3. Local Coordinate System

We wish to express vector fields (and other linear objects) on a parameterized surface. Each vector is in the plane tangent to the surface, at each point. These tangent planes can be expressed in terms of basis vectors. The basis vectors define the local coordinate system.

In the case of 2-d Cartesian space, we can use the standard basis $\hat{x}_1$ and $\hat{x}_2$ as the basis for the local coordinate system at each point.

Suppose 2-d Cartesian space is parametrized by alternative coordinates (u^1, u^2) (possibly polar coordinates). This can be expressed as the function $\vec{x}(\vec{u})$.

An alternative local bases is given by $\left[\frac{\partial \vec{x}}{\partial u^1}, \frac{\partial \vec{x}}{\partial u^2}\right]$

The jth basis vector is the jth column of the Jacobian matrix. It is expressed in Einstein summation notation as $\frac{\partial x^i}{\partial u^j}$.

Polar Coordinate Example

We use familiar notation identifying (x^1, x^2) with (x, y) and (u^1, u^2) with (r, θ) .

As before, $x = r\cos\theta$, $y = r\sin\theta$.

Calculating partial derivatives gives:

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

So we have local basis vectors

$$\frac{\partial(x, y)}{\partial r} = (\cos\theta, \sin\theta) \quad \text{and} \quad \frac{\partial(x, y)}{\partial \theta} = r\,(-\sin\theta, \cos\theta)$$

We will discuss related functionals, expressed as dr and $d\theta$ in the future.

Tangent Plane
A parametrized surface may be embedded in a Cartesian space of higher dimension than the parameter space. In this case we can visualize the Jacobian matrix of the parametrization as the best linear approximation to the parametrization.

The tangent plane is spanned by the basis vectors $\frac{\partial x^i}{\partial u^j}$ indexed by j. Think of a tangent plane as the embedded image of a local coordinate system from the parameter space.

Geography Example
For the geographic parameterization of the earth $(\text{lon}, \text{lat}) = (\alpha, \beta)$ we previously had $x = R\cos\alpha\cos\beta$, $y = R\sin\alpha\cos\beta$, $z = R\sin\beta$

To find a local basis tangent to the surface, we differentiate:

$$\frac{\partial x}{\partial \alpha} = -R\sin\alpha\cos\beta \qquad \frac{\partial x}{\partial \beta} = -R\cos\alpha\sin\beta$$

$$\frac{\partial y}{\partial \alpha} = R\cos\alpha\cos\beta \qquad \frac{\partial y}{\partial \beta} = -R\sin\alpha\sin\beta$$

$$\frac{\partial z}{\partial \alpha} = 0 \qquad \frac{\partial z}{\partial \beta} = R\cos\beta$$

We get tangent basis vectors:

$$\frac{\partial(x, y, z)}{\partial \alpha} = (-R\sin\alpha\cos\beta\ ,\ R\cos\alpha\cos\beta\ ,\ 0) \quad \text{pointing east}$$

and $\frac{\partial(x, y, z)}{\partial \beta} = (-R\cos\alpha\sin\beta\ ,\ -R\sin\alpha\sin\beta\ ,\ R\cos\beta)$ pointing north

Torus Example
In this example, the parameter space (on θ^1, θ^2) is 2-d and the embedding space with variables (x^1, x^2, x^3, x^4) is 4-d.

$x^1 = \cos\theta^1$, $x^2 = \sin\theta^1$, $x^3 = \cos\theta^2$, $x^4 = \sin\theta^2$

We cannot easily draw or imagine this, but we can find the embedded 2-d tangent plane through its spanning vectors. The math extends our vision.

$$\frac{\partial(x^1, x^2, x^3, x^4)}{\partial \theta^1} = (-\sin\theta^1, \cos\theta^1, 0, 0) \qquad \frac{\partial(x^1, x^2, x^3, x^4)}{\partial \theta^2} = (0, 0, -\sin\theta^2, \cos\theta^2)$$

Vectors and Functionals in the Parameter Space
It is often easier to work in the parameter space than the embedding space. The vectors and functionals that we wish to consider in the embedded tangent plane are viewed as transformed images from the parameter space.

Consider a parametrization $\vec{x}(\vec{u})$ that is essentially a change of variables like polar coordinates for the 2-d Cartesian plane. Also consider a vector field $\vec{v}(\vec{x})$ where $\vec{v}$ is expressed in terms of its local standard basis and alternative local coordinates from the parametrization.

$$\vec{v}=\sum v^i \hat{x}_i=\sum v'^k \frac{\partial \vec{x}}{\partial u^k} \quad \text{equivalently} \quad v^i=v'^k \frac{\partial x^i}{\partial u^k} \quad \text{in Einstein notation.}$$

The above expressions are an application of the transformation rules for vectors. The same result follows from viewing the tangent plane as the best linear approximation to the parametrization. A vector Field v'^k in the parameter space is a displacement vector. The Jacobian matrix of the parameterization represents the first derivative term and the Taylor series looks like:

$$v^i=\vec{x}(\vec{u})-\vec{x}(\vec{u}_0)\approx\frac{\partial x^i}{\partial u^k}v'^k \quad \text{where} \quad v'^k=\vec{u}-\vec{u}_0$$

In polar coordinates the calculations for each component look like:

$$v^x = v'^r\frac{\partial x}{\partial r}+v'^\theta\frac{\partial x}{\partial \theta} = v'^r\cos\theta-v'^\theta r\sin\theta$$

$$v^y = v'^r\frac{\partial y}{\partial r}+v'^\theta\frac{\partial y}{\partial \theta} = v'^r\sin\theta+v'^\theta\cos\theta$$

In particular, consider the vector field in the parameter space given by $(v'^r, v'^\theta)=(1,0)$ which is a constant vector, pointing in the $\hat{r}$ direction.

The corresponding vector field on the surface (standard plane) is given by: $(v^x, v^y)=(\cos\theta, \sin\theta)$ which is not constant, because the local coordinate system is not constant. The issue of the relationship between varying local coordinates and vector fields will be addressed later with covariant differentiation.

Transforming Functionals

We have seen that we can express a vector in the tangent plane of a surface v^i by way of its parameterization in terms of a vector in the parameter space v'^k as:

$$v^i = v'^k \frac{\partial x^i}{\partial u^k}$$

Recall the previous transformation rules for vectors and functionals

$$\vec{x} = T\ \vec{x}' \quad \text{and} \quad \left(\vec{a}^t\right)' = \vec{a}^t T$$

In this case $T = \frac{\partial x^i}{\partial u^k}$, so the relationship between a functional in the parameter space a'_k and the corresponding functional in the tangent plane a_i is given by:

$$a'_k = a_i \frac{\partial x^i}{\partial u^k}$$

For a vector field or a co-vector (functional) field where the object is defined in all the local coordinate systems in the parameter space, we can use the dual transformation rules $v^i = v'^k \frac{\partial x^i}{\partial u^k}$ and $a'_k = a_i \frac{\partial x^i}{\partial u^k}$ where the derivative terms change from place to place.

Distinguishing between Vectors and Functionals

The distinction between vectors and functionals is related to the context of their application. For example, physical work is calculated as a scalar result of the dot product: $W = \vec{f} \cdot \vec{d}$

We hope that if we change coordinates for the displacement vector $\vec{d}$ and the force co-vector $\vec{f}$, that the work W won't change. We achieve this by calling $\vec{f}$ a functional and $\vec{d}$ a vector $W = f_i d^i$. Then use dual transformation rules for f_i and d^i. The interpretation of $\vec{f}$ as a functional becomes more consistent if it is expressed as the gradient of the potential energy $\vec{f} = -\nabla U$.

We will show later that the gradient of a function should be interpreted as a functional.