Chapter 15
Differentiation of Functions of a Complex Variable

Part I

15.1 *Introduction*

If you were given a function $f(x)$ of a real variable and asked to differentiate it, you would know what you were being asked to do and would know how to do it. Likewise, if you were asked to integrate it from a to b you would know what you were being asked to do and (except perhaps in a few unusually difficult cases) you would know how to do it.

On the other hand, if you were told that $f(z)$ is a function of a complex variable and you were asked to differentiate it, or perhaps to integrate it from $a + ib$ to $c + id$, and if you are new to this topic, it is possible that, not only do you not know how to do it, but you are not entirely certain exactly what is being asked of you.

15.2 *Differentiation of* $f(z)$. *Numerical discussion*

Imagine a complex number $z = x + iy$ in the z-plane. Let $f(z) = w = u + iv$.

Now suppose that z were to change by δz in the z-plane. There would be a resulting change δw in the w-plane. We would like somehow to define the derivative $\frac{dw}{dz}$ as the limit of $\frac{\delta w}{\delta z}$ as $\delta z \to 0$.

The difficulty is that both $\delta z = \delta x + i\delta y$ and $\delta w = \delta u + i\delta v$ are complex numbers with both magnitude (modulus) and direction (argument). Does the resulting δw depend on the direction of δz? Or do we get the same answer for δw regardless of the direction (argument) of z?

Another question will probably arise in the mind of a newcomer. If $w = f(z)$, do the ordinary rules of differentiation, with which we are familiar, apply? For example, if $w = \sin z$, is the derivative, whatever that means, just $\cos z$ as usual?

It will take a little while to answer these questions. In the meantime let us try an empirical example. I'll write a computer program as I go to do the numerical work. Let us suppose that $w = \sin z$, and we'll start with a complex number

$$z = 0.6 + 0.9i.$$

Then $w = u + iv$ with (see previous chapter) $u = \sin 0.6 \cosh 0.9$ and $v = \cos 0.6 \sinh 0.9$.
Hence

$$w = 0.8091814413 + 0.8472208130i$$

Now let

$$\delta z = 0.0002 - 0.0003i.$$

Then $u + \delta u$ = sin0.6002 cosh0.8997 and v = cos0.6002 sinh0.8997.

Hence

$$w + \delta w = 8092440816 + 0.8467501266i$$

so that

$$\delta w = 0.0000626404 - 0.0004706864i$$

From this we find $$\frac{\delta w}{\delta z} = 1.1825692533 - 0.5795782690i$$

[For the details of this numerical cakulation, see Appendix at the end of Chapter 15 Part I.]

Now let us try, starting from the original $z = 0.6 + 0.9i$, instead,

$$\delta z = 0.0003 + 0.0005i.$$

After going through the same procedure (*i.e.* not doing any more work, but merely substituting this new z in the computer program) I obtain

$$\frac{\delta w}{\delta z} = 1.1828676637 - 0.5799443961i$$

The two derivatives are nearly equal. We would not expect them to be exactly equal unless we had made infinitesimal changes in z. As it is, these results suggest that, at least in this example, the derivative is independent of the direction (argument) of z. We cannot, however, assume that this will necessarily be the case for functions other than sin z.

The second question that we asked was: Can we obtain the derivative from the same rules that we are accustomed to when dealing with real variables? That is, is the derivative of sin z equal to cos z? :Let us see:

From the previous chapter,

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

I find:

$$\cos z = 1.1827772332 - 0.5796149430i$$

Thus apparently we *can* just use the ordinary rules of differentiation that we are used to! But read Section 15.3 before coming to any hasty general conclusions.

Appendix to Part I

$$\frac{\delta w}{\delta z} = \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y}$$

As always, whenever there is a complex number in the denominator of an expression, immediately multiply top and bottom by the complex conjugate:

$$\frac{\delta w}{\delta z} = \frac{(\delta u + i\,\delta v)(\delta x - i\,\delta y)}{(\delta x)^2 + (\delta y)^2} = \frac{\delta u\delta x + \delta v\delta y + i(\delta v\delta x - \delta u\delta y)}{(\delta x)^2 + (\delta y)^2}$$

Part II

15.3 *Differentiation of* $f(z)$. *Analytical discussion*

In Section 15.2 of Part I we found, by a purely empirical numerical example, that the function $f(z)$ seemed to have the property that the derivative dw/dz is independent of the direction (argument) of δz, and therefore at a given point z in the z-plane, there exists a uniquely defined derivative. We didn't prove this – it just turned out approximately this way in a random numerical example with the particular function sin z. It also seemed that we could differentiate sin z in the usual way that we are familiar with in the calculus of real variables, so that the derivative of sin z is cos z.

But this isn't necesarily the case for all functions. If a function has these properties that sin z seems to have, then the function is called, by various authors, an *analytic,* or a *regular* or a *homomorphic* function, or, in more common parlance, a "well-behaved" function – well-behaved in the sense that it and its derivative are finite, single-valued and continuous. Such functions are not rare; with luck most of the functions you are interested in will conveniently turn out to be analytic.

If you followed the numerical calculation in Section 15.2 by computer, it will be very easy for you to try the same calculation with other functions, to see if they are analytic or not. You just have to change the function at the beginning of the program, and the three solutions for the derivative will come out in seconds. For example, try it for the functions

$$w \;=\; z^2 , \quad 1/z , \; \sqrt{z} , \; \ln z , \quad \cos z , \quad e^z$$

I haven't tried it, but I expect they will all turn out to be probably analytic, except that $1/z$ obviously won't be analytic at the origin, and $\sqrt{z}$ will be potentially analytic only if, by the use of the $\sqrt{z}$ symbol we specifically mean the positive square root of z – a convention that is used by many, but not all, authors. If your computer program is working smoothly (I have occasionally known this to happen) you should have the answers for all these functions in a few minutes.

I have used the words "probably" and "potentially", because the numerical calculation does not prove the analyticity or otherwise of a function. There is, however, a test that will prove, without numerical calculation, whether or not a function is analytic. We shall derive this test as follows:

Let $w \;=\; f\,(z) \;=\; f(x+iy) \;=\; u \;+\; iv.$
If δz is <u>wholly real</u> *i.e.* $\delta z = \delta x$, what is δw ?

Answer: $$\delta w \;\simeq\; \frac{\partial w}{\partial x}\delta x \;=\; \left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right)\delta x$$

Therefore $$\frac{dw}{dx} \;=\; \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}$$

and therefore

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \qquad \text{A}$$

If δ*z* is <u>wholly imaginary</u> *i.e.* δz = *i* δ*y* and δ*y* = – *i* δ*z* what is δ*w* ?

Answer: $\delta w \simeq \frac{\partial w}{\partial y}\,\delta y = \left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right)\delta y$

Therefore $\frac{dw}{dy} = \frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}$

and therefore

$$\frac{dw}{dz} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y} \qquad \text{B}$$

If $\frac{dw}{dz}$ is to be the same whether δz is real or imaginary (and hence whatever the direction of δz)

the expressions A and B for $\frac{dw}{dz}$ must be equal in their real and imaginary parts separately. Thus the conditions that the derivative is independent of the direction of δ*z* are

$$\frac{du}{dx} = \frac{dv}{dy}$$

and

$$\frac{dv}{dx} = -\,\frac{du}{dy}$$

These are the <u>Riemann-Cauchy conditions</u>.

In Chapter 14 we listed *u* and *v* as fiunctions of *x* and *y* for each of the functions

$$w = z^2\,, \quad 1/z\,, \quad \sqrt{z}\,, \quad \ln z\,, \quad \sin z\,, \quad \cos z\,, \quad e^z$$

You can now use the Riemann-Cauchy conditions to see which of these functions are analytic, and you can see if this agrees with what you found by numerical calculation.