12. Divergence Theorem Revisited

Think about a vector field flowing through a 2d surface embedded in xyz-space. When we look at approximations to the associated flux integral, we can recast the description of the vector field as a 2-form. For example consider a vector field parallel to the z-axis $(f\,\hat{z})$ and a surface restricted to the xy-plane. Then the differential form for the flux integral would be $f\,dx\wedge dy$. For a general vector field, the recasting looks like this:

$$(F_x, F_y, F_z) \rightarrow F_x dy\wedge dz + F_y dz\wedge dx + F_z dx\wedge dy$$

It is easy to see how this differential form works for surfaces aligned with the axes, such as a standard cell. A corrugated approximation works for flux through more general surfaces, even though the corrugated approximation does not work for surface area. Also, any smooth region can be represented by stacked contiguous images of standard cells.

Application of the general Stokes' theorem to closed surfaces results in the traditional Divergence theorem involving the divergence of the vector field. For a vector field (F_x, F_y, F_z) we use $\omega = F_x dy\wedge dz + F_y dz\wedge dx + F_z dx\wedge dy$.

For a closed volume V:

$$\int_V d\omega = \int_{\partial V} \omega$$

We have $$\int_V d\omega = \int_V \frac{\partial F_x}{\partial x} dx\wedge dy\wedge dz + \frac{\partial F_y}{\partial y} dy\wedge dz\wedge dx + \frac{\partial F_z}{\partial z} dz\wedge dx\wedge dy$$
$$= \int_V \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\wedge dy\wedge dz$$

$$\Rightarrow \quad \int_V \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\wedge dy\wedge dz = \int_{\partial V} F_x dy\wedge dz + F_y dz\wedge dx + F_z dx\wedge dy$$

which has the appearance of the traditional Divergence theorem.

Vector Fields as Coordinate Charts

Consider two independent vector fields $\vec{\alpha}$ and $\vec{\beta}$ on a 2-d manifold. We can define a chart with variables s and t by sequentially following the vector fields $\vec{\alpha}$ and $\vec{\beta}$ to a point identified by (s,t). In more detail, start at a base point and following a path γ_α to $\gamma_\alpha(s)$ where the derivative of γ_α is $\vec{\alpha}$, then start at $\gamma_\alpha(s)$ and follow a path γ_β to $\gamma_\beta(t)$ where the derivative of γ_β is $\vec{\beta}$.

The fields in the new chart look orthonormal, but they aren't when considering the metric. The metric in this chart satisfies $\left[g_{ij}\right] \approx \begin{bmatrix} \langle \vec{\alpha}, \vec{\alpha} \rangle & \langle \vec{\alpha}, \vec{\beta} \rangle \\ \langle \vec{\alpha}, \vec{\beta} \rangle & \langle \vec{\beta}, \vec{\beta} \rangle \end{bmatrix}$ and $\sqrt{\left|g_{ij}\right|} \approx \sqrt{\|\vec{\alpha}\|^2 \|\vec{\beta}\|^2 - \langle \vec{\alpha}, \vec{\beta} \rangle}$ in (s,t) coordinates (in a neighborhood of the base point).

Lie Bracket

We previously encountered the lie bracket as a notational convenience when calculating the exterior derivative of a 1-form, $[\vec{\alpha}, \vec{\beta}] \equiv \nabla_{\vec{\alpha}} \vec{\beta} - \nabla_{\vec{\beta}} \vec{\alpha}$. In our previous example of creating a chart from a pair of vector fields, we could have followed $\vec{\beta}$ first and then $\vec{\alpha}$ to arrive at a possibly different point. It turns out that either path arrives at the same point if $[\vec{\alpha}, \vec{\beta}] = \vec{0}$ in the relevant region. To see this, we express the parametrized paths as taylor series and gather up the third order error terms as bounded coeffecients $\vec{B}$'s and $\vec{E}$'s

We calculate P(s,t) by following $\vec{\alpha}$ and then $\vec{\beta}$ as

$$P(s,t) = \left(\vec{\alpha}(0)s + \frac{1}{2}(\nabla_{\vec{\alpha}}\vec{\alpha})\big|_0 s^2 + \vec{B}_1\ s^3\right) + \left(\vec{\beta}(0) + \left(\nabla_{\vec{\alpha}}\vec{\beta}\right)\big|_0 s + \vec{B}_2 s^2\right)t + \frac{1}{2}\left(\nabla_{\vec{\beta}}\vec{\beta}\right)\big|_0 t^2 + \vec{B}_2\ t^3$$

Similarly calculate Q(s,t) by following $\vec{\beta}$ and then $\vec{\alpha}$ as

$$Q(s,t) = \left(\vec{\beta}(0)t + \frac{1}{2}\left(\nabla_{\vec{\beta}}\vec{\beta}\right)\big|_0 t^2 + \vec{B}_3\ t^3\right) + \left(\vec{\alpha}(0) + \left(\nabla_{\vec{\beta}}\vec{\alpha}\right)\big|_0 t + \vec{B}_4 t^2\right)s + \frac{1}{2}(\nabla_{\vec{\alpha}}\vec{\alpha})\big|_0 s^2 + \vec{B}_5\ s^3$$

Then

$$(P-Q)(s,t) = \left(\left(\nabla_{\vec{\alpha}}\vec{\beta}\right)\big|_0 - \left(\nabla_{\vec{\beta}}\vec{\alpha}\right)\big|_0\right)st + \vec{E}_s s^3 + \vec{E}_t t^3 = [\vec{\alpha}, \vec{\beta}]_0 st + \vec{E}_s s^3 + \vec{E}_t t^3$$

We see that (P-Q) and its first derivatives are $\vec{0}$ at the origin, and its second partial derivatives are given by the lie bracket.

Curvature

The elementary idea of curvature refers to a manifold embedded in a higher dimensional euclidean space.

For example, we can embed the circle of radius R in 2-d space. This embedded 1-d manifold has curvature $K = \frac{1}{R}$.

The curvature of a 1-d manifold at a point is the same as that of a circle tangent at the point and having the same second derivative when measuring the distance from the tangent plane (line).

Embedded 2-d Surface Curvature

At any point on a 2-d surface embedded in 3-d euclidean space, there is a spray of geodesic curves. We describe the curvature of the surface in terms of the minimum and maximum curvature of the collection of geodesics. These are called principle curvatures.

A cylinder with radius R has principle curvatures $K_1 = \frac{1}{R}$ and $K_2 = 0$.

A paraboloid $z = \frac{\lambda_1}{2} x^2 + \frac{\lambda_2}{2} y^2$ has principle curvatures at the origin given by $K_1 = \lambda_1$ and $K_2 = \lambda_2$. But, not simple away from the origin where it flattens out.

For a sphere of radius R, the principle curvatures are the same $K_1 = \frac{1}{R} = K_2$.

Derivative of the Unit Normal

Consider the field of unit radius vectors along a circle. The tangential derivative of this vector field has the same magnitude as the curvature.

We can study the curvature of an embedded geodesic by considering the tipping of the unit normal while moving along the geodesic. The directional derivative of the unit normal is tangent to the surface.

In the case of a principle direction (which is an extrema), it is plausible that the result of a directional derivative is parallel to the direction of differentiation. Imagine the unit normal of a cylinder differentiated along a circumference vs some other direction. In other words, the directional derivative is an eigenvector of the directional derivative operator. The eigenvalues of the directional derivative correspond to the principle curvatures.

Gaussian Curvature

We define Gaussian curvature of a 2-d surface embedded in 3-d Cartesian space by the product of the principle curvatures $K_G \equiv \lambda_1 \lambda_2$ which is the product of the eigenvalues of the directional derivative operator of the unit normal.

With this definition, a plane or cylinder has zero curvature. A sphere of radius R has curvature given by $K_G = \frac{1}{R^2}$.

At the origin of a paraboloid like $z=-\frac{\lambda_1}{2}x^2-\frac{\lambda_2}{2}y^2$, the unit normal is given by $\hat{n}=\frac{(\lambda_1 x,\lambda_2 y,1)}{\sqrt{1+\lambda_1^2x^2+\lambda_2^2y^2}}$ and its derivative at the origin is given by

$$\nabla\hat{n}\big|_{\vec{0}}=\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

The curvature at the origin is $K_G=\lambda_1\lambda_2$. We don't worry about the lower right corner of the matrix, because it represents differentiation away from the surface.

Although we have defined Gaussian curvature in terms of its embedding, it turns out that it only depends on its Riemannian metric (Gauss's Theorema Egregium). The curvature tensor field is especially convenient because it is a scalar field and hence remains unchanged, when changing coordinate systems.

We next consider curvature ideas that do not depend on an embedding.

Change in a Vector after Flattening

Consider a smooth vector field $\vec{v}$ on a cone. We cut along a line through the vertex coplanar with the axis. Then flatten it by opening a sector at the cut. Two neighboring field vectors that are on opposite sides of the cut are almost parallel. After cutting and flattening, their orientations have an angular separation equal to the angle of the missing sector (deficit angle). For a cone with a smoothed vertex, we can imagine retrieving curvature information from the deficit angle.

We can do a similar operation on any smooth manifold and vector field. For an earth example, consider the northern hemisphere. Take twin field vectors on the arctic circle on either side of and close to the Greenwich meridian. Then we can cut the earth along the the Greenwich meridian and then flatten it. We flatten the earth along the arctic circle by imagining a cone tangent to the arctic circle and then cutting and flattening the cone. The twin vectors then show a deficit angle. The deficit angle will be the same for any vector field, but it will depend on the path (arctic circle in this case).

We can mathematically flatten along any non-closed path by creating a collection of geodesic coordinate systems along the path and then correlating them.

In flattened coordinates, we can find the twin vector by integrating the derivative of the vector field from one twin to the other. This is equivalent to integrating the covariant derivative in the original coordinates. We don't have to flatten along the loop or even cut it. The terminal twin is found by integrating the covariant derivative of the vector field around the loop. The deficit angle is the angle from the terminal twin to the initial twin. This is the same rotation experienced by the original vector undergoing parallel transport around the loop.