Chapter 2. Albedo

1. Introduction.

Albedo is a measurement, expressed as a fraction, of the amount of radiation scattered from a surface or an object. In this chapter we describe those albedos most commonly used and describe methods to calculate them in cases where analytical solutions are difficult, if not impossible, to obtain. In order to do this we introduce two more photometric quantities, namely *exitance* M and *intensity* I. A comprehensive summary of the photometric quantities is also presented.

2. Scattering and Absorption.

The reduction in radiance when a beam passes through a given medium by any process in which the radiation is converted to heat or excitation energy is called *absorption*. A process by which the radiance is reduced by redirection of part of the radiation (by reflection, refraction or diffraction, or by being absorbed and immediately re-radiated in all directions) is called *scattering*, for which *reflectance* is often used as a synonym. The total effect of absorption and scattering is called *extinction*, although the author prefers the less often used alternative, *attenuation*.

3. Absorption, Scattering and Attenuation Coefficients.

The decrease in radiance $-dL$ as a beam of radiance L passes through a medium of thickness ds as a result of absorption is

$$-dL = a\ L\,ds \tag{1}$$

where a is the *linear absorption coefficient* With similar equations we can define the *linear scattering coefficient* s and the *linear attenuation (extinction) coefficient* e. The SI units of a, s and e are $\mathrm{m^{-1}}$ and $e = s + a$.

The *mass absorption coefficient, mass scattering coefficient* and *mass extinction coefficient* each with units $\mathrm{m^2\ kg^{-1}}$ are defined respectively as a/r, s/r and e/r, where r is the density $(\mathrm{kg\ m^{-3}})$ of the medium. Chandrasekhar uses k for the mass extinction coefficient, which, in the theory of stellar atmospheres, is also known as the *opacity*.

The *atomic* (or *molecular*) *absorption, scattering* and *extinction coefficients* are respectively a/N, s/N and e/N, where N is the number density (atoms or molecules per unit volume), with units of $\mathrm{m^2/atom}$ (or $\mathrm{molecule}$). Because of these units the coefficients are often referred to as *cross-sections*.

4. Surfaces - Single Scattering Albedo

We have already encountered a bare-boned, but nonetheless adequate, definition of single scattering albedo in Chapter 1. The loss of radiance from a beam of radiance L traversing a thickness ds of a medium is

$$dL = -e\, L\, ds = -(a + s)\, L\, ds$$

and the single scattering albedo *is that fraction of the loss which can be attributed to scattering alone. i.e.*

$$v_0 = \frac{s}{a + s} = \frac{s}{e} \qquad (2)$$

and the single scattering albedo is thus the ratio of the scattering coefficient to the extinction coefficient.

Single scattering albedo is the property of a *surface* or a *layer*, and may be regarded as the fundamental albedo, since all albedos that will be derived here from a given definition or reflectance rule will contain at least one instance of v_0.

5. Surfaces - Normal Albedo

If a lossless (conservative) Lambertian reflector ($v_0 = 1$) is irradiated normally with flux density F, then its radiance in any direction will be F$/p$. The *normal albedo* p_n of a point on a surface is the ratio of the normally observed radiance to that of the Lambertian surface, so that

$$p_n = p\, f_r(m = m_0 = 1)\,. \qquad (3)$$

The author has found two definitions of normal albedo in the literature. In one, the surface must be radiated normally and observed normally ($m = m_0 = 1$) and the other in which it can be irradiated from any direction, in which case p_n is a function of m_0.

6. Net Flux and Exitance

Formerly known as *emittance*, the exitance M refers to a point on a reflecting or emitting surface and is defined as the total power emitted in all directions per unit *physical* area, so that

$$M = \int_0^{2p} \int_0^{p/2} L(J, j) \sin J \cos J \, dJ\, dj \qquad (4)$$

where it may be seen from the limits of integration that “in all directions” means over a *hemisphere*. The factor $\sin J\, dJ\, dj$ is an element of solid angle, dw, and the factor $\cos J$ is needed to convert the projected area of radiance back into physical area. Using the notation of Chapter 1., *i.e.* let $m = \cos J$, $dm = -\sin J\, dJ$, we have

$$M = \int_0^{2p} \int_0^{1} L(m, j)\, m\, dm\, dj\ . \qquad (5)$$

If we compare M to Chandrasekhar's quantity the *net fluxp* F, which, in particular, he uses for a plane parallel beam of radiation

$$\begin{aligned} p\,F &= \int_0^{2p}\int_0^{p} L(J, j)\sin J \cos J \, dJ\, dj \\ &= \int_0^{2p}\int_{-1}^{1} L(m, j)\, m \, dm \, dj \end{aligned} \tag{6}$$

we see that the net flux is indeed the result of integration over all directions, *i.e.* over a *sphere*. It follows that net flux and exitance are *not the same thing* (although there may be situations in which they amount to the same), and nor does $p\,F$ always mean the strength of a plane parallel beam of radiant flux density F. Indeed, we can calculate the net flux of a plane parallel beam incident on a surface in the direction (m_0, j_0), using the radiance of a plane parallel beam given by Chapter1, equation (7), as

$$p\,F = \int_0^{2p}\int_{-1}^{1} \mathsf{F}\, d(m - m_0)\, d(j - j_0)\, m \, dm \, dj \ , \tag{7}$$

which results in

$$p\,F = \mathsf{F} m_0 , \tag{8}$$

this result being the irradiance E of the surface, as we knew it should be!

7. *Surfaces - Hemispherical Albedo*

Also known as the *directional hemispherical reflectance*, the hemispherical albedo r refers to a point on a reflecting surface, and is defined as the ratio of the exitance M to the irradiance E, so that

$$r(m_0, j_0) = \frac{M}{E(m_0, j_0)} , \tag{9}$$

and in terms of the BRDF, we have

$$r(m_0, j_0) = \int_0^{2p}\int_0^{1} f_r(m, j; m_0, j_0)\, m \, dm \, dj \ . \tag{10}$$

Unlike the single scattering albedo, r and the other albedos that we will encounter do not necessarily have in principle a maximum possible value of unity. (See *A Brief History of the Lommel-Seeliger Law*). The scattering properties of the surfaces that we have studied so far are summarised in Table I, from which, for the Lommel-Seeliger law, it can be seen that the maximum possible value for r is ½ and 0.125 for the normal albedo.

Table I. Properties of Surfaces		
	Lambertian	Lommel-Seeliger
f_r	v_0/p	$\frac{v_0}{4p}\frac{1}{m_0+m}$
r	v_0	$\frac{v_0}{2}[1-m_0\ln(1+1/m_0)]$
p_n	v_0	$\frac{v_0}{8}$

8. *Intensity*

The *intensity* of a source in a given direction is the power radiated per unit solid angle about the specified direction, *i.e.*

$$I = dP/dw. \tag{11}$$

The SI units are watts per steradian ($W\ sr^{-1}$). The intensity of an element of area is the product of its radiance and its *projected* area., and the intensity of a surface in a given direction is the integral of the radiance over the projected area of the surface. As an example, the shape of an irregularly shaped asteroid can be approximated as a set of connected planar triangular facets; two such facets are shown in figure 1.

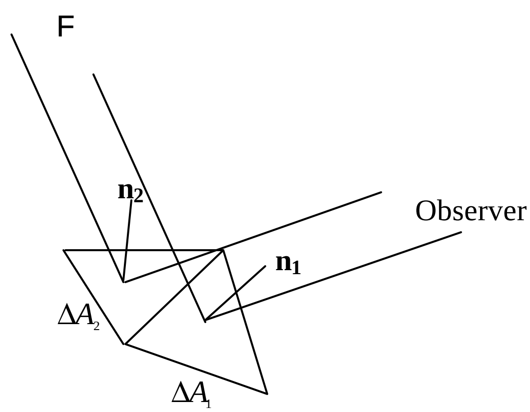

Fig.1.

For each facet of area ΔA_k the contribution to the intensity in the direction of the observer is

$$\Delta I_k = L_{obs,\,k}\,\Delta A_k \cos q_k \tag{12}$$

where q_k is the angle between the surface normal vector $\mathbf{n}_k$ and the (fixed) direction to the observer. The total intensity (in the direction toward the observer) of the asteroid is then

$$I = \sum_{k=1}^{N} \Delta A_k \tag{13}$$

where N is the total number of facets both irradiated and visible to the observer.

Of particular interest is the intensity of a *sphere* as a function of solar phase angle a. If we consider a sphere of radius a centred in an Oxyz frame with directional spherical coordinates (Θ, Φ) irradiated from the x-direction with flux density F, an element of surface area is $a^2 \sin\Theta\, d\Theta\, d\Phi$ and its projected area in the direction m is $m\, a^2 \sin\Theta\, d\Theta\, d\Phi$.

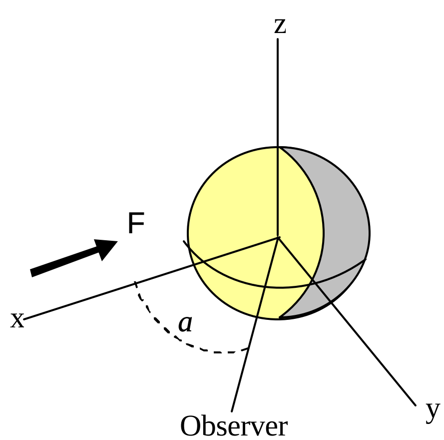

Fig. 2.

The irradiance of a point (a, Θ, Φ) of a point on the surface is $E = \mathsf{F} m_0$, where it may be shown that

$$m_0 = \sin\Theta \cos\Phi, \tag{14}$$

and for an observer at phase angle a in the xy-plane

$$m = \sin\Theta \cos(a - \Phi), \tag{15}$$

in which case the intensity as a function of phase angle is given by

$$I(a) = a^2 \mathsf{F} \int_{a-p/2}^{p/2} \int_0^p f_r m_0 m \sin\Theta\, d\Theta\, d\Phi\,. \tag{16}$$

We will return to this equation, with more detail, in **§9**.

9. Spheres - Bond Albedo, Phase Integral & Geometrical Albedo

Originally defined for a sphere, the Bond albedo is defined as the ratio of the total power P_r scattered by the sphere to the total power P_i intercepted by it.

If we let the intensity of the sphere as a function of solar phase angle a be $I(a)$ watts per steradian, then the total scattered flux may be obtained by multiplying by $2p \sin a \, da$ and integrating over a from 0 to p

$$P_r = 2p \int_0^p I(a) \sin a \, da \ , \tag{17}$$

which can be expressed in terms of the normalised *phase law* $y(a) = I(a)/I(0)$

$$P_r = 2p \, I(0) \int_0^p y(a) \sin a \, da \ . \tag{18}$$

For a sphere of radius a, the intercepted flux is simply $P_i = p \, a^2 F$, so that the Bond albedo may be expressed as

$$A = \frac{I(0)}{a^2 F} \times 2 \int_0^p y(a) \sin a \, da = p \times q \tag{19}$$

in which it may be seen as the product of two factors, the second of which,

$$q = 2 \int_0^p y(a) \sin a \, da \ , \tag{20}$$

is called the *phase integral*, which depends only on the directional reflecting properties of the planet. The first factor

$$p = \frac{I(0)}{a^2 F} \tag{21}$$

depends only on the geometrical and photometric properties of the planet when observed at full phase. The quantity p is itself a (kind of) albedo since $a^2 F$ can be seen as the intensity, scattered back towards the source, of a normally irradiated *lossless* $(v_0 = 1)$ Lambertian disc of the same radius as the planet. The factor p is called the *geometrical albedo*. [When albedo is used without qualification in the context of the photometry of asteroids it (usually) means geometrical albedo, in particular that observed in the Johnson V-band, p_V, the *visual geometrical albedo*].

For the reflectance rules we have considered so far, *i.e.* Lambert's law and the Lommel-Seeliger law, analytical expressions for A, p and q are readily found, as summarised in Table II.

Table II Properties of Spheres		
	Lambertian	Lommel-Seeliger
q	$\frac{3}{2}$	$\frac{16}{3}(1-\ln 2)$
p	$2v_0/3$	$v_0/8$
A	v_0	$\frac{2}{3}v_0(1-\ln 2)$

More complicated reflectance laws, in particular those which address the problem of the *opposition effect* for atmosphereless bodies do not readily lend themselves to analytical solutions. In general, such laws exhibit a BRDF which depends on phase angle a and a possible set of *reflectance parameters*, symbolised by the ellipsis, so that the BRDF would be generally expressed in the form

$$f_r = f_r(m_0, m, a;\ldots), \tag{22}$$

where the dependence on j and j_0 has been replaced by a, the angle between the incident and scattered radiation, *i.e.* a does not always mean *solar* phase angle.

10. A, p and q for General Reflectance Rules $f_r(m_0, m, a;\ldots)$.

Again, consider a sphere of radius a centred in an O*xyz* frame with corresponding directional spherical coordinates (Θ, Φ), and let the sphere be irradiated with flux density F from the z-direction. For the geometrical albedo the phase angle a is zero and the incident and reflected radiation are given by $m_0 = m = \cos\Theta$, so that

$$p = \int_0^{2p}\int_0^{p/2} f_r(\cos\Theta, \cos\Theta, 0;\ldots)\cos^2\Theta\sin\Theta d\Theta d\Phi, \tag{23}$$

resulting in

$$p = 2p\int_0^1 f_r(m, m, 0;\ldots)m^2 dm. \tag{24}$$

Using the same geometry for the Bond albedo, for each point on the irradiated hemisphere we have $m_0 = \cos\Theta$, so that the directional hemispherical reflectance is

$$r(m_0) = \int_0^{2p}\int_0^1 f_r(m_0, m, a;\ldots)m\,dm\,d f_r, \tag{25}$$

where the phase angle is that between the incident and reflected radiation at each stage of the integral,

$$\cos a = m_0 m + \sqrt{(1 - m_0^2)(1 - m^2)} \cos f_r \ , \qquad (26)$$

where f_r is the azimuth of the reflected radiation. The Bond albedo is then given by

$$A = \frac{1}{p} \int_0^{2p} \int_0^{p/2} r(\cos\Theta) \cos\Theta \sin\Theta d\Theta d\Phi \ , \qquad (27)$$

which reduces to

$$A = 2\int_0^1 r(m) m\, dm \ . \qquad (28)$$

The phase integral, equation (20), may be computed from equations (14), (15) and (16), the factor $a^2 F$ disappearing in the process, so that we may write, for the purposes of computation, equation (16) as

$$I(a) = \int_{a - p/2}^{p/2} \int_0^p f_r(m_0, m, a; \ldots) m_0 m \sin\Theta d\Theta d\Phi \ , \qquad (29)$$

where it can be seen that for Φ the range of integration is from $a - p/2$, the limb, to $p/2$, the terminator.

In these equations it can be seen that the geometrical albedo is just a single integral and thus may be quickly and accurately integrated numerically with just about any method. The Bond albedo and the phase integral are, however, triple integrals, so that a method which combines the advantages of speed and accuracy is required; for this reason *Gaussian Quadrature* is the chosen method. In the following section we present this method in algorithmic form and discuss its application to the integrals at hand.

For the theory and examples of Gaussian Quadrature, its performance compared to other methods of integration as well as tabulations of the *roots* and *coefficients* needed, the reader is referred to `astrowww.phys.uvic.ca/~tatum/` *`Celestial Mechanics, Chap. 1`*.

11. Gaussian Triple Integral Algorithm.

To approximate the integral

$$\mathrm{I} = \int_a^b \int_c^d \int_e^f \mathrm{F}(x,y,z)\,dz\,dy\,dx$$

where it is assumed that the roots R and coefficients C are stored in two-dimensional arrays.

```
BEGIN
      h1 = (b – a)/2
      h2 = (b + a)/2
      I = 0

      FOR i = 1, 2,..., m DO
            Ix = 0
            x = h1*R[m][i] + h2
            k1 = (d – c)/2
            k2 = (d + c)/2

            FOR j = 1, 2,..., n DO
                  Iy = 0
                  y = k1*R[n][i] + k2
                  l1 = (f – e)/2
                  l2 = (f + e)/2

                  FOR k = 1, 2,..., p DO
                        z = l1*R[p][k] + l2
                        Iy = Iy + C[p][k]*F(x, y, z)
                  END FOR { k-loop }

                  Ix = Ix + C[n][j]*l1*Iy

            END FOR { j-loop }

            I = I + C[m][i]*k1*Ix

      END FOR { i-loop }

      I = h1*I
      PRINT I
END.
```

This algorithm may be generalised further by allowing limits e and f to be functions e(x,y) and f(x,y) and c and d to be functions c(x) and d(x). For our purposes the limits of integration are fixed values.

Applying this algorithm to equation (28) for the Bond albedo and identifying m with x, we see that

$$\frac{A}{2} = \int_0^1 \int_0^{2p} \int_0^1 \mathrm{x}\, f_r(\mathrm{x}, m, a\,;\ldots)\, m\, dm\, df\, d\mathrm{x}, \qquad (30)$$

and by further identifying $\mathtt{z}$ with m and $\mathtt{y}$ with f

$$\mathtt{F}(\mathtt{x},\mathtt{y},\mathtt{z}) = 2\mathtt{x}\mathtt{z}\, f_r(\mathtt{x},\mathtt{z},a;\ldots), \tag{31}$$

where a is itself a function of $\mathtt{x}$, $\mathtt{y}$ and $\mathtt{z}$ [*cf*. equation (26)]

$$a = \cos^{-1}[\mathtt{x}\mathtt{z} + \sqrt{(1-\mathtt{x}^2)(1-\mathtt{z}^2)}\cos\mathtt{y}]\ . \tag{32}$$

For the phase integral, there is no need to invoke the likes of equation (32) since the intensity $I(a)$ is explicitly expressed in terms of a and one stage of the integration is with respect to a. The parameters, … , are, of course, not variables since they retain their values for the duration of the integration.

When applying these integrals it is strongly suggested that A, p and q each be calculated independently in order to verify that the relationship $A = p\,q$ holds. Taking shortcuts may bury insidious bugs, some possibly as simple as a typo., inside a program and result in at least two undetected erroneous results.

12. Summary of Photometric Quantities

With this chapter we have completed the description of the basic photometric quantities used in planetary photometry (although we have yet to embrace *magnitude*). These are summarised in Table III, in which those names in the first column correspond to those in *standard usage*, the exception being flux density $\mathtt{F}$. The third and fourth columns correspond to standard symbols and units. In the second column may be found some names commonly, and not so commonly, used in astronomical literature.

Table III. Photometric Quantities

Name	Synonyms	Symbol	SI Units
Radiant Flux	Radiant Power	P, Φ	W
Radiant Flux Density	Collimated Intensity	$\mathtt{F}$	$\mathrm{W \cdot m^{-2}}$
Irradiance	Insolation	E	$\mathrm{W \cdot m^{-2}}$
Exitance	Emittance	M	$\mathrm{W \cdot m^{-2}}$
Radiance	Surface Brightness Specific Intensity Intensity	L	$\mathrm{W \cdot m^{-2} sr^{-1}}$
Intensity	Integrated Brightness	I	$\mathrm{W \cdot sr^{-1}}$

The author has seen the term “collimated intensity” used by only one author (Hapke) when referring to a plane parallel beam, and he finds it a more meaningful term than “flux density”, so much so that in standard usage the term “collimated radiance” would make a splendid alternative.

The symbols have been used in their most general sense, without any subscripting or other embellishments so that *e.g.* L could mean L_l, the radiance in the wavelength interval $[\, l\, ,\ l + d\, l\ ]$, or L_V, the "visual radiance" in the Johnson V-band or indeed it could mean the radiance integrated over all wavelengths, the "bolometric radiance".

Reference Notes.

Much of the content of this chapter is an adaptation from, and an extension to, the *Theory of Planetary Photometry* by

1. Lester. P. L., McCall, M. L. & Tatum, J. B., 1979, *J. Roy. Astron. Soc. Can*., **73**, 233.

Further definitions, and interesting insights into the photometric quantities and standard usage may be found in the above reference, as well as in

2. astrowww.phys.uvic.ca/~tatum/ *Stellar Atmospheres, Chap. 1.*

Sections 9 and 10 are based on an article by the author

3. Fairbairn, M. B., 2004, *J. Roy. Astron. Soc. Can.*, **98**, 149

in which a numerical example may be found in the appendix.