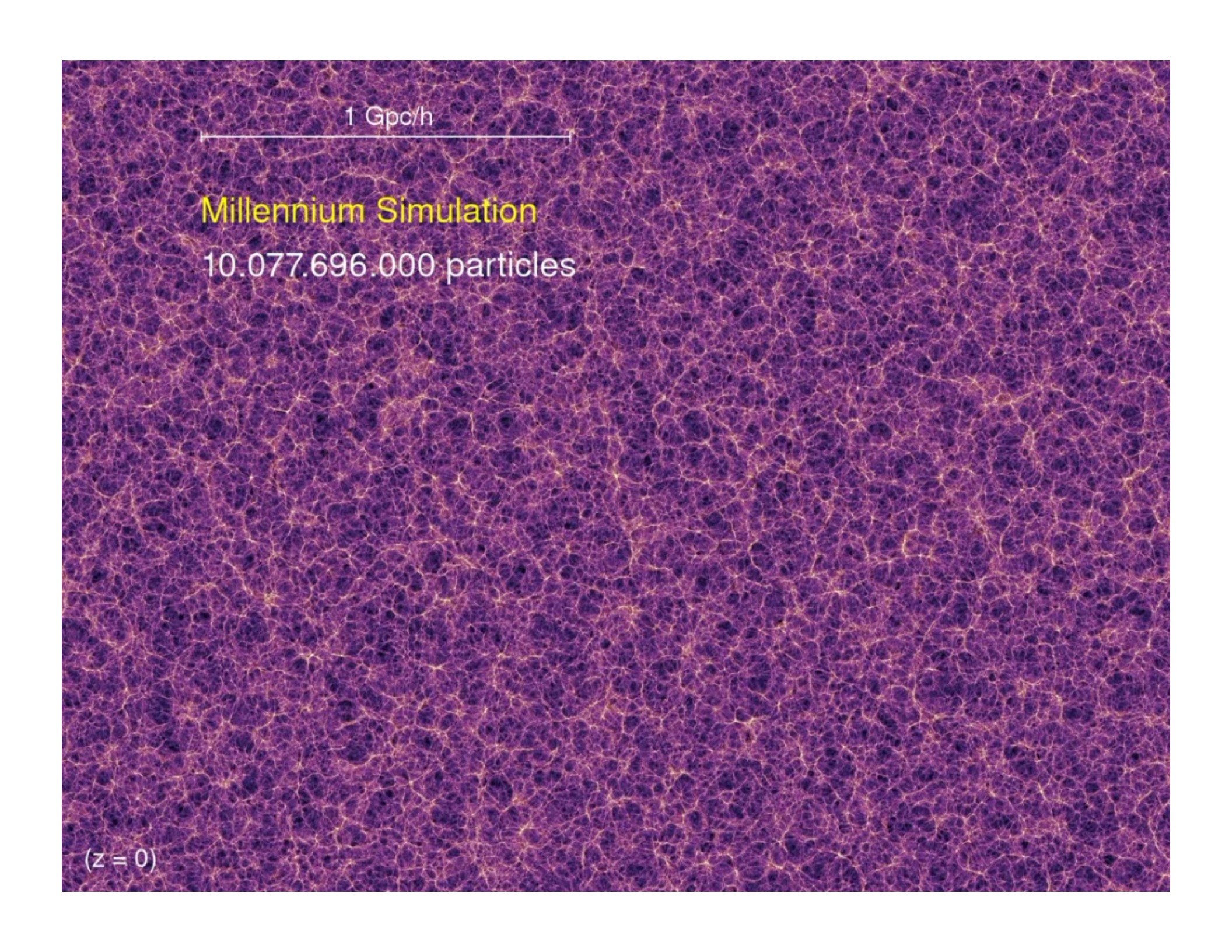
www.durham.ac.uk/virgo

UK, Germany, Canada,
Japan, US collaboration

$z = 20.0$

50 Mpc/h

A visualization of a cosmic web simulation at redshift $z = 20.0$. The image shows a dense network of dark, filamentary structures (galaxies and gas) against a lighter, reddish-brown background. The structures are interconnected, forming a complex web. A scale bar at the bottom left indicates a length of 50 Mpc/h.



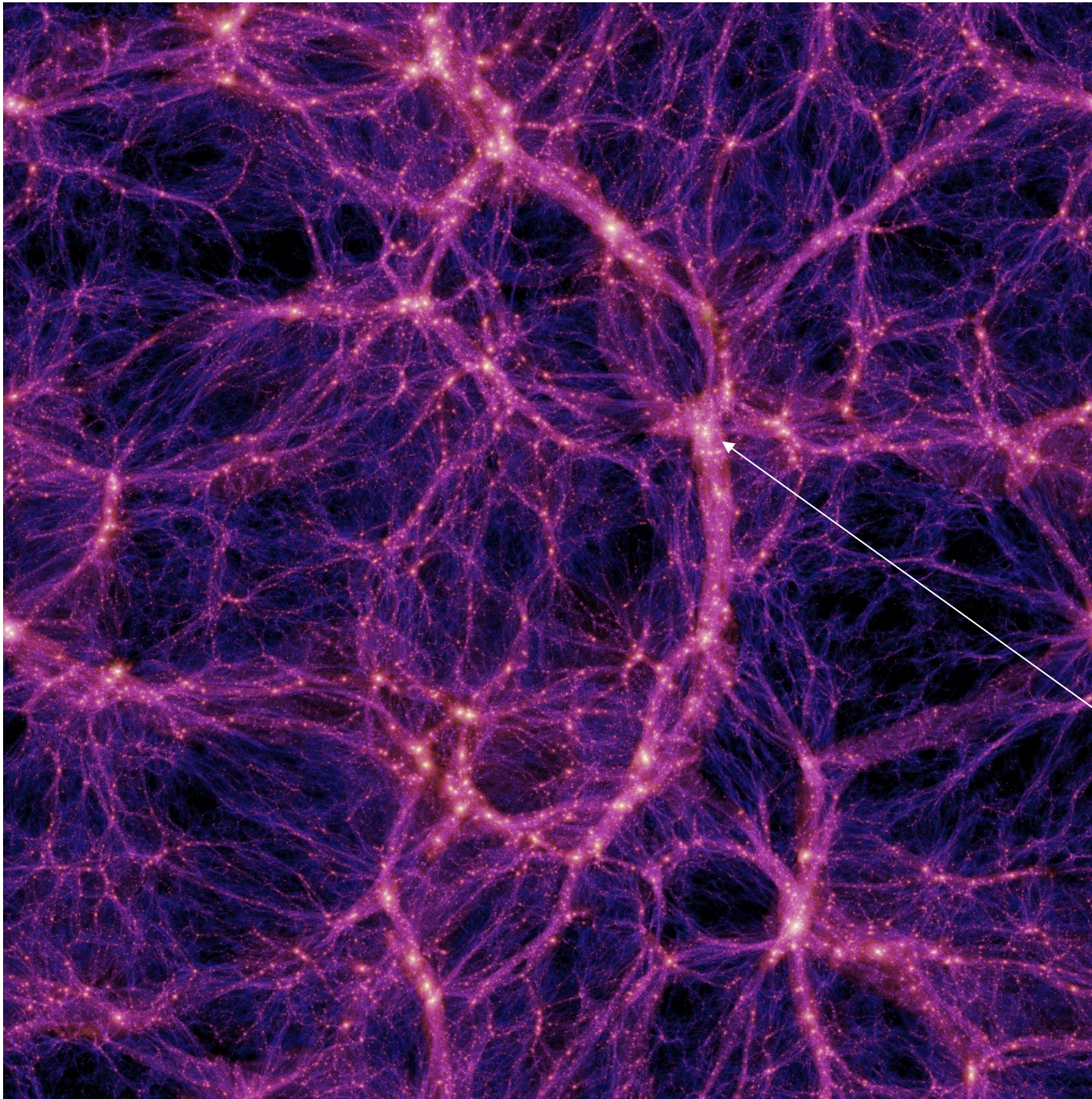
1 Gpc/h

Millennium Simulation

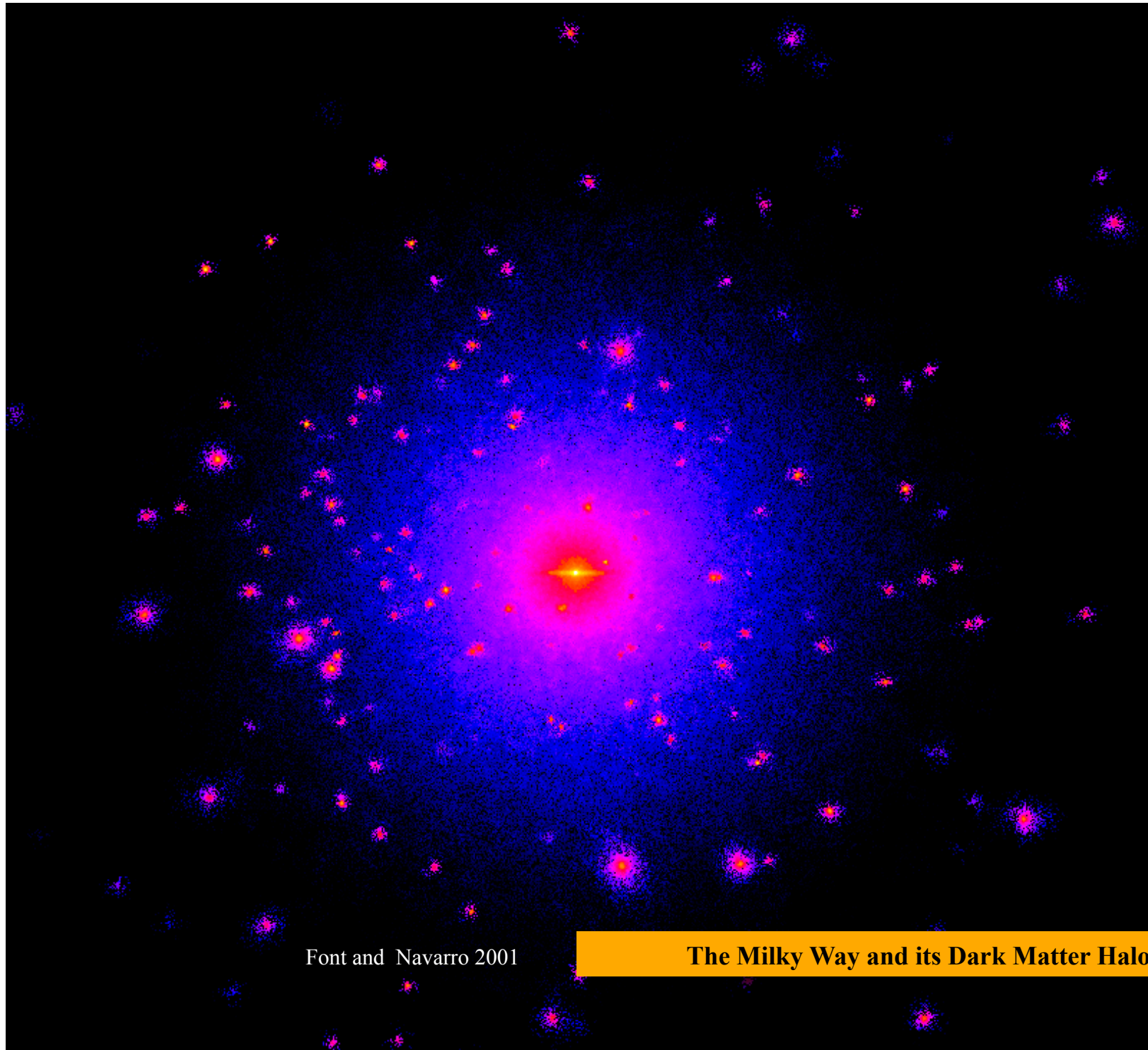
10.077.696.000 particles

($z = 0$)

**The
Aquarius
Project:
Cold Dark
Matter
Halos
under a
Numerical
Microscope**



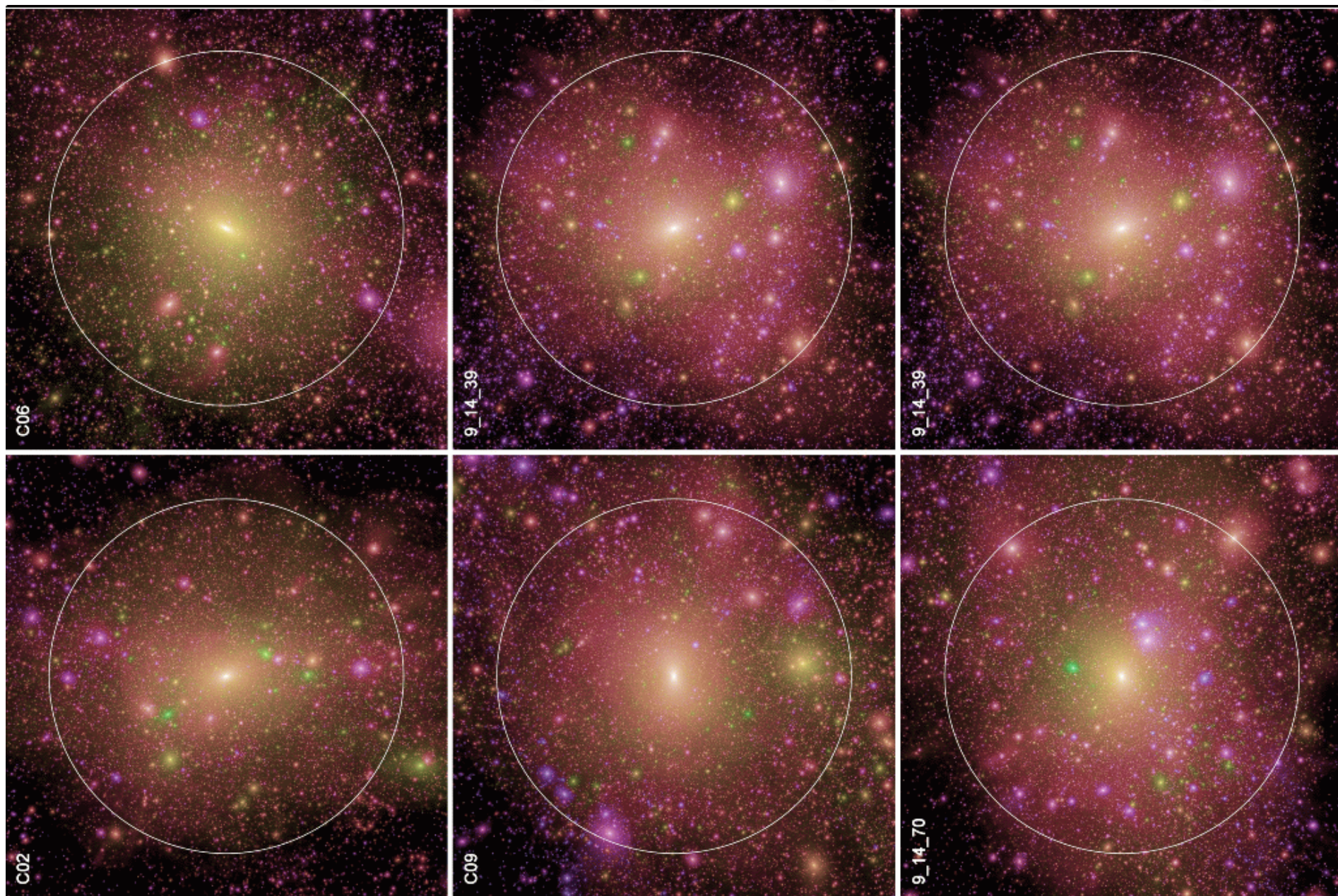
One of the Aquarius
halos in the
100Mpc/h box
parent simulation.



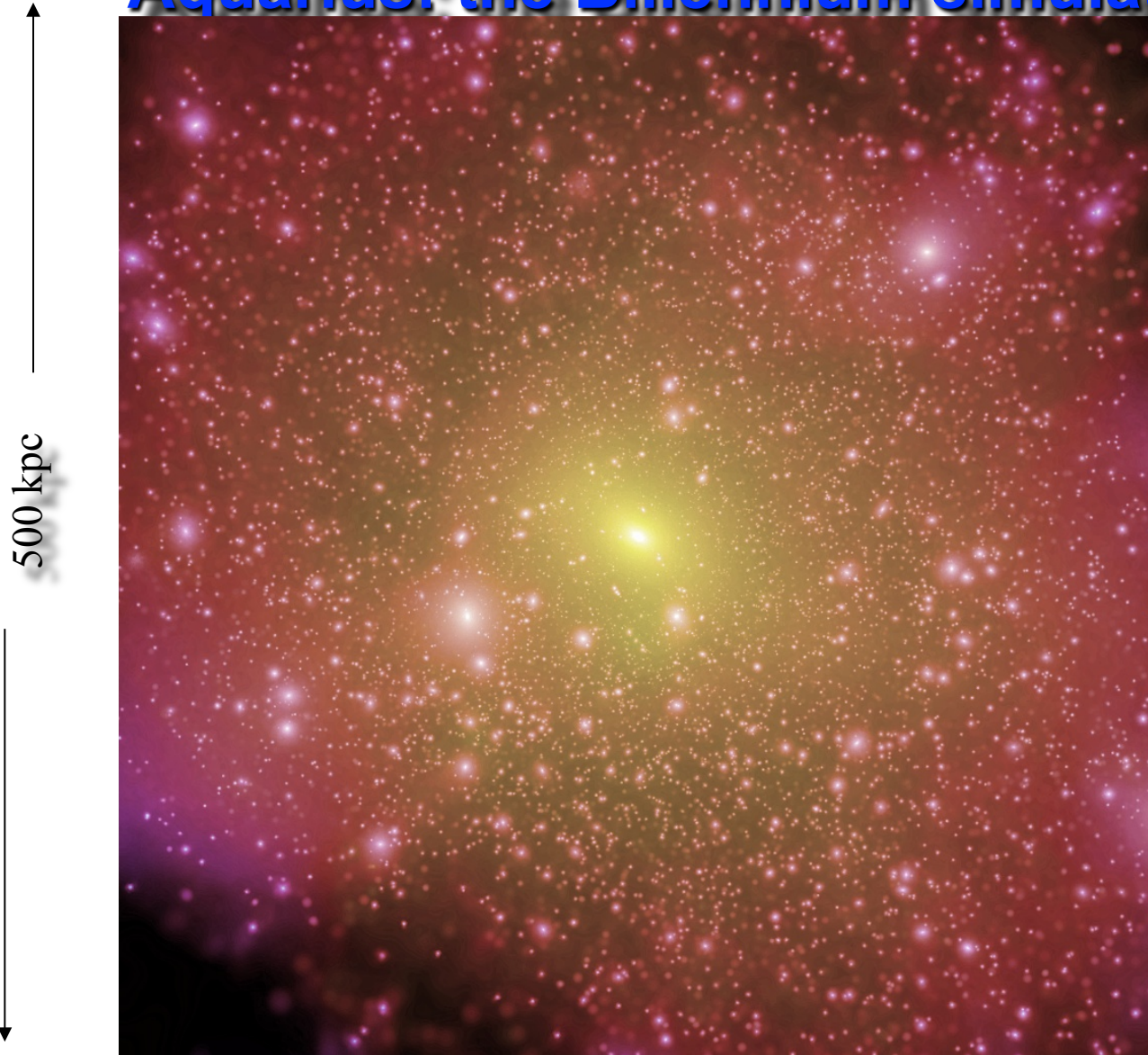
Font and Navarro 2001

The Milky Way and its Dark Matter Halo

Pictures of all Aquarius halos (level-2 resolution)



Aquarius: the Billennium simulation



The Aquarius
“Billennium”
halo simulation.
A dark matter
halo with 1
billion particles
within the virial
radius.

$z = 48.4$

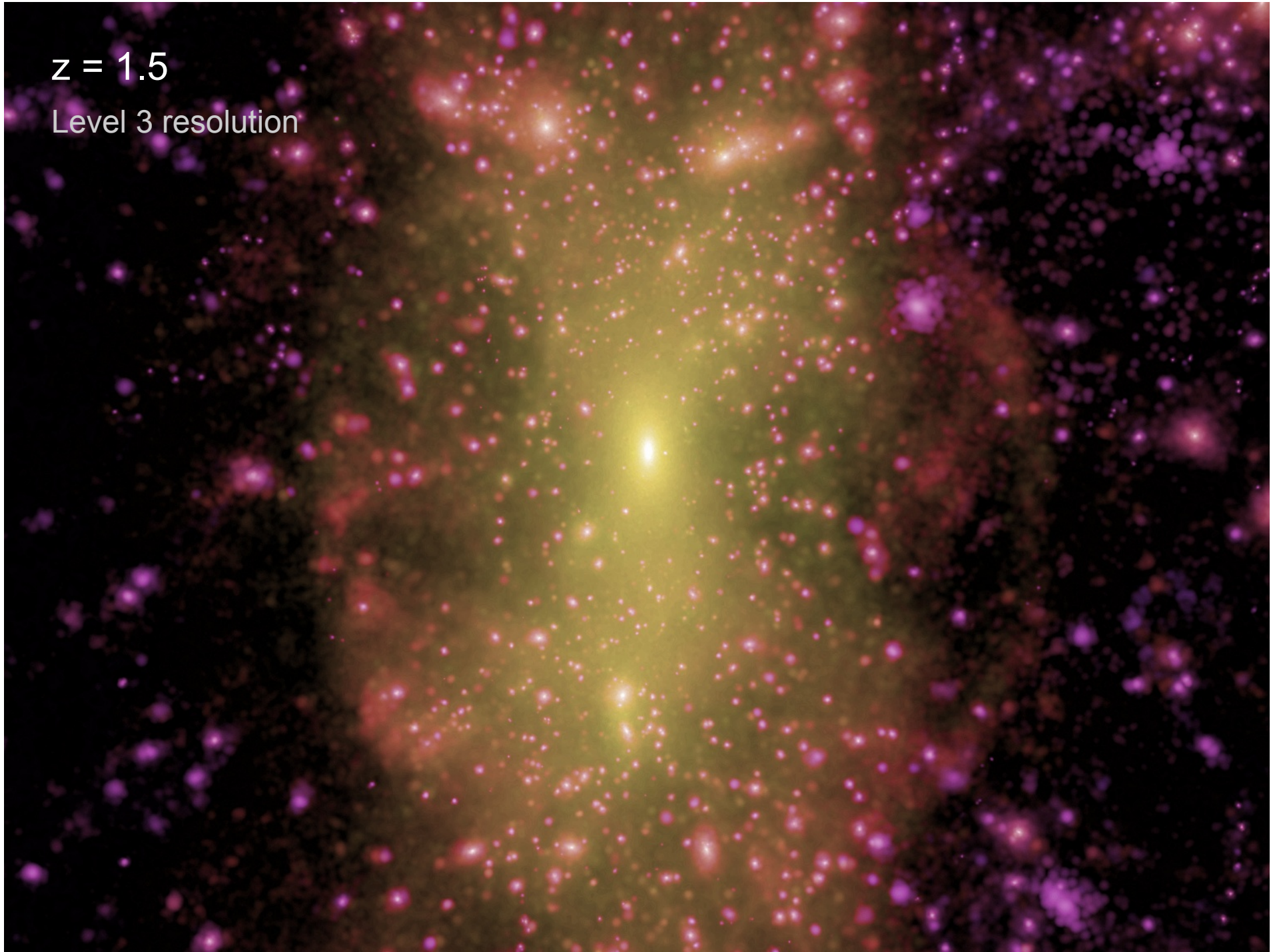
$T = 0.05 \text{ Gyr}$

500 kpc

The image shows a dark, textured field of purple and black, representing a simulated galaxy at a very early stage. The texture is grainy and noisy, with some faint, irregular patterns of slightly lighter purple. In the top left corner, the text 'z = 48.4' is displayed. In the top right corner, the text 'T = 0.05 Gyr' is displayed. At the bottom center, there is a horizontal scale bar with vertical end caps, and the text '500 kpc' is written below it.

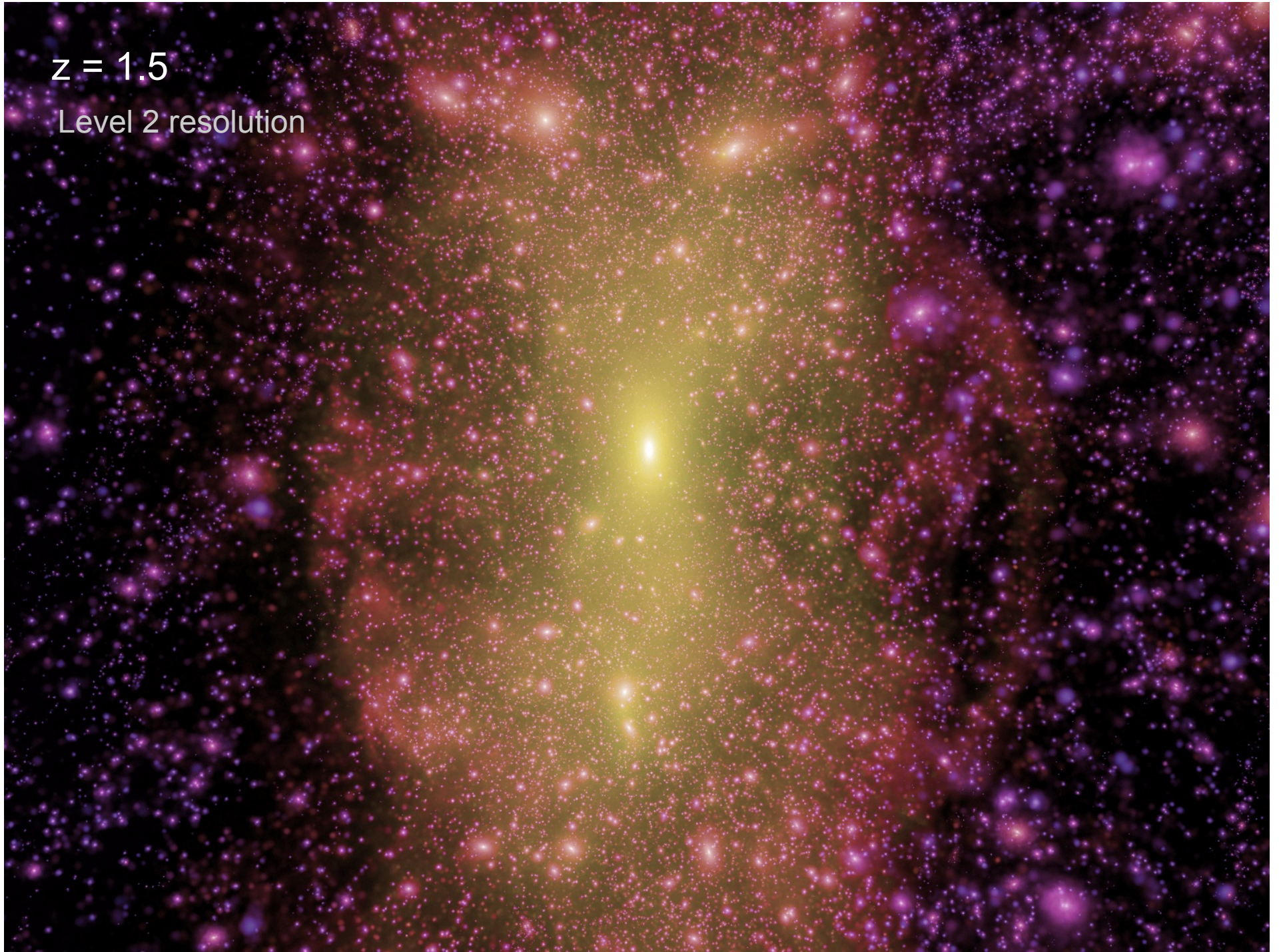
$z = 1.5$

Level 3 resolution



$z = 1.5$

Level 2 resolution



$z = 1.5$

Level 1 resolution



$z = 0.1$

Level 1



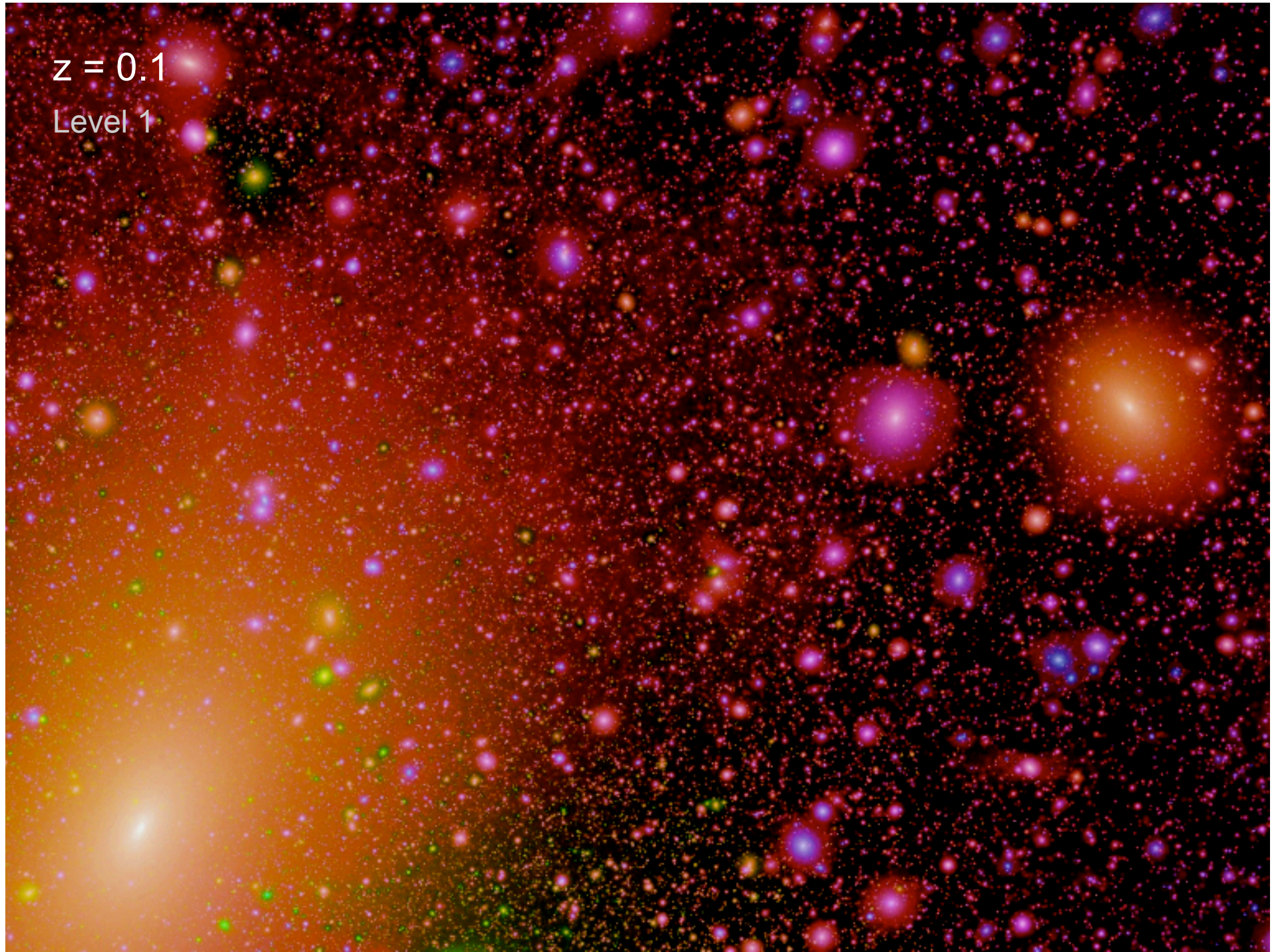
$z = 0.1$

Level 1

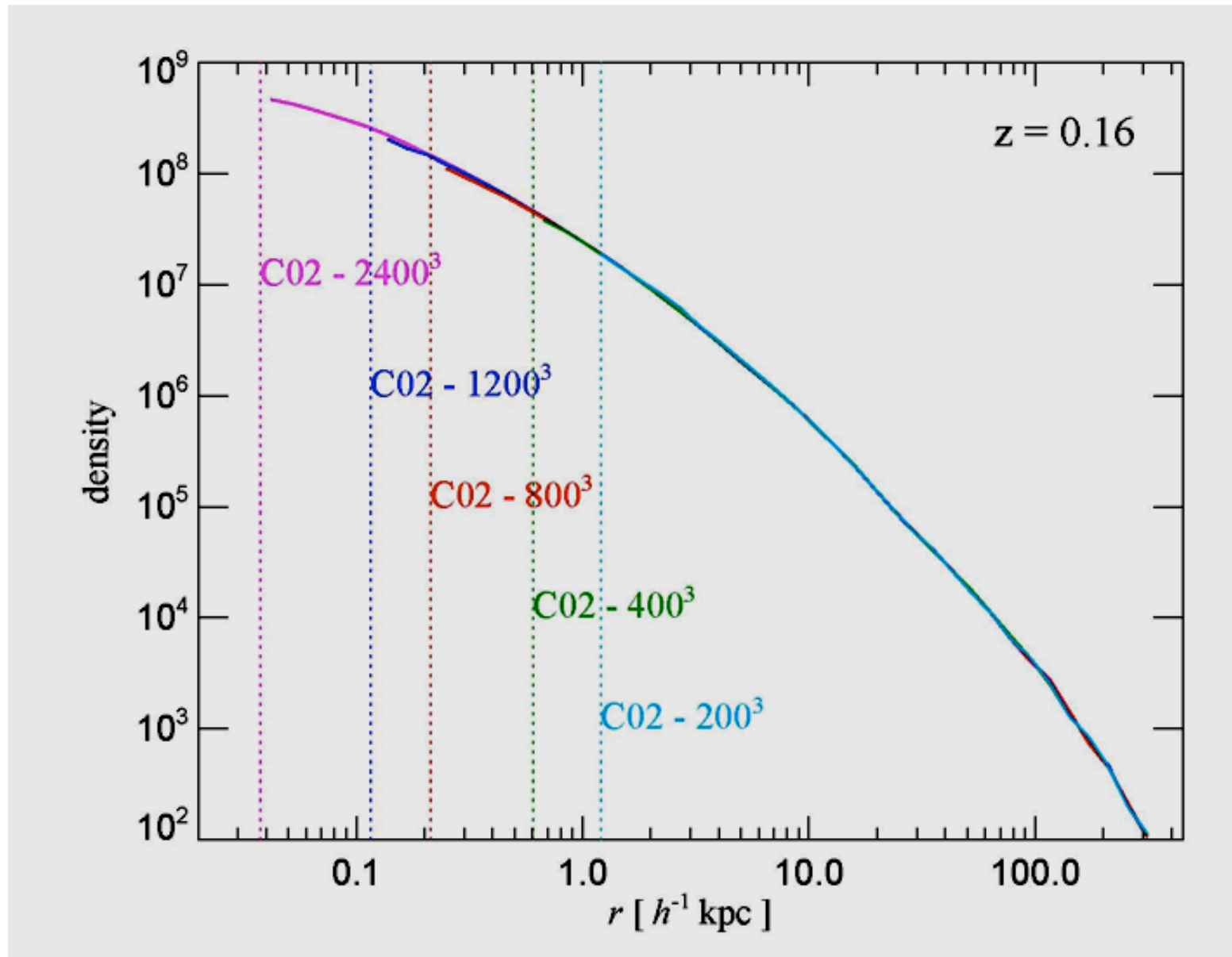


$z = 0.1$

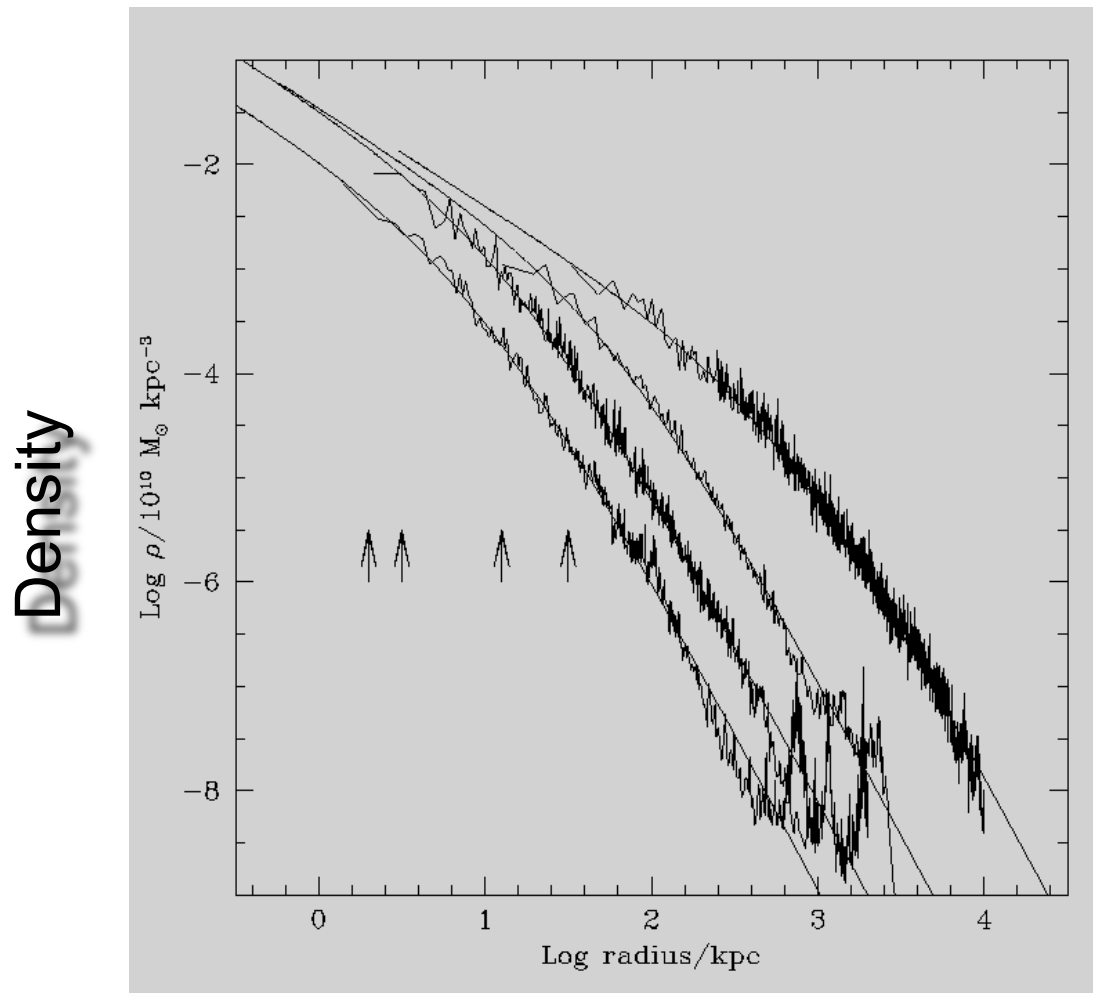
Level 1



The Density Profile: numerical convergence



The Density Profile of Cold Dark Matter Halos

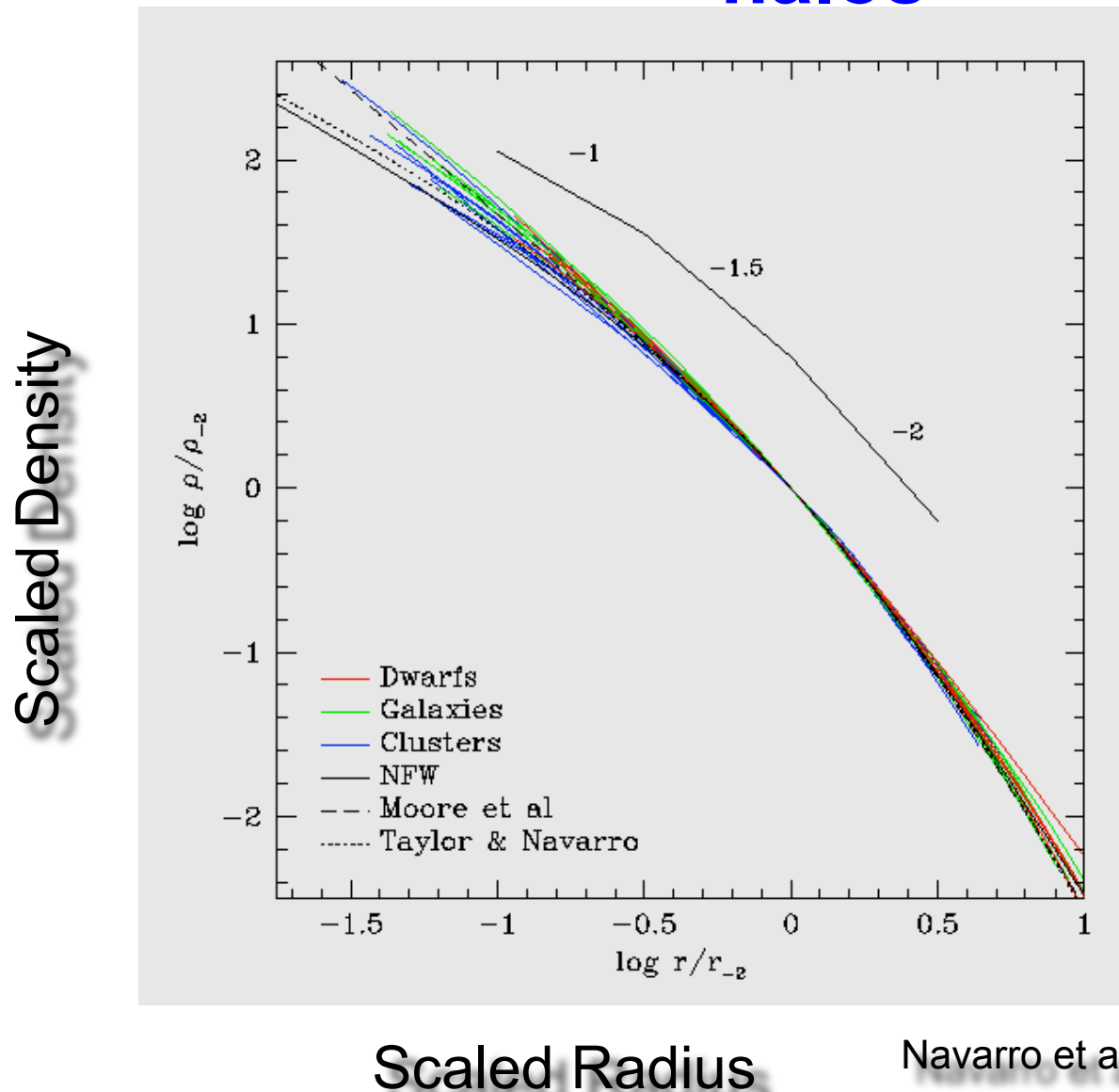


The profiles of dark matter halos are independent of mass and cosmological parameters. The profile is shallower (Navarro, Frenk & White, NFW) than the isothermal profile.

- How much shallower?

$$\frac{\rho(r)}{\rho_{crit}} = \frac{\delta_c}{(r/r_s)(1+r/r_s)^2}$$

The Universal Mass Profile of Λ CDM halos



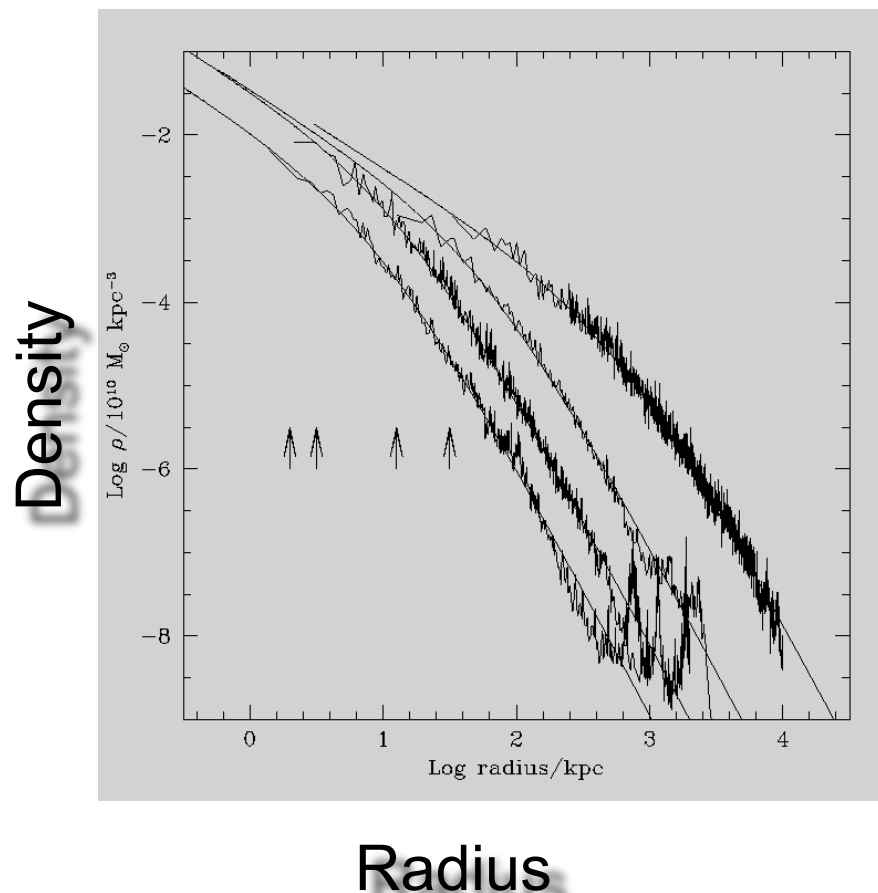
• Properly scaled, all halos look alike:

CDM halo structure appears to be approximately “universal”

• Usually characterized by a “concentration” parameter: $c = r_{\text{vir}} / r_{-2}$

Navarro et al 2004

The Navarro-Frenk-White Profile



Mass profiles of dark halos are independent of halo mass and cosmological parameters (Navarro, Frenk & White 1996, 1997)

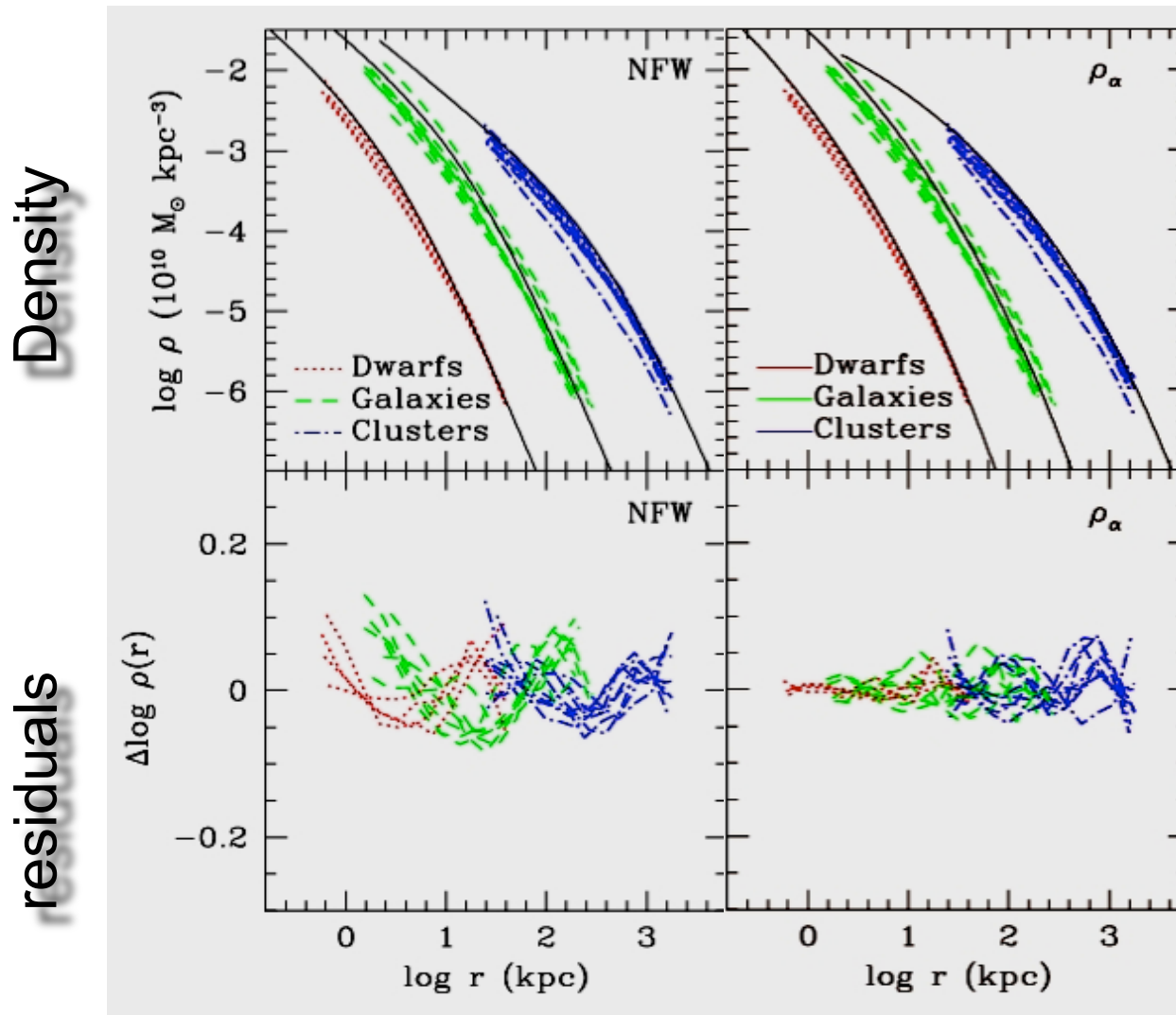
Problem 3.7 The *Navarro–Frenk–White* (NFW) model describes the halos of cold dark matter that form in simulations like that of Figure 7.16. Show that the potential corresponding to the density

$$\rho_{\text{NFW}}(r) = \frac{\rho_N}{(r/a_N)(1+r/a_N)^2} \text{ is } \Phi_{\text{NFW}}(r) = -\sigma_N^2 \frac{\ln(1+r/a_N)}{(r/a_N)}, \quad (3.24)$$

where $\sigma_N^2 = 4\pi G \rho_N a_N^2$. The density rises steeply at the center, but less so than in the singular isothermal sphere; at large radii $\rho(r) \propto r^{-3}$. Show that the speed V of a circular orbit at radius r is given by

$$V^2(r) = \sigma_N^2 \left[\frac{\ln(1+r/a_N)}{(r/a_N)} - \frac{1}{(1+r/a_N)} \right]. \quad (3.25)$$

An improved fitting formula



A law where the logarithmic slope of the density profile is a simple power-law of radius fits the dark halos to better than 5% at all radii.

$$\frac{d \log \rho_{\alpha}}{d \log r} = -2 \left(\frac{r}{r_{-2}} \right)^{\alpha}$$

Remarkably, this is the same radial behaviour (a Sersic law) of the stellar distribution in elliptical galaxies.

Navarro et al 2004

Radius

CDM halos: Main results

- CDM mass profiles are nearly **universal**
 - shape is independent of mass
- CDM density profiles are **cuspy**
 - no evidence for a constant-density central “core”
- CDM halos are **clumpy**
 - Abundant but non-dominant substructure
- CDM halos are **triaxial**
 - Preference for prolate configuration, asphericity increasing toward the center.



CDM halos: Outstanding issues

- **The Structure of the Central Cusp**
 - Power-law divergent slope ($\rho \propto r^{-1}$ or $\rho \propto r^{-1.2}$ or $\rho \propto r^{-1.5}$?)
 - Annihilation signal
 - Disk galaxy rotation curves (cusp vs core vs triaxiality)
- **The Structure of Substructure**
 - Mass profile and abundance of Local Group satellites
 - Annihilation signal from substructures and “boost factors”
 - Abundance, spatial distribution and kinematics
 - lensing flux ratio anomaly, satellite distribution + orbits
- **The Phase-Space Distribution of Dark Matter**
 - Implications for direct dark matter detection experiments
- **The Origin of a Universal Density Profile**
 - Theoretical interest
 - Important to understand baryon-induced transformations of dark halo structure

Stellar Orbits

- If the mass distribution in a galaxy is static (i.e., the galaxy is in equilibrium), the energy of a star is constant

$$\frac{d}{dt}(m\mathbf{v}) = -m \nabla \Phi(\mathbf{x}), \quad \text{with } \Phi(\mathbf{x}) = - \sum_{\alpha} \frac{Gm_{\alpha}}{|\mathbf{x} - \mathbf{x}_{\alpha}|} \quad \text{for } \mathbf{x} \neq \mathbf{x}_{\alpha}, \quad (3.3)$$

$$\mathbf{v} \cdot \frac{d}{dt}(m\mathbf{v}) + m\mathbf{v} \cdot \nabla \Phi(\mathbf{x}) = 0 = \frac{d}{dt} \left[\frac{1}{2} m\mathbf{v}^2 + m\Phi(\mathbf{x}) \right]. \quad (3.26)$$

Thus

$$\mathcal{E} \equiv \frac{1}{2} m\mathbf{v}^2 + m\Phi(\mathbf{x}) = \text{constant along the orbit.} \quad (3.27)$$

- A star can “escape” (i.e., move arbitrarily far away from the galaxy) if its speed exceeds the local “escape speed”, given by

$$v_e^2 = -2\Phi(\mathbf{x}). \quad (3.28)$$

- Note that the escape speed depends on the global distribution of matter, whereas the circular speed depends mainly on the mass interior to a given radius.

Stellar Orbits

- The angular momentum of a star changes according to:

The star's *angular momentum* $\mathcal{L} = \mathbf{x} \times m\mathbf{v}$ changes according to

$$\frac{d\mathcal{L}}{dt} = \mathbf{x} \times \frac{d}{dt}(m\mathbf{v}) = -m\mathbf{x} \times \nabla\Phi \quad (3.29)$$

- Depending on the symmetry of a galaxy one or more components of the angular momentum of stars may be conserved in an equilibrium galaxy
 - In a spherically-symmetric galaxy all component of L are conserved
 - In a galaxy with **axial symmetry** (around a rotation axis, z , for example) only the component along that axis, L_z , is conserved)

The total energy of a star cluster

- In a star cluster, in general the orbital energy of star will change, but the total energy of the cluster remains constant. We can show this by taking the scalar product of eq. 3.2 with $\mathbf{v}_\alpha$

$$\frac{d}{dt}(m\mathbf{v}) = -\frac{Gm\mathcal{M}}{r^3}\mathbf{r}, \quad (3.1)$$

where G is Newton's gravitational constant. In a cluster of N stars with masses m_α ($\alpha = 1, 2, \dots, N$), at positions $\mathbf{x}_\alpha$, we can add the forces on star α from all the other stars:

$$\frac{d}{dt}(m_\alpha \mathbf{v}_\alpha) = - \sum_{\substack{\beta \\ \alpha \neq \beta}} \frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3} (\mathbf{x}_\alpha - \mathbf{x}_\beta). \quad (3.2)$$

$$\sum_{\alpha} \mathbf{v}_\alpha \cdot \frac{d}{dt}(m_\alpha \mathbf{v}_\alpha) = \frac{d}{dt} \mathcal{KE} = - \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} \frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3} (\mathbf{x}_\alpha - \mathbf{x}_\beta) \cdot \mathbf{v}_\alpha. \quad (3.30)$$

But we could have started with the equation for the force on star β , and taken the scalar product with $\mathbf{v}_\beta$ to find

$$\frac{1}{2} \sum_{\beta} \frac{d}{dt}(m_\beta \mathbf{v}_\beta \cdot \mathbf{v}_\beta) = - \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} \frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3} (\mathbf{x}_\beta - \mathbf{x}_\alpha) \cdot \mathbf{v}_\beta. \quad (3.31)$$

Adding the right-hand sides of the last two equations gives

$$- \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} \frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3} (\mathbf{x}_\alpha - \mathbf{x}_\beta) \cdot (\mathbf{v}_\alpha - \mathbf{v}_\beta) = \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} \frac{d}{dt} \left(\frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|} \right). \quad (3.32)$$

The cluster's potential energy $\mathcal{PE}$ is the sum of contributions from pairs of stars:

$$\mathcal{PE} = -\frac{1}{2} \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} \frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|} = \frac{1}{2} \sum_{\alpha} m_\alpha \Phi(\mathbf{x}_\alpha) \quad \text{or} \quad \frac{1}{2} \int \rho(\mathbf{x}) \Phi(\mathbf{x}) dV; \quad (3.33)$$

dividing by two means that each pair contributes only one term to the sum. On adding Equations 3.30 and 3.31, we see that

$$2 \frac{d}{dt} \left[\mathcal{KE} - \frac{1}{2} \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} \frac{Gm_\alpha m_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|} \right] = 0. \quad (3.34)$$

Thus the total energy $\mathcal{E} = \mathcal{KE} + \mathcal{PE}$ of the cluster is constant.

The virial theorem

- In an equilibrium system where gravity dominates, another relation is satisfied by the kinetic and potential energy
 - Conservation of energy implies $KE + PE = \text{constant}$
 - The virial theorem states that $2\langle KE \rangle + \langle PE \rangle = 0$.
- The virial theorem is more useful.
 - This is because $\langle KE \rangle \sim M \langle V^2 \rangle$
 - On the other hand, $\langle PE \rangle \sim M^2 / \langle R \rangle$
 - This means that $M \sim \langle V^2 \rangle R$, which gives as a way of estimating the mass of a system if we can measure the typical velocity (squared) of stars in a system ($\langle V^2 \rangle$), together with its (unchanging?) size (R).
 - Note that for the latter we need the distance to the object
- This is the way in which *most* mass estimates are carried out.
- Note that the virial theorem is satisfied in a circular orbit.
- It is also true for an elliptical orbit if KE and PE are averaged over a full orbit

The virial theorem

- To prove the virial theorem in a general way, we start with eq. 3.2, but add an external force, to represent, for example, the gravitational pull of a galaxy on a star cluster. We then take the scalar product with $\mathbf{x}_\alpha$ and sum over all stars

$$\sum_{\alpha} \frac{d}{dt} (m_{\alpha} \mathbf{v}_{\alpha}) \cdot \mathbf{x}_{\alpha} = - \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} \frac{G m_{\alpha} m_{\beta}}{|\mathbf{x}_{\alpha} - \mathbf{x}_{\beta}|^3} (\mathbf{x}_{\alpha} - \mathbf{x}_{\beta}) \cdot \mathbf{x}_{\alpha} + \sum_{\alpha} \mathbf{F}_{\text{ext}}^{\alpha} \cdot \mathbf{x}_{\alpha}. \quad (3.38)$$

We would have had a similar equation if we had started with the β force:

$$\sum_{\beta} \frac{d}{dt} (m_{\beta} \mathbf{v}_{\beta}) \cdot \mathbf{x}_{\beta} = - \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} \frac{G m_{\alpha} m_{\beta}}{|\mathbf{x}_{\alpha} - \mathbf{x}_{\beta}|^3} (\mathbf{x}_{\beta} - \mathbf{x}_{\alpha}) \cdot \mathbf{x}_{\beta} + \sum_{\beta} \mathbf{F}_{\text{ext}}^{\beta} \cdot \mathbf{x}_{\beta}. \quad (3.39)$$

The left-hand sides of these two equations are the same; each is equal to

$$\frac{1}{2} \sum_{\alpha} \frac{d^2}{dt^2} (m_{\alpha} \mathbf{x}_{\alpha} \cdot \mathbf{x}_{\alpha}) - \sum_{\alpha} m_{\alpha} \mathbf{v}_{\alpha} \cdot \mathbf{v}_{\alpha} = \frac{1}{2} \frac{d^2 I}{dt^2} - 2\mathcal{KE}, \quad (3.40)$$

where I is the *moment of inertia* of the system:

$$I \equiv \sum_{\alpha} m_{\alpha} \mathbf{x}_{\alpha} \cdot \mathbf{x}_{\alpha}. \quad (3.41)$$

The virial theorem

- Averaging eqs 3.38 and 3.39 we see that the 1st term on the right hand side is the potential energy, and then

$$\frac{1}{2} \frac{d^2 I}{dt^2} - 2\mathcal{KE} = \mathcal{PE} + \sum_{\alpha} \mathbf{F}_{\text{ext}}^{\alpha} \cdot \mathbf{x}_{\alpha}. \quad (3.42)$$

Now we average this equation over the time interval $0 < t < \tau$:

$$\frac{1}{2\tau} \left[\frac{dI}{dt}(\tau) - \frac{dI}{dt}(0) \right] = 2 \langle \mathcal{KE} \rangle + \langle \mathcal{PE} \rangle + \sum_{\alpha} \langle \mathbf{F}_{\text{ext}}^{\alpha} \cdot \mathbf{x}_{\alpha} \rangle, \quad (3.43)$$

where the angle brackets are used to represent this long-term average. As long as all the stars are bound to the cluster, the products $|\mathbf{x}_{\alpha} \cdot \mathbf{v}_{\alpha}|$, and hence $|dI/dt|$, never exceed some finite limits. Thus the left-hand side of this equation must tend to zero as $\tau \rightarrow \infty$, giving

$$2 \langle \mathcal{KE} \rangle + \langle \mathcal{PE} \rangle + \sum_{\alpha} \langle \mathbf{F}_{\text{ext}}^{\alpha} \cdot \mathbf{x}_{\alpha} \rangle = 0. \quad (3.44)$$

Orbits in Spherical Potentials

- In spherical systems, stars move in a centrally directed force field. Because of the rotational symmetry the angular momentum is conserved. So, if the position vector is written as

$$\mathbf{r} = r\hat{\mathbf{e}}_r$$

And the radial acceleration as

$$\mathbf{F} = F(r)\hat{\mathbf{e}}_r,$$

Then the equation of motion is

$$\frac{d^2\mathbf{r}}{dt^2} = F(r)\hat{\mathbf{e}}_r.$$

Which implies that the vector

$$\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{L}.$$

remains constant, since

$$\frac{d}{dt} \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt} \right) = \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = F(r)\mathbf{r} \times \hat{\mathbf{e}}_r = 0.$$

Orbits in Spherical Potentials

- The conservation of L means that the motion occurs in a **plane**, defined by the position vector and velocity at any time. We can choose polar coordinates in that plane to write the equations of motion as

$$\ddot{r} - r\dot{\psi}^2 = F(r)$$

$$2\dot{r}\dot{\psi} + r\ddot{\psi} = 0.$$

Multiplying the second of this by r shows that

$$r^2\dot{\psi} = \text{constant} \equiv L.$$

With the usual interpretation that $L=2 \times$ rate at which the radius vector sweeps area

To obtain the shape of the orbit we replace time by angle in the above equations using

$$\frac{d}{dt} = \frac{L}{r^2} \frac{d}{d\psi},$$

Orbits in Spherical Potentials

- And we can write the eq.

$$\ddot{r} - r\dot{\psi}^2 = F(r)$$

as

$$\frac{L^2}{r^2} \frac{d}{d\psi} \left(\frac{1}{r^2} \frac{dr}{d\psi} \right) - \frac{L^2}{r^3} = F(r).$$

Which can be simplified using the substitution $u=1/r$ to give

$$\frac{d^2 u}{d\psi^2} + u = -\frac{F(1/u)}{L^2 u^2}.$$

- This equation admits two types of solutions, depending on whether u may reach 0 (r goes to infinity=**unbound** orbits) or whether u oscillates between finite limits (**bound** orbits).
- The general solution to this equation must be typically found numerically, unless $F(1/u)$ has a simple form
- If we multiply the above equation by $du/d\psi$ and integrate over ψ

Orbits in Spherical Potentials

- We get that

$$\left(\frac{du}{d\psi}\right)^2 + \frac{2\Phi}{L^2} + u^2 = \text{constant} \equiv \frac{2E}{L^2},$$

where Φ is defined by $F = -d\Phi/dr$ and we have introduced E to characterize the constant of integration. One may rewrite the above equation as

$$E = \frac{1}{2} \left(\frac{dr}{dt}\right)^2 + \frac{1}{2} \left(r \frac{d\psi}{dt}\right)^2 + \Phi(r),$$

Which shows that E is just the usual energy per unit mass of the star

For bound orbits the condition $du/d\psi=0$ is satisfied, which implies that

$$u^2 + \frac{2[\Phi(1/u) - E]}{L^2} = 0$$

The roots of this equation, $u_1=1/r_1$ and $u_2=1/r_2$ define the pericenter and apocenter of the orbit, which “contain” radially the orbit. The star oscillates between this inner and outer radius while revolving in ψ

Orbits in Spherical Potentials

- The radial period T_r of the orbit is the time the star takes to travel from pericenter to apocenter and back. Using the definition of $L=r^2 d\psi/dt$ we can rewrite

$$E = \frac{1}{2} \left(\frac{dr}{dt} \right)^2 + \frac{1}{2} \left(r \frac{d\psi}{dt} \right)^2 + \Phi(r),$$

- as

$$\left(\frac{dr}{dt} \right)^2 = 2(E - \Phi) - \frac{L^2}{r^2}.$$

- Which may be rewritten as

$$\frac{dr}{dt} = \pm \sqrt{2[E - \Phi(r)] - \frac{L^2}{r^2}}.$$

- The radial period is then

$$T_r = 2 \int_{r_1}^{r_2} \frac{dr}{\sqrt{2[E - \Phi(r)] - L^2/r^2}}.$$

- During a radial period the azimuthal angle increases by

$$\Delta\psi = 2 \int_{r_1}^{r_2} \frac{d\psi}{dr} dr = 2 \int_{r_1}^{r_2} \frac{L}{r^2} \frac{dt}{dr} dr.$$

Orbits in Spherical Potentials

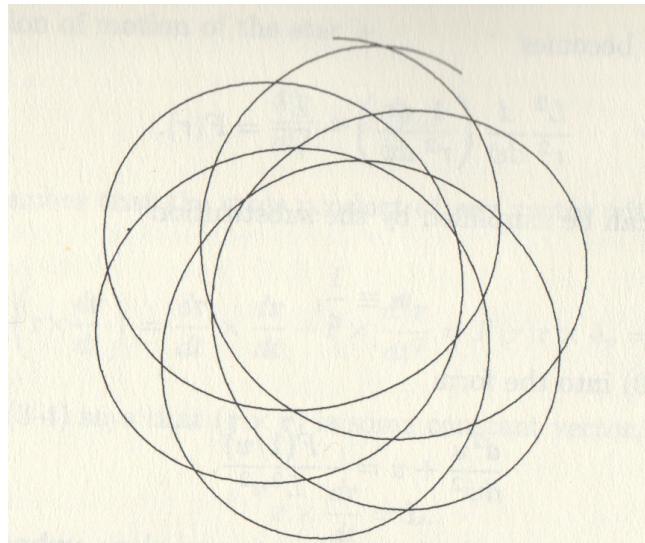
- or

$$\Delta\psi = 2L \int_{r_1}^{r_2} \frac{dr}{r^2 \sqrt{2[E - \Phi(r)] - L^2/r^2}}.$$

- The azimuthal period is then

$$T_\psi = \frac{2\pi}{\Delta\psi} T_r;$$

- In general $\frac{2\pi}{\Delta\psi}$ won't be a rational number: the orbit is **not closed**
- The typical orbit is a **rosette**



Examples of Orbits in Spherical Potentials

- The Spherical Harmonic Oscillator (a homogeneous sphere)

$$\Phi(r) = \frac{1}{2}\Omega^2 r^2 + \text{constant}$$

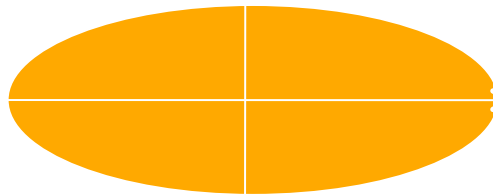
- In this case the equations of motion are easiest to solve in Cartesian coordinates,

$$\ddot{x} = -\Omega^2 x \quad ; \quad \ddot{y} = -\Omega^2 y,$$

With solutions

$$x = X \cos(\Omega t + \epsilon_x) \quad ; \quad y = Y \cos(\Omega t + \epsilon_y),$$

- Every orbit is closed, since the period of oscillation about x and y are identical.
- The orbits are ellipses centered on the center of attraction.



Examples of Orbits in Spherical Potentials

- The **Kepler potential**, where the force law is $F = -GM/r^2$. The orbit equation is now

$$\frac{d^2 u}{d\psi^2} + u = \frac{GM}{L^2},$$

- With a general solution

$$u(\psi) = C \cos(\psi - \psi_0) + \frac{GM}{L^2},$$

- This may be rewritten as

$$r(\psi) = \frac{a(1 - e^2)}{1 + e \cos(\psi - \psi_0)}.$$

- defining the eccentricity and the semimajor axis by:

$$e = \frac{CL^2}{GM}$$

$$a \equiv \frac{L^2}{GM(1 - e^2)},$$

Examples of Orbits in Spherical Potentials

- In the **Kepler potential**, for $e \geq 1$ the orbit is unbound
- For $e < 1$ the orbit is bound; the radius is a periodic function of ψ with period 2π .
- Bound orbits are closed and form ellipses with the attracting center at one focus

$$r(\psi) = \frac{a(1 - e^2)}{1 + e \cos(\psi - \psi_0)}.$$

- In a **Kepler potential**, the star completes a radial oscillation in the time required for ψ to increase from 0 to 2π .
- This is different from the **spherical harmonic oscillator**, where a radial oscillation is completed in the time it takes ψ to increase from 0 to π .

Orbits in Spherical Potentials

- Realistic galaxy models are less concentrated than a point mass and more concentrated than a homogeneous sphere, so we expect radial oscillations to take less time than required for a complete azimuthal cycle.
- It is sometimes useful to think of orbits in non-Keplerian potentials as forming an approximate ellipse, in a rotating frame that precesses by

$$\psi_p = \Delta\psi - 2\pi$$

- The precession is in the sense opposite to the rotation of the star itself. Thus, in a rotating frame with angular velocity

$$\Omega_p = \frac{\psi_p}{T_r} = \frac{\Delta\psi - 2\pi}{T_r}.$$

the orbit appears as a closed ellipse.

Orbits in Axisymmetric Potentials

- If the potential is only a function of the cylindrical coordinates (R, z) stellar motions in this potential will conserve the z -component of the angular momentum, L_z .
- Stars originally in the symmetry plane $z=0$ and with $v_z=0$ will remain on the plane. Their motions will be **rosettes** just like the ones we described before.
- Stars that move away from the equatorial plane are more interesting.
- In cylindrical coordinates, the eq. of motion will be

$$\frac{d^2 \mathbf{r}}{dt^2} = -\nabla \Phi(R, z).$$

- Writing

$$\mathbf{r} = R\hat{\mathbf{e}}_R + z\hat{\mathbf{e}}_z$$
$$\nabla \Phi = \frac{\partial \Phi}{\partial R} \hat{\mathbf{e}}_R + \frac{\partial \Phi}{\partial z} \hat{\mathbf{e}}_z.$$

Orbits in Axisymmetric Potentials

- We obtain, in cylindrical coordinates, the following equations

$$\begin{aligned}\ddot{R} - R\dot{\phi}^2 &= -\frac{\partial\Phi}{\partial R}, \\ \frac{d}{dt} (R^2\dot{\phi}) &= 0, \\ \ddot{z} &= -\frac{\partial\Phi}{\partial z}.\end{aligned}$$

- The second equation is just the conservation of L_z . Using that to eliminate the azimuthal dependence we can write the other two equations as

$$\begin{aligned}\ddot{R} &= -\frac{\partial\Phi_{\text{eff}}}{\partial R} \quad ; \quad \ddot{z} = -\frac{\partial\Phi_{\text{eff}}}{\partial z}, \\ \Phi_{\text{eff}} &\equiv \Phi(R, z) + \frac{L_z^2}{2R^2}.\end{aligned}$$

Orbits in Axisymmetric Potentials

- This reduces the problem to the motion of a star in a (R,z) plane that revolves **non-uniformly** around the z-axis. This plane is known as the “meridional plane”.
- The total energy of a star is given by

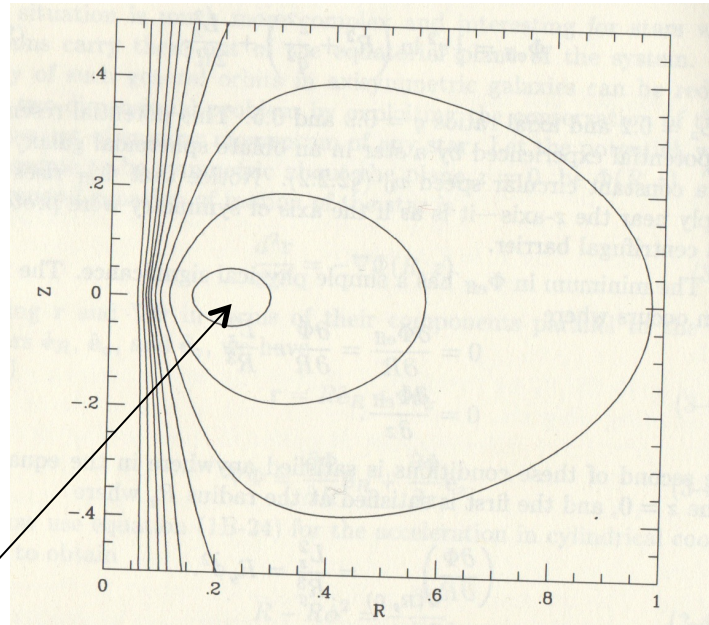
$$E = \frac{1}{2}[\dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2] + \Phi = \frac{1}{2}(\dot{R}^2 + \dot{z}^2) + \left(\Phi + \frac{L_z^2}{2R^2} \right)$$
$$= \frac{1}{2}(\dot{R}^2 + \dot{z}^2) + \Phi_{\text{eff}}.$$

so the “effective potential” is just the sum of the potential energy plus the kinetic energy in azimuthal motion.

- Since the kinetic energy is non-negative, the motion of a bound orbit is confined to a region of the meridional plane where $E > \Phi_{\text{eff}}$. This region is limited by the “zero velocity curve” of a star.

Orbits in Axisymmetric Potentials

- Some examples of “zero velocity curves” in a given potential.



- The minimum in ϕ_{eff} has a simple physical significance. It occurs where

$$\left(\frac{\partial \Phi}{\partial R} \right)_{(R_g, 0)} = \frac{L_z^2}{R_g^3} = R_g \dot{\phi}^2.$$

which is the condition for a circular orbit.

Epicycles

- Stars like the Sun are on nearly circular orbits, so it is useful to derive approximate solutions to the equations of motion under that condition
- Defining

$$x \equiv R - R_g,$$

- We have that $(x,z)=(0,0)$ describe the location of the minimum of Φ_{eff} .
- Expanding in a Taylor series, we obtain

$$\Phi_{\text{eff}} = \frac{1}{2} \left(\frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right)_{(R_g, 0)} x^2 + \frac{1}{2} \left(\frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \right)_{(R_g, 0)} z^2 + O(xz^2) + \text{constant.} \quad (3-55)$$

- The equations of motion are now

$$\ddot{x} = -\kappa^2 x,$$

$$\ddot{z} = -\nu^2 z.$$

Epicycles

- Where

$$\kappa^2 \equiv \left(\frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right)_{(R_g, 0)} ; \quad \nu^2 \equiv \left(\frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \right)_{(R_g, 0)},$$

- are the epicycle frequency and the vertical frequency, respectively.
- After some algebra, we can write

$$\kappa^2 = \left(R \frac{d\Omega^2}{dR} + 4\Omega^2 \right)_{R_g}.$$

- Where the circular frequency is given by

$$\Omega^2(R) = \frac{1}{R} \left(\frac{\partial \Phi}{\partial R} \right)_{(R, 0)} = \frac{L_z^2}{R^4},$$

Epicycles

- For the Kepler potential we have

$$\Omega \propto R^{-3/2}, \quad \kappa = \Omega.$$

- For the homogeneous sphere
 - $\Omega = \text{constant}$ and therefore $\kappa = 2\Omega$

- Therefore in general for a realistic galaxy

$$\Omega \lesssim \kappa \lesssim 2\Omega.$$

Epicenter and epicycles

- The solution to the harmonic equations are

$$x(t) = X \cos(\kappa t + \psi),$$

$$z = Z \cos(\nu t + \zeta),$$

so the star oscillates backwards and forwards and up and down

- Since

$$\begin{aligned}\dot{\phi} &= \frac{L_z}{R^2} = \frac{L_z}{R_g^2} \left(1 + \frac{x}{R_g}\right)^{-2} \\ &\simeq \Omega_g \left(1 - \frac{2x}{R_g}\right).\end{aligned}$$

- We obtain

$$\phi = \Omega_g t + \phi_0 - \frac{2\Omega_g X}{\kappa R_g} \sin(\kappa t + \psi).$$

- And, in Cartesian coordinates

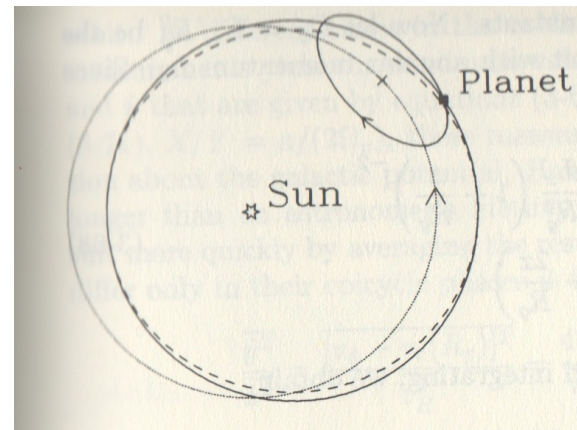
$$\begin{aligned}y &= -\frac{2\Omega_g}{\kappa} X \sin(\kappa t + \psi) \\ &\equiv -Y \sin(\kappa t + \psi).\end{aligned}$$

Epicenter and epicycles

- Since the oscillation in x and y have the same frequency the orbit is closed around the **guiding center** defined by the circular orbit
- In the (x,y) plane the star moves on an ellipse called “**epicycle**”
- The shape of the ellipse is given by

$$\frac{X}{Y} = \frac{\kappa}{2\Omega_g}.$$

- Note that for the Kepler problem $X/Y=1/2$ and that the motion around the ellipse is retrograde (**why?**).
- These are the main reasons why Hipparchus’ **circular** epicycles could not describe the motion of planets



Kinematic density waves

- Typically, the orbit of a star in an axisymmetric galaxy is a rosette, where a radial oscillation is completed in the time it takes for the azimuthal angle to change by $\Delta\phi$.
- In general $\Delta\phi/2\pi$ is irrational and the orbit does not close.
- In a rotating frame with angular speed Ω_p the change in azimuth after one radial period would be

$$\Delta\phi_p = \Delta\phi - \Omega_p T_r.$$

- Therefore we can choose a frame where the orbit closes. In particular, if

$$\Delta\phi_p = 2\pi n/m,$$

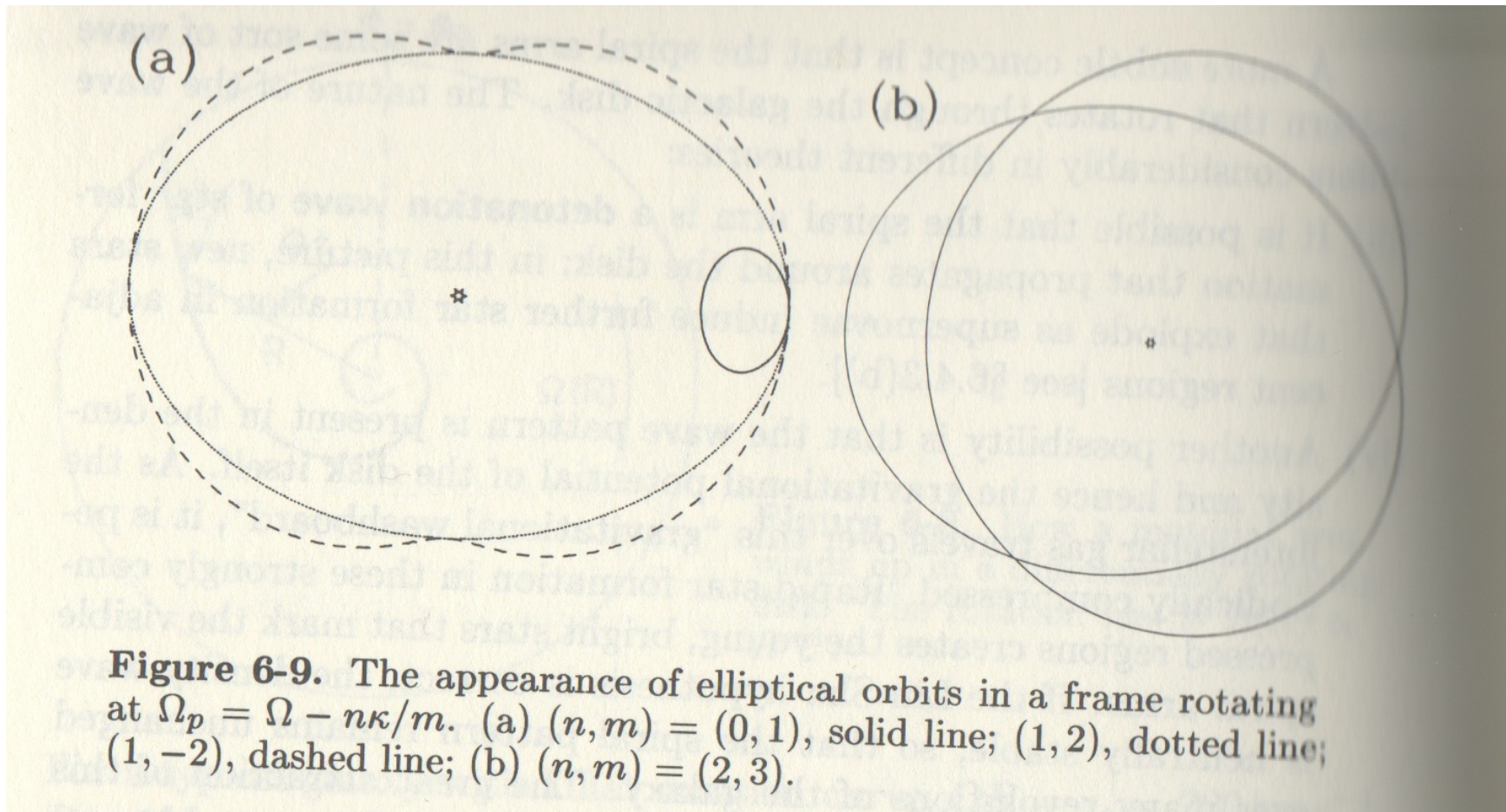
- The orbit closes after m radial oscillations. In this case

$$\Omega_p = \omega_a - \frac{n\omega_r}{m} \simeq \Omega - \frac{n\kappa}{m},$$

where ω_a and ω_r are the angular and radial oscillation frequencies

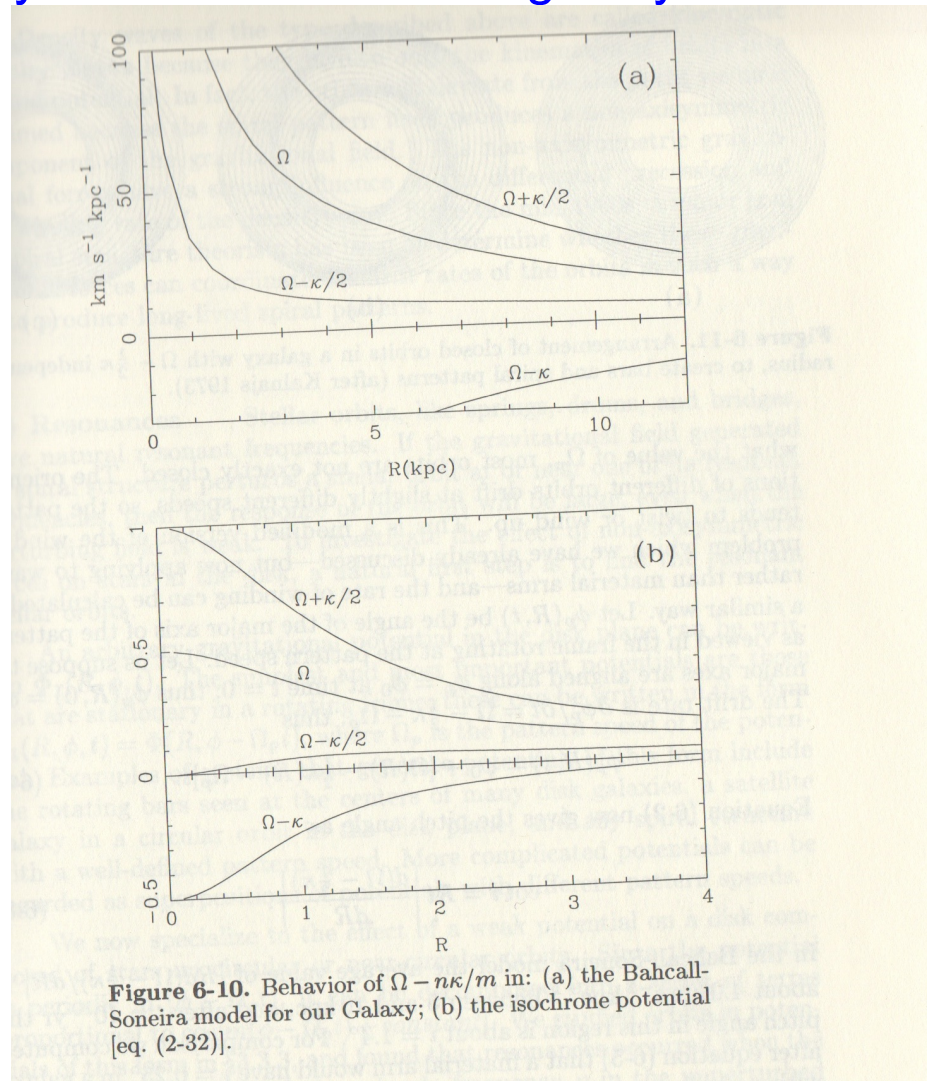
Kinematic density waves

- These are examples of what the closed orbits look like in various reference frames



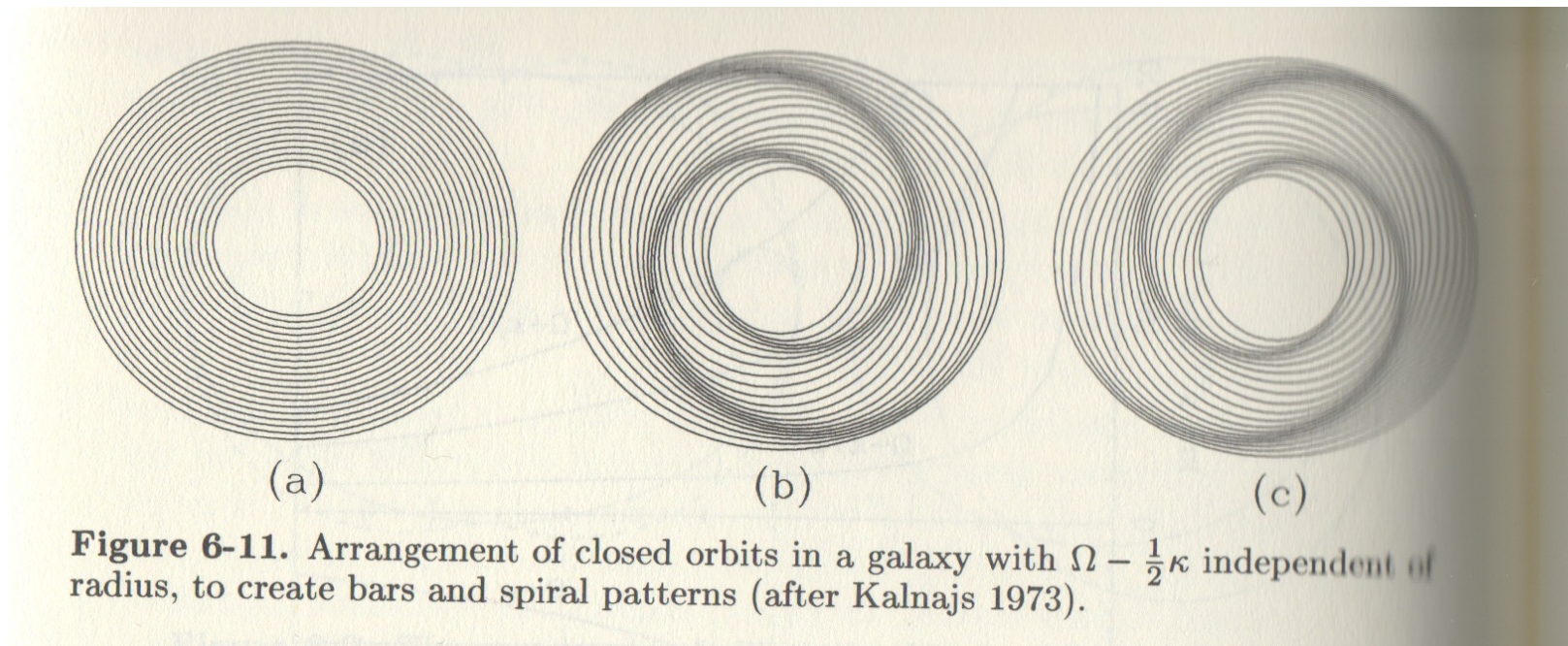
Kinematic density waves

- One surprising fact about realistic disk galaxy models is that $\Omega - \kappa/2$ is approximately constant across the galaxy.



Kinematic density waves

- Hence we can set up a nested aligned set of these orbits covering a range of radii in order to create patterns such as a bar-like wave pattern, or leading/trailing spiral patterns.
- Seen from an inertial frame, these patterns rotate at angular speed Ω_p —the pattern speed
- Note that the patterns arise even if the potential is axisymmetric!
- These patterns are much longer lived than “physical spiral arms” which suffer from the winding problem.

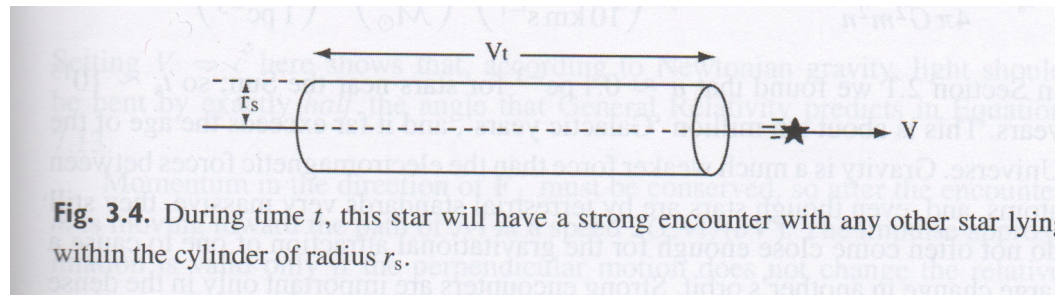


Two-body Relaxation

- Strong close encounters between stars are rare. These happen when the gravitational potential at closest approach is comparable to the kinetic energy

$$\frac{Gm^2}{r} \gtrsim \frac{mV^2}{2}, \quad \text{which means } r \lesssim r_s \equiv \frac{2Gm}{V^2};$$

- If a star moves at speed V for a time t it will have a strong encounter with stars in a cylinder of radius r_s and length Vt



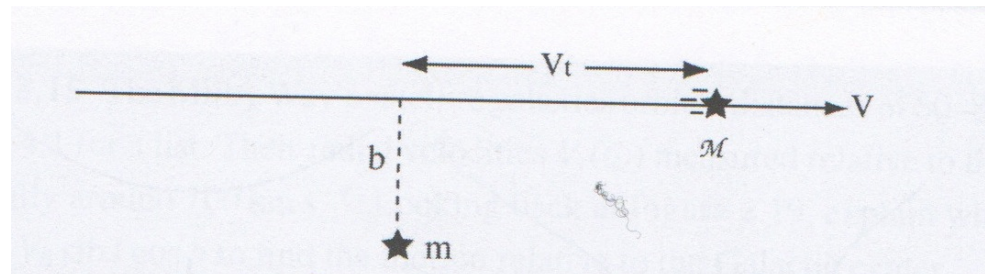
- The average time between encounters is thus given by

$$t_s = \frac{V^3}{4\pi G^2 m^2 n} \approx 4 \times 10^{12} \text{ yr} \left(\frac{V}{10 \text{ km s}^{-1}} \right)^3 \left(\frac{m}{\mathcal{M}_\odot} \right)^{-2} \left(\frac{n}{1 \text{ pc}^{-3}} \right)^{-1}.$$

- For $n \sim 0.1/\text{pc}^3$ and $V \sim 10 \text{ km/s}$ this is $\sim 10^{13}$ yrs, much longer than the age of the Universe—strong encounters are rare!

Two-body Relaxation

- The effect of weak distant encounters may be estimated by considering the change in the direction of the velocity of a star, as influenced by another.
- We can use the impulse approximation to obtain a crude estimate. Here we assume that one star is stationary and the other moves on a straight line.



- The velocity along the path will be unchanged, but the star will experience a net deviation due to the force in the perpendicular direction

$$\mathbf{F}_{\perp} = \frac{Gm\mathcal{M}b}{(b^2 + V^2t^2)^{3/2}} = \mathcal{M} \frac{dV_{\perp}}{dt}.$$

- Integrating over time we obtain

$$\Delta V_{\perp} = \frac{1}{\mathcal{M}} \int_{-\infty}^{\infty} \mathbf{F}_{\perp}(t) dt = \frac{2Gm}{bV};$$

Two-body Relaxation

- Although over time $\langle \Delta V_T \rangle = 0$ (stars tug in different direction), the average of $\langle \Delta V_T^2 \rangle$ will grow to become comparable to the original velocity V^2 of the star
- When that happens the star has “lost” memory of its original direction.

$$\langle \Delta V_{\perp}^2 \rangle = \int_{b_{\min}}^{b_{\max}} n V t \left(\frac{2Gm}{bV} \right)^2 2\pi b \, db = \frac{8\pi G^2 m^2 n t}{V} \ln \left(\frac{b_{\max}}{b_{\min}} \right).$$

- This happens after the “relaxation time”

$$t_{\text{relax}} = \frac{V^3}{8\pi G^2 m^2 n \ln \Lambda} = \frac{t_s}{2 \ln \Lambda}$$

$$\approx \frac{2 \times 10^9 \text{ yr}}{\ln \Lambda} \left(\frac{V}{10 \text{ km s}^{-1}} \right)^3 \left(\frac{m}{\mathcal{M}_{\odot}} \right)^{-2} \left(\frac{n}{10^3 \text{ pc}^{-3}} \right)^{-1}.$$

- Note that this is still of order 10^{13} yrs, for $n \sim 0.1/\text{pc}^3$
- Therefore collisions are unimportant for a galaxy

Two-body Relaxation

- For an isolated cluster of N stars of mass m , average speed V and size R we have, according to the virial theorem

$$\frac{1}{2}NmV^2 \sim \frac{G(Nm)^2}{2R}, \quad \text{so } \Lambda = \frac{R}{r_s} \sim \frac{GmN}{V^2} \cdot \frac{V^2}{2Gm} \sim \frac{N}{2}.$$

The crossing time $t_{\text{cross}} \sim R/V$; since $N = 4n\pi R^3/3$, we have

$$\frac{t_{\text{relax}}}{t_{\text{cross}}} \sim \frac{V^4 R^2}{6NG^2m^2 \ln \Lambda} \sim \frac{N}{6 \ln(N/2)}.$$

- Thus the effects of collisions are important in systems with few stars or with short crossing times.
- They are important, for example, in the centers of globular clusters, and also in clusters with small numbers of stars

Effects of two-body relaxation

- Just like air molecules, collisions will tend to redistribute the energy of stars and drive them to equipartition. The velocity distribution will tend to a Maxwellian distribution

$$f_M(\mathcal{E}) \propto \exp\left(\frac{-\mathcal{E}}{k_B T}\right) = \exp\left\{-\left[m\Phi(\mathbf{x}) + \frac{mv^2}{2}\right] / (k_B T)\right\},$$

- of “temperature” T

$$\frac{1}{2}m\langle v^2(\mathbf{x}) \rangle = \frac{3}{2}k_B T.$$

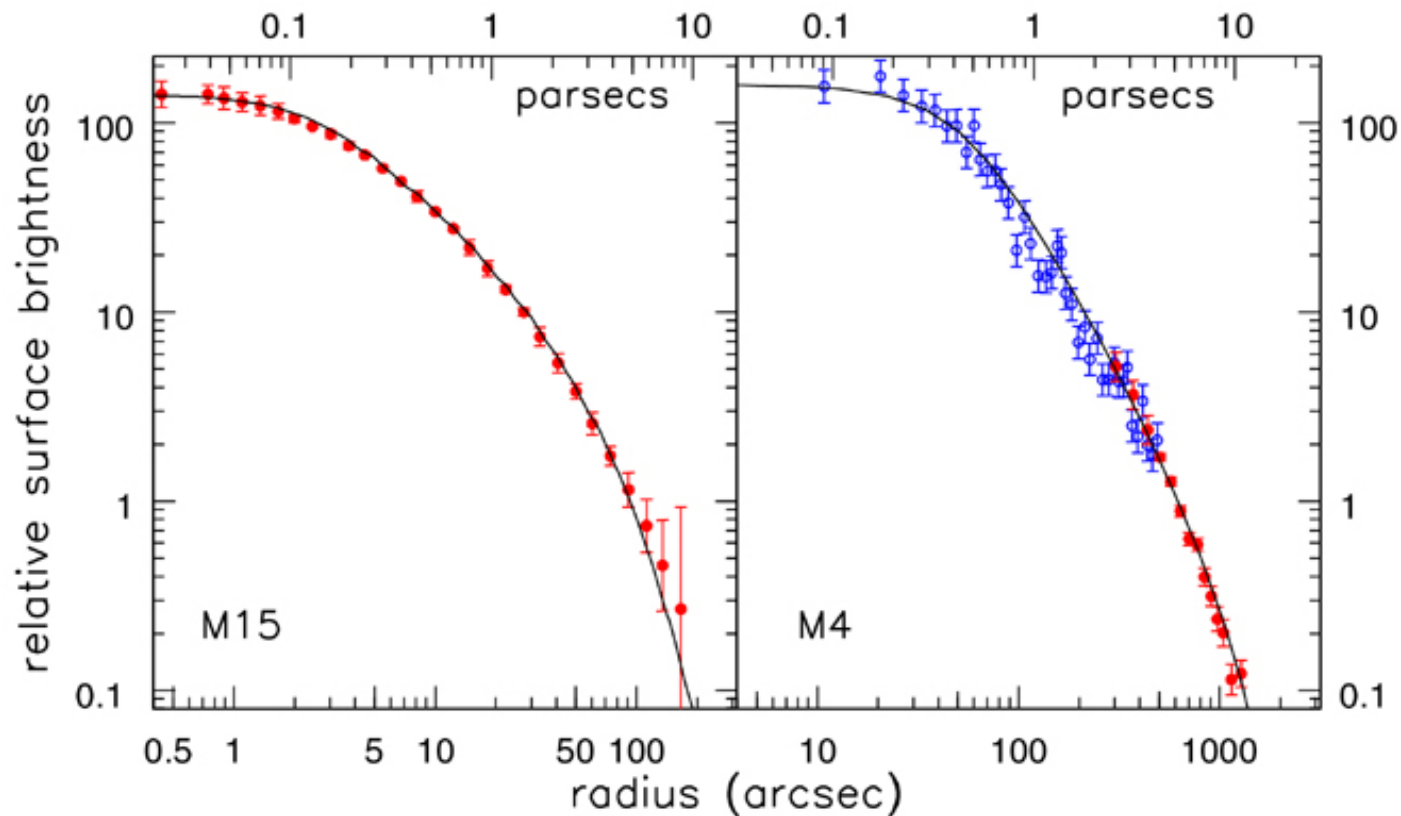
- One effect of relaxation is the evaporation of stars, which occurs on a timescale

$$t_{\text{evap}} \sim 136 t_{\text{relax}}.$$

- Another effect is mass segregation, which leads more massive stars to “sink” to the center of a star cluster

Effects of two-body relaxation

- In a globular cluster, the evaporation of stars leads to the phenomenon of “core collapse”



- What is the ultimate “thermodynamic” fate of a stellar system?