1.

$$\ln\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} = e^x e^y$$

$$\underline{\underline{e^{-y} = C - e^x}}$$

2.

$$\frac{dy}{dx} + \frac{xy}{x^2 - 3xy + y^2} = 0$$

Let $y = ux$ $\quad \frac{dy}{dx} = x\frac{du}{dx} + u$

$$x\frac{du}{dx} + u + \frac{u}{1 - 3u + u^2} = 0$$

$$x\frac{du}{dx} + \frac{u^3 - 3u^2 + 2u}{u^2 - 3u + 1} = 0$$

$$\int \frac{u^2 - 3u + 1}{u^3 - 3u^2 + 2u}du + \int \frac{dx}{x} = 0$$

You have succeeded in separating the variables. It is now just a matter of integrating the two sides. From this point you are on your own.

$$\underline{\underline{\frac{(x - y)^2 y}{2x - y} = C}}$$

3.

$$(2\sqrt{xy} - x)\frac{dy}{dx} + y = 0$$

$$\int \frac{2u^{1/2} - 1}{2u^{3/2}}du + \int \frac{dx}{x} = 0$$

$$\frac{1}{u^{1/2}} + \ln u + \ln x = A$$

$$\underline{\underline{\sqrt{\frac{x}{y}} + \ln y = A}}$$

4. $xy + (y^4 - 3x^2)\dfrac{dy}{dx} = 0$

This one is not homogeneous, and it will require a bit of inventiveness. Multiply throughout b y:

Multiply by y:

$$xy^2 + (y^4 - 3x^2)y\frac{dy}{dx} = 0$$

Now let $z = y^2$, so that $y\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{dz}{dx}$

The equation then becomes

$$2xz + (z^2 - 3x^2)\frac{dz}{dx} = 0$$

$$\frac{dz}{dx} + \frac{2xz}{z^2 - 3x^2} = 0$$

The equation is now homogeneous and can be solved in the usual way by letting $z = ux$ so that $\dfrac{dz}{dx} = x\dfrac{du}{dx} + u$, which results in separation of the variables:

$$\frac{u^2 - 3}{u^3 - u} + \frac{dx}{x} = 0$$

This is now easily integrable to give

$$\frac{u^3 x}{1 - u^2} = C$$

Then recall that $u = \dfrac{z}{x} = \dfrac{y^2}{x}$ to obtain finally

$$\underline{\underline{y^6 = C(x^2 - y^4)}}$$

5. $y^3\dfrac{dy}{dx} + x + y^2 = 0$

This can be made homogenous as follows:

Let $w = x + y^2$ $\qquad \dfrac{dw}{dx} = 1 + 2y\dfrac{dy}{dx}$

Then $2y^3\dfrac{dy}{dx} = y^2 \times 2y\dfrac{dy}{dx} = (w - x)\left(\dfrac{dw}{dx} - 1\right)$

The equation to be solved becomes

$$(w - x)\left(\frac{dw}{dx} - 1\right) + 2w = 0.$$

Thus is now homogenous, and can be made separable by letting $w = ux$, whence

$$\frac{u-1}{u^2+1}\int du + \int \frac{dx}{x} = 0$$

$$\ln(u^2+1) - 2\tan^{-1} u + 2\ln x = C$$

$$\underline{\underline{\ln(y^4 + 2xy^2 + 2x^2) - 2\tan^{-1}\left(\frac{y^2 + x}{x}\right) = C}}$$

6. This one is straightforward:

$$\frac{dy}{dx} = \frac{2x + 4y + 8}{x - y - 2}$$

Let $x = X + h$ and $y = Y + k$.

Then $\dfrac{dY}{dX} = \dfrac{2X + 4Y + 2h + 4k + 8}{X - Y + h - k - 2}$.

This is homogeneous in X and Y if $h = 0$, $k = 2$.

Then $\dfrac{dY}{dX} = \dfrac{2X + 4Y}{X - Y}$.

Now let $Y = UX$, $\dfrac{dY}{dX} = X\dfrac{dU}{dX} + U$.

Then $X\dfrac{dU}{dX} + U = \dfrac{2 + 4U}{1 - U}$

$$\int \frac{U-1}{U^2 + 3U + 2} dU + \int dX = 0$$

It is then routine to obtain finally

$$\underline{\underline{(2x + y + 2)^3 = C(x + y + 2)^2}}$$

7.

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{x+1}{x}$$

The integrating factor is $e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$

[See the text to undstand why we need not add "+C" to 3ln$x$.

Multiply the original equation by x^3:

$$x^3\frac{dy}{dx} + 3x^2 y = x^3 + x$$

That is, $\dfrac{d}{dx}(yx^3) = x^3 + x$

$\underline{\underline{12yx^3 = 3x^4 + 4x^3 + C}}$

8. $\dfrac{dy}{dx} + y = xy^3$

This is of Bernoulli form. Divide throughout by y^3 and proceed as shown:

$-\dfrac{1}{y^3}\dfrac{dy}{dx} - \dfrac{1}{y^2} = -x$

Let $u = \dfrac{1}{y^2}$, $\quad \dfrac{du}{dx} = -\dfrac{2}{y^3}\dfrac{dy}{dx}$

$\dfrac{du}{dx} - 2u = -2x$

IF $= e^{-2x}$

$ue^{-2x} = -2\int xe^{-2x}dx = \tfrac{1}{2}e^{-2x}(2x+1) + C$

$u = \tfrac{1}{2}(2x+1) + Ce^{2x}$

$\underline{\underline{\dfrac{1}{y^2} = x + \tfrac{1}{2} + Ce^{2x}}}$

9. This can be solved in the same way as example **6** - probably the easiest way. However, it will be seen that the equation is exact, so let's for practice solve it that way. That is to say we suppose that the solution is of the form $H(x, y) = 0$, where

$$H(x, y) = \int(x - y + 2)dx = \tfrac{1}{2}x^2 - (y - 2)x + \phi(y).$$

Here $\phi(y)$ is to be found from

$$\frac{d\phi}{dy} = -x - y + 2 - \frac{\partial}{\partial y}(\tfrac{1}{2}x^2 - xy + 2x) = -x - y + 2 + x = -y + 2.$$

That is, $\phi = -\tfrac{1}{2}y^2 + 2y - \tfrac{1}{2}C.$

The solution is therefore $\tfrac{1}{2}x^2 - (y - 2)x - \tfrac{1}{2}y^2 + 2y - \tfrac{1}{2}C = 0,$

or $\underline{\underline{x^2 - 2x(y - 2) - y^2 + 4y = C}}$

We could also write this as
$x^2 - 2x(y-2) - (y-2)^2 = D.$