CHAPTER 16
THE RESTRICTED THREE-BODY PROBLEM

[An earlier version of these notes included material on the theory of the equivalent potential. Much of the material was not immediately relevant to our subject of celestial mechanics, and it has now been moved, and expanded, to my notes on Classical Mechanics (http://orca.phys.uvic.ca/~tatum/classmechs.html), as Chapter 21.]

16.1 *The Collinear Lagrangian Points*

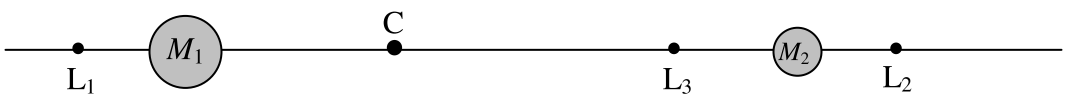


FIGURE XVI.4

We are going to consider the following problem. Two masses, M_1 and M_2 are revolving around their mutual centre of mass C in circular orbits, at a constant distance a apart. The orbital period is given by

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)} \qquad 16.1.1$$

and the angular orbital speed is given by

$$\omega^2 = \frac{G(M_1 + M_2)}{a^3}\,. \qquad 16.1.2$$

I establish the following notation.

Mass ratio: $$\frac{M_1}{M_2} = q. \qquad 16.1.3$$

Mass fraction: $$\frac{M_1}{M_1 + M_2} = \mu. \qquad 16.1.4$$

They are related by $$q = \frac{\mu}{1 - \mu} \qquad 16.1.5$$

and
$$\mu = \frac{q}{1+q} \cdot \qquad 16.1.6$$

We note the following distances:

$$M_1C = (1-\mu)a, \quad M_2C = \mu a. \qquad 16.1.7$$

We ask ourselves the following question: Are there any points on the line passing through the two masses where a third body of negligible mass could orbit around C with the same period as the other two masses; i.e. it would remain on the line joining the two main masses?

In fact there are three such points, and they are known as the collinear lagrangian points. (The collinear points were discussed by Euler before Lagrange, but Lagrange took the problem further and discovered an additional two points not collinear with the masses, and the five points today are generally all known as the lagrangian points. We shall discuss the additional points in section 16.2.) I have marked the three points in figure XVI.4 with the letters L_1, L_2 and L_3. There are evidently 3! = 6 ways in which I could choose the subscripts. Often today, the inner lagrangian point is labelled L_1 and the outer points are labelled L_2 and L_3. This seems to me to lack logic, and I choose to label the inner point L_3, and the outer points associated with M_1 and M_2 are then L_1 and L_2 respectively. Incidentally, I am not making any assumption about which of the two main bodies is the more massive.

Let us deal first with L_1. Let us suppose that the distance from C to L_1 is xa.

A particle of mass m at L_1 is subject (in a co-rotating reference frame) to three forces, namely the gravitational attractions from the two main bodies, and the centrifugal force acting away from C. If this body is to be in equilibrium, we must have

$$\frac{GM_1m}{[(x-1+\mu)a]^2} + \frac{GM_2m}{[(x+\mu)a]^2} = mxa\omega^2. \qquad 16.1.8$$

On making use of equations 16.1.2 and 16.1.4, we find that this equation becomes

$$\frac{\mu}{(x-1+\mu)^2} + \frac{1-\mu}{(x+\mu)^2} = x. \qquad 16.1.9$$

After manipulation, this becomes

$$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + x^5 = 0, \qquad 16.1.10$$

where $$a_0 = -1 + 3\mu - 3\mu^2, \qquad 16.1.11$$

$$a_1 = 2 - 4\mu + \mu^2 - 2\mu^3 + \mu^4, \qquad 16.1.12$$

$$a_2 = -1 + 2\mu - 6\mu^2 + 4\mu^3, \qquad 16.1.13$$

$$a_3 = 1 - 6\mu + 6\mu^2 \qquad 16.1.14$$

and $$a_4 = -2 + 4\mu. \qquad 16.1.15$$

Although equation 16.1.10 is a quintic equation, it has just one real root for positive μ. It is also worth noting, that, although I re-wrote equation 16.1.9 in the quintic form of equation 16.1.10, my experience is that it is easier to solve the original form, 16.1.9, by the Newton-Raphson process, than to set up and solve the quintic version of the equation.

The positions of L_2 and L_3 can be found by exactly similar arguments – you just have to take care with the directions and distances of the two gravitational forces.

For L_2, the coefficients are the same as for L_1, except

$$a_1 = -2 + 4\mu + \mu^2 - 2\mu^3 + \mu^4, \qquad 16.1.16$$

$$a_2 = -1 - 2\mu + 6\mu^2 - 4\mu^3 \qquad 16.1.17$$

and $$a_4 = 2 - 4\mu. \qquad 16.1.18$$

For L_3, the coefficients are

$$a_0 = 1 - 3\mu + 3\mu^2 - 2\mu^3, \qquad 16.1.19$$

$$a_1 = 2 - 4\mu + 5\mu^2 - 2\mu^3 + \mu^4, \qquad 16.1.20$$

$$a_2 = 1 - 4\mu + 6\mu^2 - 4\mu^3, \qquad 16.1.21$$

$$a_3 = 1 - 6\mu + 6\mu^2 \qquad 16.1.22$$

and $$a_4 = 2 - 4\mu. \qquad 16.1.23$$

(Reminder: When computing any of these polynomials, write them in terms of nested parentheses. See Chapter 1, Section 1.5.)

It is also of interest to see the equivalent potential (gravitational plus centrifugal). The expression for gravitational potential energy is, as usual, $-GMm/r$, where r is the distance from the mass M. The expression for the centrifugal potential energy is $-\frac{1}{2}m\omega^2r^2$, where r is the distance from the centre of mass. The negative of the derivative of this expression is $m\omega^2r$, which is the usual expression for the centrifugal force. When we apply these principles to the system of two masses under consideration, we obtain the following expression for the equivalent potential (which, in this section, I'll just call V rather than V').

$$V = -\frac{GM_1}{|x+1-\mu|a} - \frac{GM_2}{|x-\mu|a} - \tfrac{1}{2}x^2a^2\omega^2. \qquad 16.1.24$$

On making use of equations 16.1.2 and 16.1.4, we find that this equation becomes

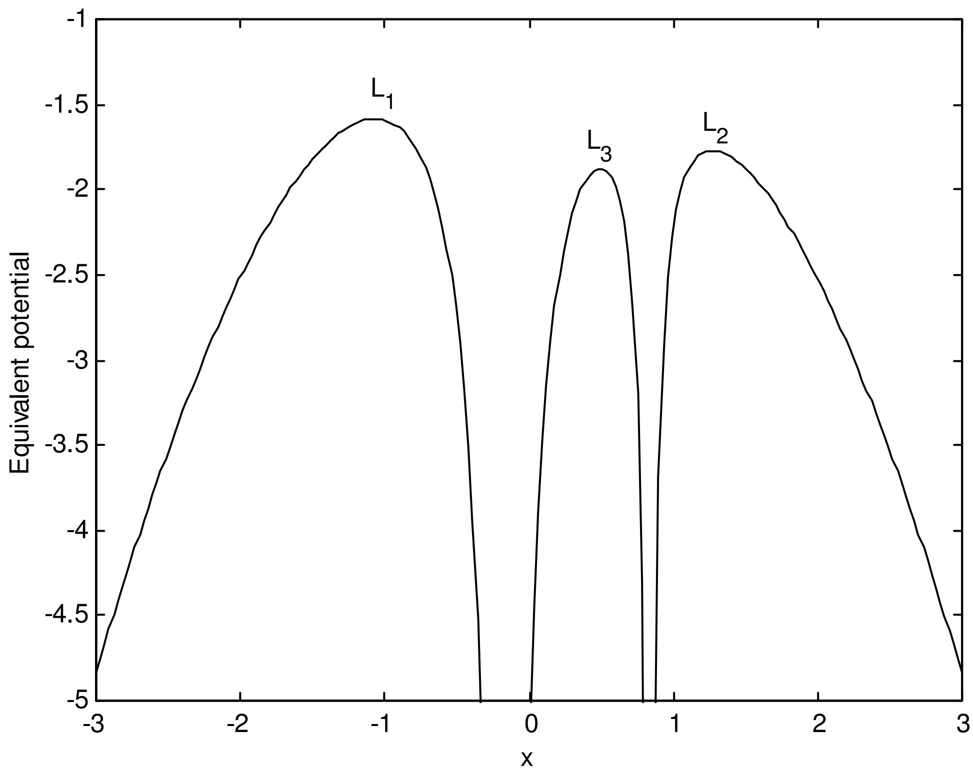
$$W = -\frac{\mu}{|x+1-\mu|} - \frac{1-\mu}{|x-\mu|} - \frac{x^2}{2}, \qquad 16.1.25$$

where $$W = V \div \left(\frac{G(M_1+M_2)}{a}\right). \qquad 16.1.26$$

Setting the derivatives of this expression to zero gives, of course, the positions of the lagrangian points, for these are equilibrium points where the derivative of the potential is zero. Figure XVI.5 shows the potential for a mass ratio $q = 5$. Note that, in the line joining the two masses, the equivalent potential at the lagrangian points is a maximum, and therefore these points, while equilibrium points, are unstable. We shall see in section 16.6 that the points are actually saddle points. While several spacecraft are in orbit or are planned to be in orbit around the collinear lagrangian points (e.g. SOHO at the interior lagrangian point, and MAP at L_2), continued small expenditure of fuel is presumably needed to keep them there.

It will be of interest to see how the positions of the lagrangian points vary with mass fraction. Indeed mass can be transferred from one member of a binary star system to the other during the evolution of a binary star system. We shall discuss a little later how this can happen. For the time being, without worrying about the exact mechanism, we'll just vary the mass fraction and see how the positions of the lagrangian points vary as we do so. However, if mass is transferred from one member of a binary star system to the other,

FIGURE XVI.5



and if there are no external torques on the system, the angular momentum L of the system will be conserved, and, to ensure this, the separation a of the two stars changes with mass fraction.

Exercise. Show that, for a given orbital angular momentum L of the system, the separation a of the components varies with mass fraction according to

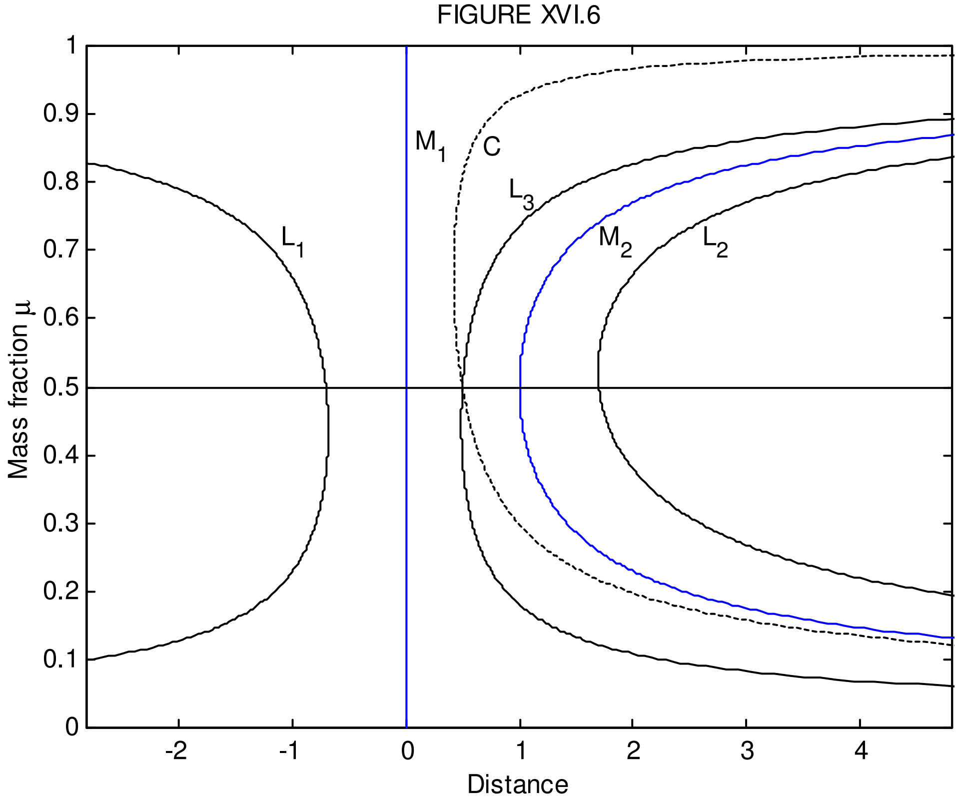
$$a = \frac{L^2}{GM^3\mu^2(1-\mu)^2}\,. \qquad 16.1.27$$

Here $M = M_1 + M_2$ is the total mass of the system. In figure XVI.6 I have used this equation, plus equations 16.1.10 and 16.1.7, to compute the distances of M_2, C, and the three lagrangian points from M_1 as a function of mass fraction. The unit of distance in figure XVI.6 is $16L^2/(GM^3)$, which is the separation of the two masses when the two masses are equal. Each of these distances has a minimum value for a particular mass fraction. These minimum distances, and the mass fractions for which they occur, are as follows:

	Least value	Mass fraction
M_1C	0.421875*	$0.66666\dot{6}$
M_1L_2	1.690392	0.524579
M_1M_2	1.000000	0.500000
M_1L_3	0.489038	0.446273
M_1L_1	0.677756	0.436062

*0.421875 = 27/64 exactly

FIGURE XVI.6



How can mass transfer actually occur in a binary star system? Well, stars are not points – they are large spherical bodies. When the hydrogen is exhausted in the core by thermonuclear reactions, a star expands hugely ("leaves the main sequence") and when it expands so much that the outer layers of its atmosphere reach the inner lagrangian point, matter from the large star spills over into the other star. The more massive of the two stars in a binary system generally evolves faster; it is the first to leave the main sequence and to expand so that its atmosphere reaches the inner lagrangian points. One can imagine the more massive star gradually filling up its potential well of figure XVI.5, until it overflows and drips over the potential hill of the inner point, and then falls into the potential well of its companion.

One way of interpreting figure XVI.6 is to imagine that M_1 starts with a large mass fraction close to 1, and therefore near the top of figure XVI.6. Now imagine that this star loses mass to its companion, so that the mass fraction decreases. We start moving down the M_1 line of figure XVI.6. We see the inner point L_3 coming closer and closer. If the surface of the star meets L_3 while L_3 is still approaching (i.e. if the mass fraction is still greater than 0.446273), then further mass transfer will make L_3 approach ever faster, and mass transfer will therefore be rapid. When the mass fraction is less than 0.5, the star that was originally the more massive star is by now less massive than its companion. When the mass fraction has been reduced below 0.446273, further mass transfer will push L_3 *away*, and therefore further mass transfer will be slow.

In these calculations I assumed that the stars can be treated gravitationally as if they are point sources – and so they can be, however large they are, as long as they are spherically symmetric. By the onset of mass transfer, the mass-losing star is quite distorted and is far from spherical. However, this distortion affects mostly the outer atmosphere of the star, and, provided that the greater bulk of the star is contained within a roughly spherically-symmetric volume, the point source approximation should continue to be good. The other assumption I made was that orbital angular momentum is conserved. There are two reasons why this might not be so – but for both of them there is likely to be very little loss of orbital angular momentum. One possibility is that mass might be lost from the system – through one or other or both of the external collinear lagrangian points. But figure XVI.5 shows that the potentials of these points are appreciably higher than the internal point; therefore mass transfer takes place well before mass loss. Another reason why orbital angular momentum might be conserved is as follows. When matter from the mass-losing star is transferred through the inner point to the mass-gaining star, or flows over the inner potential hill, it does not move in a straight line directly towards the second star. This entire analysis has been referred to a co-rotating reference frame, and when matter moves from M_1 towards M_2, it is subject to a *Coriolis force* (see section 4.9 of Classical Mechanics), which sends it around M_2 in an *accretion disc*. During this process the total angular momentum of the system is conserved (provided no mass is lost from the system) but this must now be shared between the orbital angular momentum of the two stars and the angular momentum of the accretion disc. However, as long as the latter is a relatively small contribution to the total angular momentum, conservation of orbital angular momentum remains a realistic approximation.

16.2 *The Equilateral Lagrangian Points*

There is no general analytical solution in terms of simple algebraic functions for the problem of three gravitating bodies of comparable masses. Except in a few very specific cases the problem has to be solved numerically. However in the *restricted* three-body problem, we imagine that there are two bodies of comparable masses revolving around their common centre of mass C, and a third body of negligible mass moves in the field of the other two. We considered this problem partially in section 16.1, except that we restricted our interest yet further in confining our attention to the line joining to two principal masses. In this section we shall widen our attention. One question that we asked in section 16.1 was: Are there any points where a third body of negligible mass could orbit around C with the same period as the other two masses? We found three such points, the collinear lagrangian points, on the line joining the two principal masses. In this section we shall discover two additional points, the fourth and fifth lagrangian points. They are not collinear with M_1 and M_2, but are such that the three masses are at the corners of an equilateral triangle.

We shall work in a co-rotating reference frame in which there are two deep hyperbolic potential wells of the form $-GM_1/r_1$ and $-GM_2/r_2$ from the gravitational field of the two principal masses sunk into the nose-up paraboloidal potential of the form $-\frac{1}{2}\rho^2\omega^2$, whose negative derivative is the centrifugal force per unit mass. Here ρ is the usual cylindrical coordinate, and $\omega^2 = G(M_1 + M_2)/a^3$.

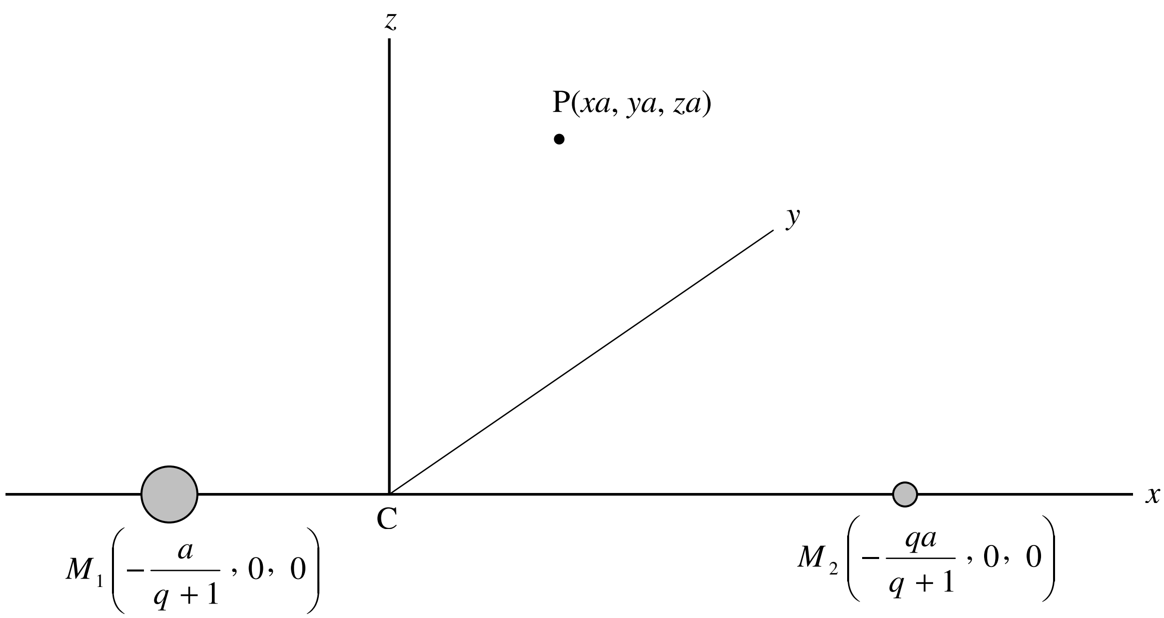


FIGURE XVI.7

In figure XVI.7 we see a coordinate system which is rotating about the z-axis, in such a manner that the two principal masses remain on the x-axis, and the origin of coordinates is the centre of mass C. The mass ratio $M_1/M_2 = q$, so the coordinates of the two masses are as shown in the figure. The constant distance between the two masses is a. P is a point whose coordinates are (xa, ya , za), x, y and z being dimensionless. The gravitational-plus-centrifugal effective potential V at P is

$$V = -\frac{GM_1}{a\left[\left(x+\frac{1}{q+1}\right)^2 + y^2+z^2\right]^{1/2}} - \frac{GM_2}{a\left[\left(x-\frac{q}{q+1}\right)^2 + y^2+z^2\right]^{1/2}} - \frac{G(M_1+M_2)(x^2+y^2)}{2a}. \qquad 16.2.1$$

Let $W = \frac{Va}{G(M_1+M_2)}$ (dimensionless). Then

$$W = -\frac{q}{\left[(1+x(q+1))^2 + (y^2+z^2)(q+1)^2\right]^{1/2}} - \frac{1}{\left[(q-x(q+1))^2 + (y^2+z^2)(q+1)^2\right]^{1/2}} - \frac{x^2+y^2}{2}. \qquad 16.2.2$$

I shall write this for short:

$$W = -Aq - B - \tfrac{1}{2}(x^2+y^2). \qquad 16.2.3$$

Here A and B are functions with obvious meaning from comparison with equation 16.2.2.

We are going to need the first and second derivatives, so I list them here, in which, for example, W_{xy} is short for $\partial^2 W/\partial x\partial y$.

$$W_x = -(q+1)[-q(1+x(q+1))A^3 + (q-x(q+1))B^3] - x, \qquad 16.2.4$$

$$W_y = (q+1)^2 y[qA^3 + B^3] - y, \qquad 16.2.5$$

$$W_z = (q+1)^2 z[qA^3 + B^3], \qquad 16.2.6$$

$$W_{xx} = -(q+1)^2[3q(1+x(q+1))^2A^5 - qA^3 + 3(q-x(q+1))^2B^5 - B^3] - 1, \qquad 16.2.7$$

$$W_{yy} = -(q+1)^2[3q(q+1)^2y^2A^5 - qA^3 + 3(q+1)^2y^2B^5 - B^3] - 1, \qquad 16.2.8$$

$$W_{zz} = -(q+1)^2[3q(q+1)^2z^2A^5 - qA^3 + 3(q+1)^2z^2B^5 - B^3], \qquad 16.2.9$$

$$W_{yz} = W_{zy} = -3(q+1)^4 yz(qA^5 + B^5), \qquad 16.2.10$$

$$W_{zx} = W_{xz} = -3(q+1)^3 z[q(1+x(q+1))A^5 - (q-x(q+1))B^5], \qquad 16.2.11$$

$$W_{xy} = W_{yx} = -3(q+1)^3 y[q(1+x(q+1))A^5 - (q - x(q+1))B^5] . \qquad 16.2.12$$

It is a little difficult to draw $W(x, y, z)$, but we can look at the plane $z = 0$ and there look at $W(x, y)$. Figure XVI.8 is a contour plot of the surface, for $q = 5$, plotted by *Mathematica* by Mr Max Fairbairn of Sydney, Australia. We have already seen, in figure XVI.5, a section along the x-axis.

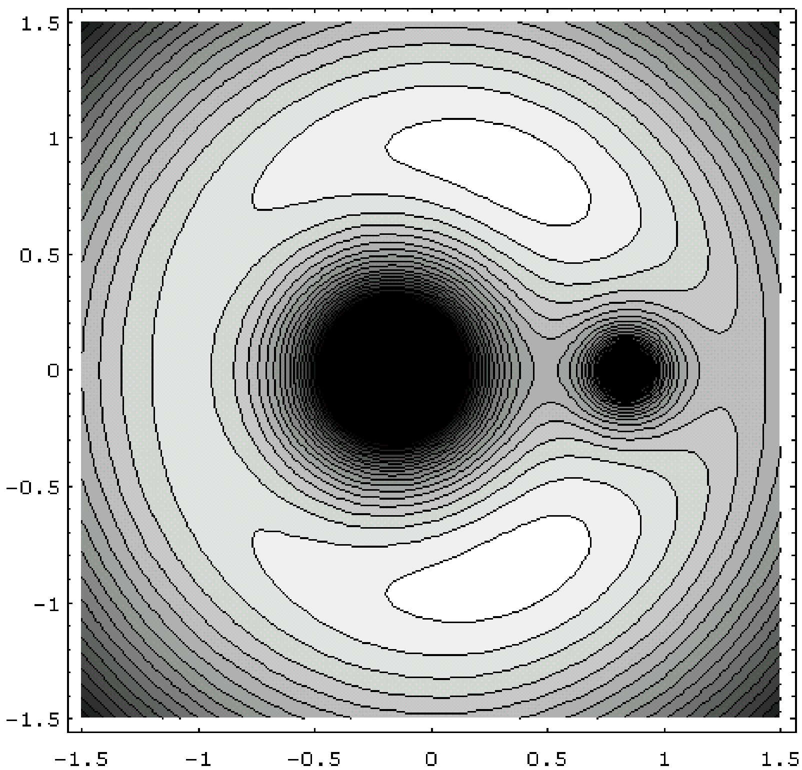


FIGURE XVI.8

Figure XVI.9a shows a three-dimensional drawing of the equivalent potential surface in the plane, also plotted by *Mathematica* by Mr Fairbairn. Figure XVI.9b is a model of the surface, seen from more or less above. This was constructed of wood by Mr David Smith of the University of Victoria, Canada, and photographed by Mr David Balam, also of the University of Victoria. The mass ratio is $q = 5$.

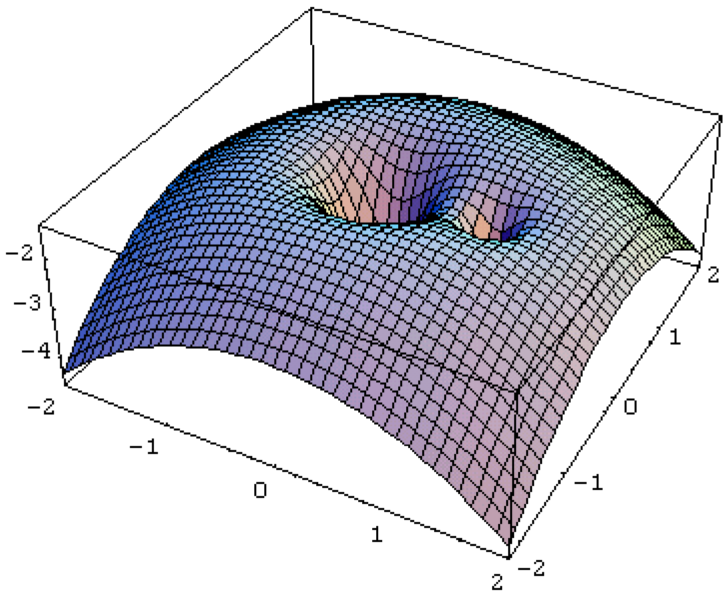
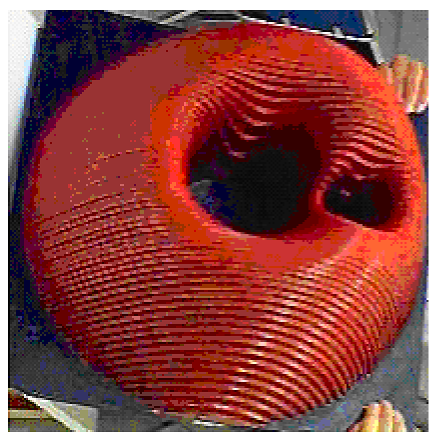


FIGURE XVI.9a

FIGURE XVI.9b



We can imagine the path taken by a small particle in the field of the two principal masses by imagining a small ball rolling or sliding on the equivalent potential surface. It might roll into one of the two deep hyperbolic potential wells representing the gravitational attraction of the two masses. Or it might roll down the sides of the big paraboloid – i.e. it might be flung outwards by the effect of centrifugal force. We must remember, however, that the surface represents the equivalent potential referred to a co-rotating frame, and that, whenever the particle moves relative to this frame, it experiences a Coriolis force at right angles to its velocity.

The three collinear lagrangian points are actually saddle points. Along the x-axis (figure XVI.5, they are maxima, but when the potential is plotted parallel to the y-axis, they are minima. However, in this section, we shall be particularly interested in the equilateral points, whose coordinates (verify this) are $x_{\rm L} = \frac{1}{2}\left(\frac{q-1}{q+1}\right)$, $y_{\rm L} = \pm\frac{\sqrt{3}}{2}\cdot$ You may verify from equations 16.2.4 and 5, (though you may need some patience to do so) that the first derivatives are zero there. Even more patience and determination would be needed to determine from the second derivatives that the equivalent potential is a maximum there – though you may prefer to look at figures XVI.8 and 9 rather than wade through that algebra. I have done the algebra and I can tell you that the first derivatives at the equilateral points are indeed zero and the second derivatives are as follows.

$$W_{xx} = -\tfrac{3}{4}, \qquad W_{yy} = -\tfrac{9}{4}, \qquad W_{zz} = +1, \qquad W_{yz} = W_{zx} = 0, \qquad W_{xy} = -\frac{3\sqrt{3}}{4}\left(\frac{q-1}{q+1}\right).$$

Because $W_{zz} = +1$, the potential at the equilateral points goes through a minimum as we cross the plane; in the plane, however, W is a maximum, and it has the value there of

$$-\frac{3q^2 + 5q + 3}{2(q+1)^2}.$$

In the matter of notation, the equilateral points are often called the fourth and fifth lagrangian points, denoted by L_4 and L_5. The question arises, then, which is L_4 and which is L_5? Most authors label the equilateral point that leads the less massive of the two principal masses by 60° L_4 and the one that trails by 60° L_5. This would be unambiguous if we were to restrict our interest, for example, to Trojan asteroids of planets in orbit around the Sun, or Calypso which leads Tethys in orbit around Saturn and Telesto which follows Tethys. There would be ambiguity, however, if the two principal bodies had equal masses, or if the two principal bodies were the members of a close binary pair of stars in which mass transfer led to the more massive star becoming the less massive one. In such special cases, we would have to be careful to make our meaning clear. For the present, however, I shall assume that the two principal bodies have unequal masses, and the equilateral point that precedes the less massive body is L_4.

In figure XVI.10 we are looking in the xy-plane. I have marked a point P, with coordinates (x, y, z); these are expressed in units of a, the constant separation of the two principal masses. The origin of coordinates is the centre of mass C, and the coordinates (in units of a) of the two masses are shown. The angular momentum vector $\boldsymbol{\omega}$ is directed along the direction of increasing z.

Now imagine a particle of mass m at P. It will be subject to a force given by the negative of the gradient of the potential energy, which is m times the potential. Thus in the x-

direction, $ma\ddot{x} = -m\frac{\partial V}{a\partial x}\cdot$ In addition to this force, however, whenever it is in motion relative to the co-rotating frame it is subject to a Coriolis force $2m\mathbf{v}\times\boldsymbol{\omega}$. Thus the x-component of the equation of motion is $ma\ddot{x} = -m\frac{\partial V}{a\partial x} + 2m\omega a\dot{y}$. Dividing through by ma we find for the equation of motion in the x-direction

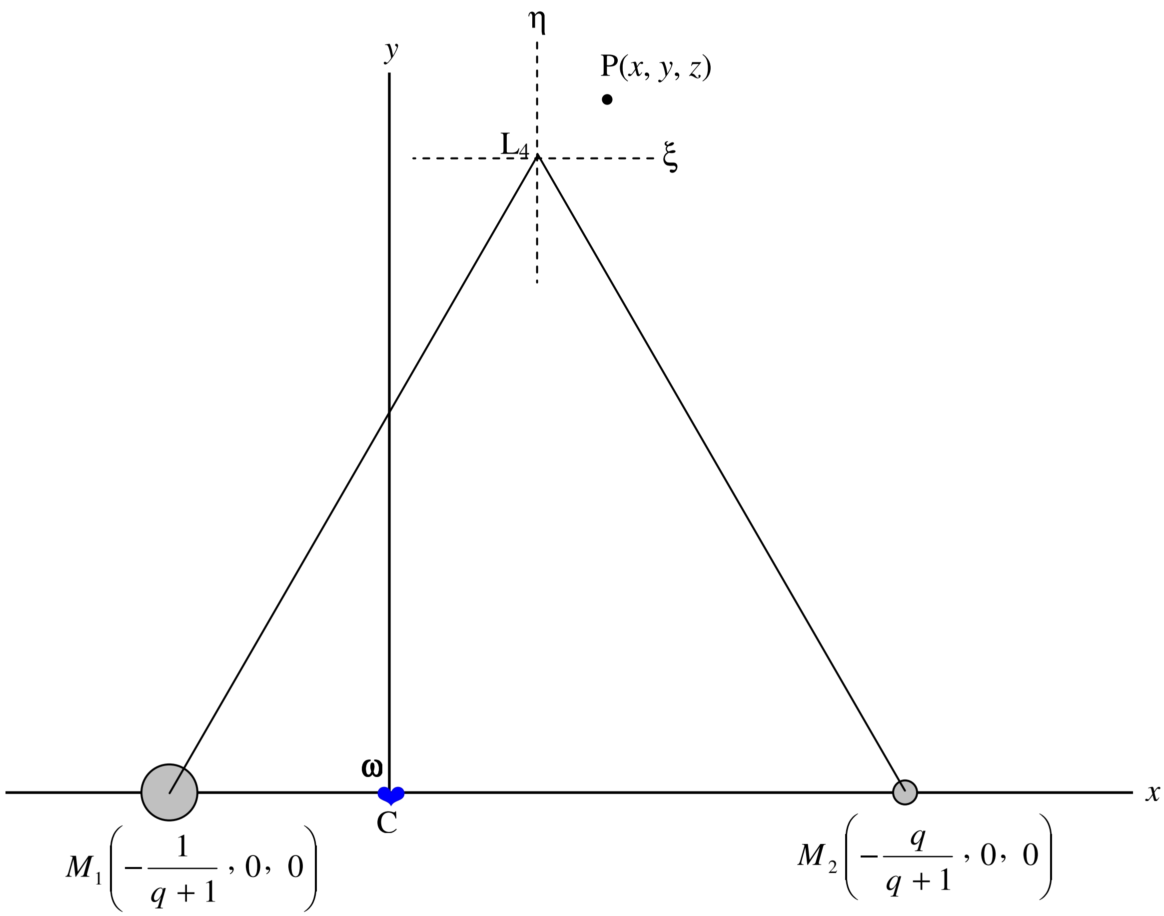


FIGURE XVI.10

$$\ddot{x} = -\frac{1}{a^2}\frac{\partial V}{\partial x} + 2\omega\dot{y}. \qquad 16.2.13$$

Similarly in the other two directions, we have

$$\ddot{y} = -\frac{1}{a^2}\frac{\partial V}{\partial y} - 2\omega\dot{x} \qquad 16.2.14$$

and $$\ddot{z} = -\frac{1}{a^2}\frac{\partial V}{\partial z}. \qquad 16.2.15$$

These, then, are the differential equations that will track the motion of a particle moving in the vicinity of the two principal orbiting masses. For large excursions, they are best solved numerically. However, solutions close to the equilateral points lend themselves to a simple analytical solution, which we shall attempt here. Let us start, then, by referring positions to coordinates with origin at an equatorial lagrangian point. The coordinates of the point P with respect to the lagrangian point are $(\xi,\ \eta,\ \zeta)$, where $\xi = x - x_L$, $\eta = y - y_L$, $\zeta = z$. Note also that $\dot{\xi} = \dot{x}$ $\ddot{\xi} = \ddot{x}$, etc. We are going to need the derivatives of the potential near to the lagrangian points, and, by Taylor's theorem (or just common sense!) these are given by

$$V_x = (V_x)_\mathrm{L} + \xi(V_{xx})_\mathrm{L} + \eta(V_{yx})_\mathrm{L} + \zeta(V_{zx})_\mathrm{L}, \qquad 16.2.16$$

$$V_y = (V_y)_\mathrm{L} + \xi(V_{xy})_\mathrm{L} + \eta(V_{yy})_\mathrm{L} + \zeta(V_{zy})_\mathrm{L}, \qquad 16.2.17$$

$$V_z = (V_z)_\mathrm{L} + \xi(V_{xz})_\mathrm{L} + \eta(V_{yz})_\mathrm{L} + \zeta(V_{zz})_\mathrm{L}. \qquad 16.2.18$$

We have already worked out the derivatives *at* the lagrangian points (the first derivatives are zero), so now we can put these expressions into equations 16.2.13,14 and 15, to obtain

$$\ddot{\xi} - 2\omega\dot{\eta} = \omega^2\left(\tfrac{3}{4}\xi + \frac{3\sqrt{3}(q-1)}{4(q+1)}\eta\right), \qquad 16.2.19$$

$$\ddot{\eta} + 2\omega\dot{\xi} = \omega^2\left(\frac{3\sqrt{3}(q-1)}{4(q+1)}\xi + \tfrac{9}{4}\eta\right) \qquad 16.2.20$$

and $$\ddot{\zeta} = -\omega^2\zeta. \qquad 16.2.21$$

The last of these equations tells us that displacements in the z-direction are periodic with period equal to the period of the two principal orbiting bodies. The motion is bounded and stable perpendicular to the plane. An orbit inclined to the plane of the orbits containing M_1 and M_2 is stable.

For ξ and η, let us seek periodic solutions of the form

$$\ddot{\xi} = n^2\xi \quad \text{and} \quad \ddot{\eta} = n^2\eta \qquad 16.2.22\text{a,b}$$

so that $$\dot{\xi} = in\xi \quad \text{and} \quad \dot{\eta} = in\eta, \qquad 16.2.23\text{a,b}$$

where n is real and therefore n^2 is positive.

Substitution of these in equations 16.2.19-21 gives

$$(n^2 + \tfrac{3}{4}\omega^2)\xi + \left(2\omega n i + \frac{3\sqrt{3}}{4}\left(\frac{q-1}{q+1}\right)\omega^2\right)\eta = 0 \qquad 16.2.24$$

and

$$\left(2\omega n i - \frac{3\sqrt{3}}{4}\left(\frac{q-1}{q+1}\right)\omega^2\right)\xi - (n^2 + \tfrac{9}{4}\omega^2)\eta = 0. \qquad 16.2.25$$

A trivial solution is $\xi = \eta = 0$; that is, the particle is stationary at the lagrangian point. While this is indeed a possible solution, it is unstable, since the potential is a maximum there. Nontrivial solutions are found by setting the determinant of the coefficients equal to zero. Thus

$$n^4 - \omega^2 n^2 + \frac{27q\omega^4}{4(q+1)^2} = 0. \qquad 16.2.26$$

This is a quadratic equation in n^2, and for real n^2 we must have $b^2 > 4ac$, or $1 > \frac{27q}{(q+1)^2}$, or $q^2 - 25q + 1 > 0$. That is, $q > 24.959\,935\,8$ or $q < 1/24.959\,935\,8 = 0.040\,064\,206$. We also require n^2 to be not only real but positive. The solutions of equation 16.2.26 are

$$2n^2 = \omega^2\left(1 \pm \sqrt{1 - 27q/(1+q)^2}\right). \qquad 16.2.27$$

For any mass ratio q that is less than 0.040 064 206 or greater than 24.959 935 8 both of these solutions are positive. Thus stable elliptical orbits (in the co-rotating frame) around the equilateral lagrangian points are possible if the mass ratio of the two principal masses is greater than about 25, but not otherwise.

If we consider the Sun-Jupiter system, for which $q = 1047.35$, we have that

$$n = 0.996\,757\omega \quad \text{or} \quad n = 0.080\,464\,5\omega.$$

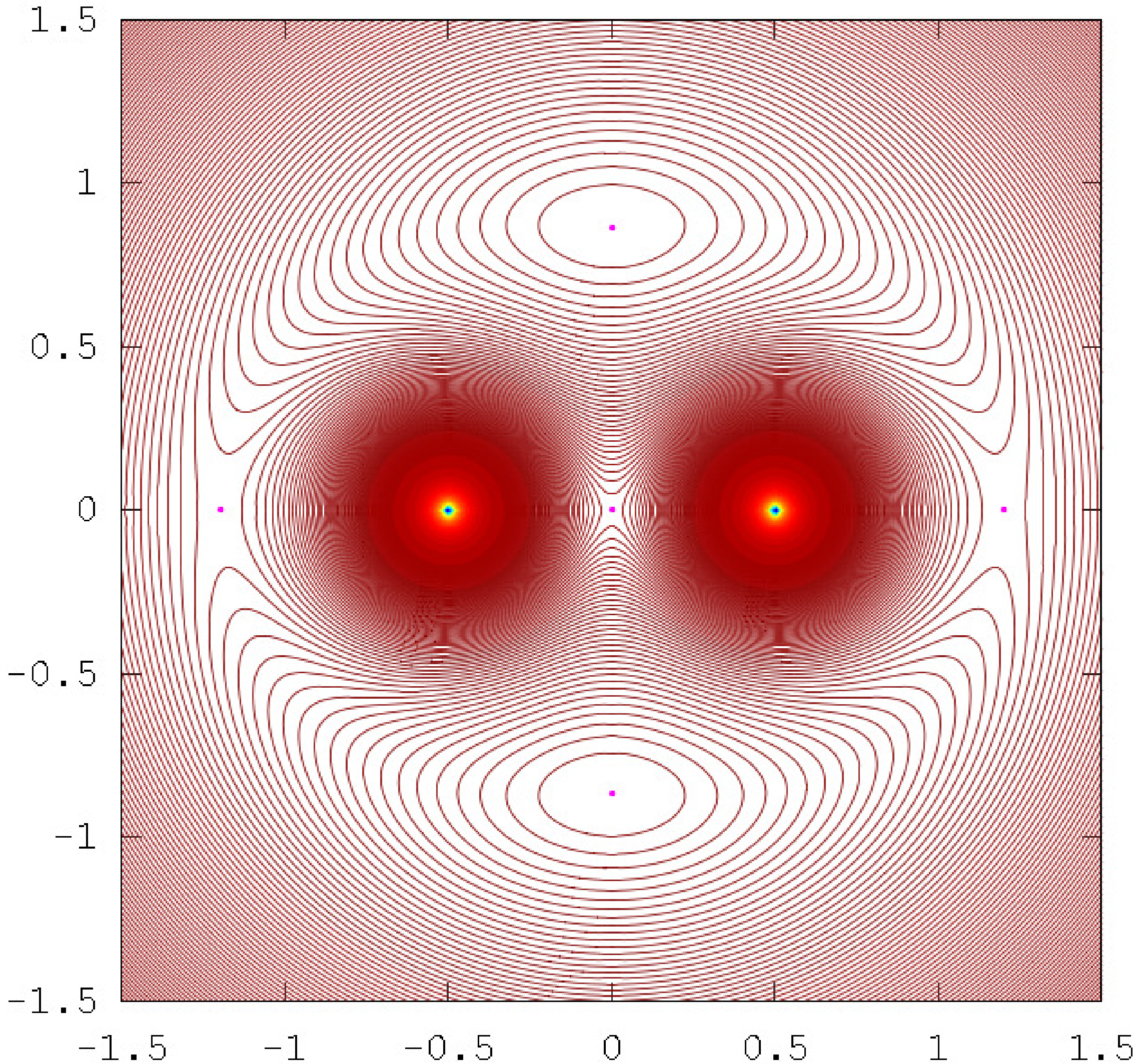
The period of the motion around the lagrangian point is then

$$P = 1.0033P_{\rm J} \quad \text{or} \quad P = 12.428P_{\rm J}.$$

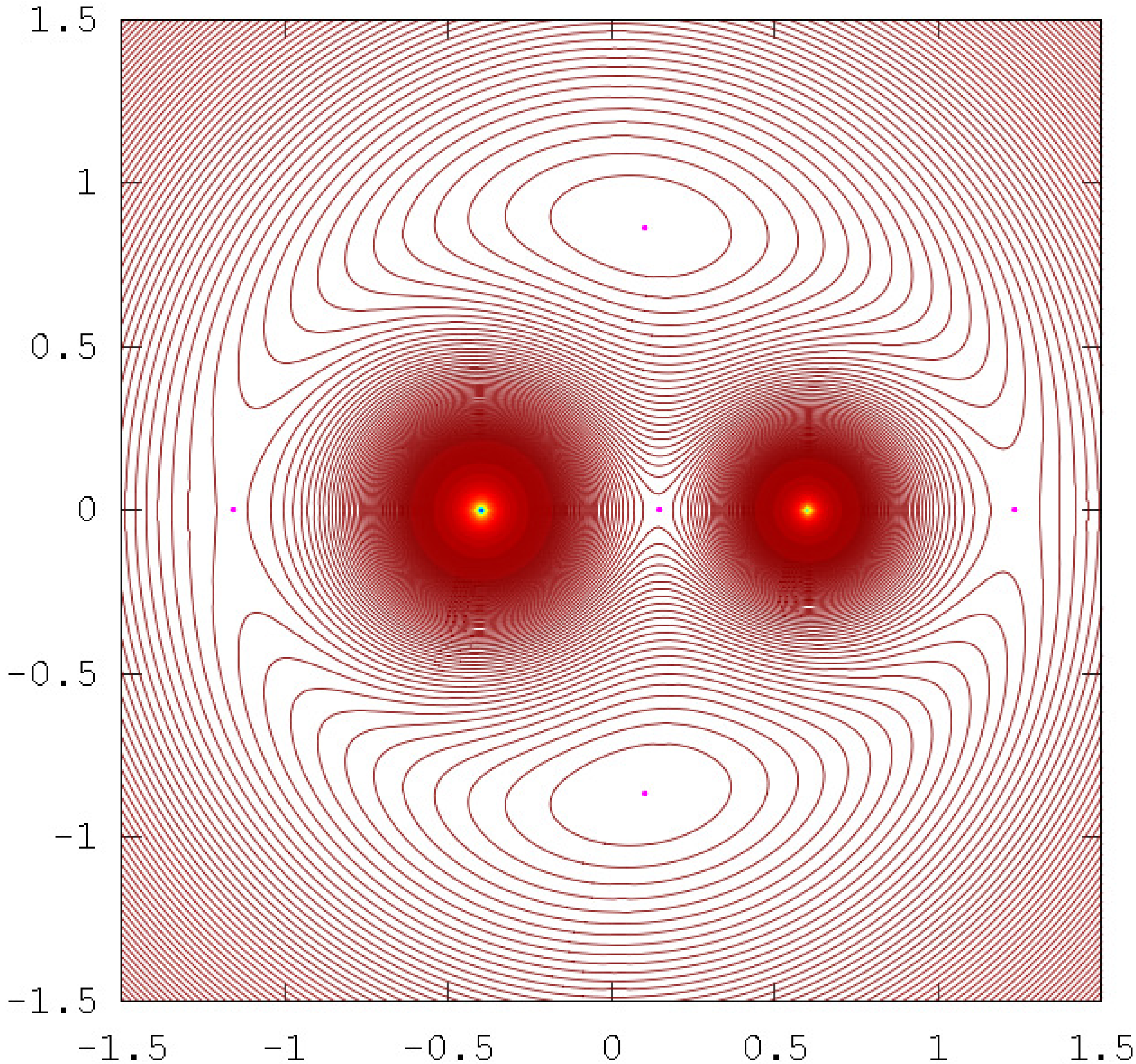
This description of the motion applies to asteroids moving closely around the equilateral lagrangian points, and the approximation made in the analysis appeared in the Taylor expansion for the potential given by equations 16.2.16-18. For more distant excursions one might try analytical solutions by expanding the Taylor series to higher-order terms (and of course working out the higher-order derivatives) or it might be easier to integrate equations 16.2.19 and 20 numerically. Many people have had an enormous amount of fun with this. The orbits do not follow the equipotential contours exactly, of course, but in general shape they are not very different in appearance from the contours. Thus, for larger excursions from the lagrangian points the orbits become stretched out with a narrow tail curving towards L_1; such orbits bear a fanciful resemblance to a tadpole shape and are often referred to as tadpole orbits. For yet further excursions, an asteroid may start near L_4 and roll downhill, veering around the back of the more massive body, through the L_1 point and then upwards towards L_5; then it slips back again, goes again through L_1 and then up to L_4 again – and so on. This is a so-called horseshoe orbit.

The drawings below show the equipotential contours for a number of mass ratios. These drawings were prepared using *Octave* by Dr Mandayam Anandaram of Bangalore University, and are dedicated by him to the late Max Fairbairn of Sydney, Australia, who prepared figures XVI.8 and XVI.9a for me shortly before his untimely death. Anand and Max were my first graduate students at the University of Victoria, Canada, many years ago. These drawings show the gradual evolution from tadpole-shaped contours to horseshoe-shaped contours. The mass-ratio q = 24.959 935 8 is the critical ratio below which stable orbits around the equilateral points L_4 and L_5 are not possible. The mass-ratios q = 81.3 and 1047 are the ratios for the Earth-Moon and Sun-Jupiter systems respectively. The reader will notice that, in places where the contours are closely-spaced, in particular close to the deep potential well of the larger mass, Moiré fringes appear. These fringes appear where the contour separation is comparable to the pixel size, and the reader will recognize them as Moiré fringes and, we think, will not be misled by them.

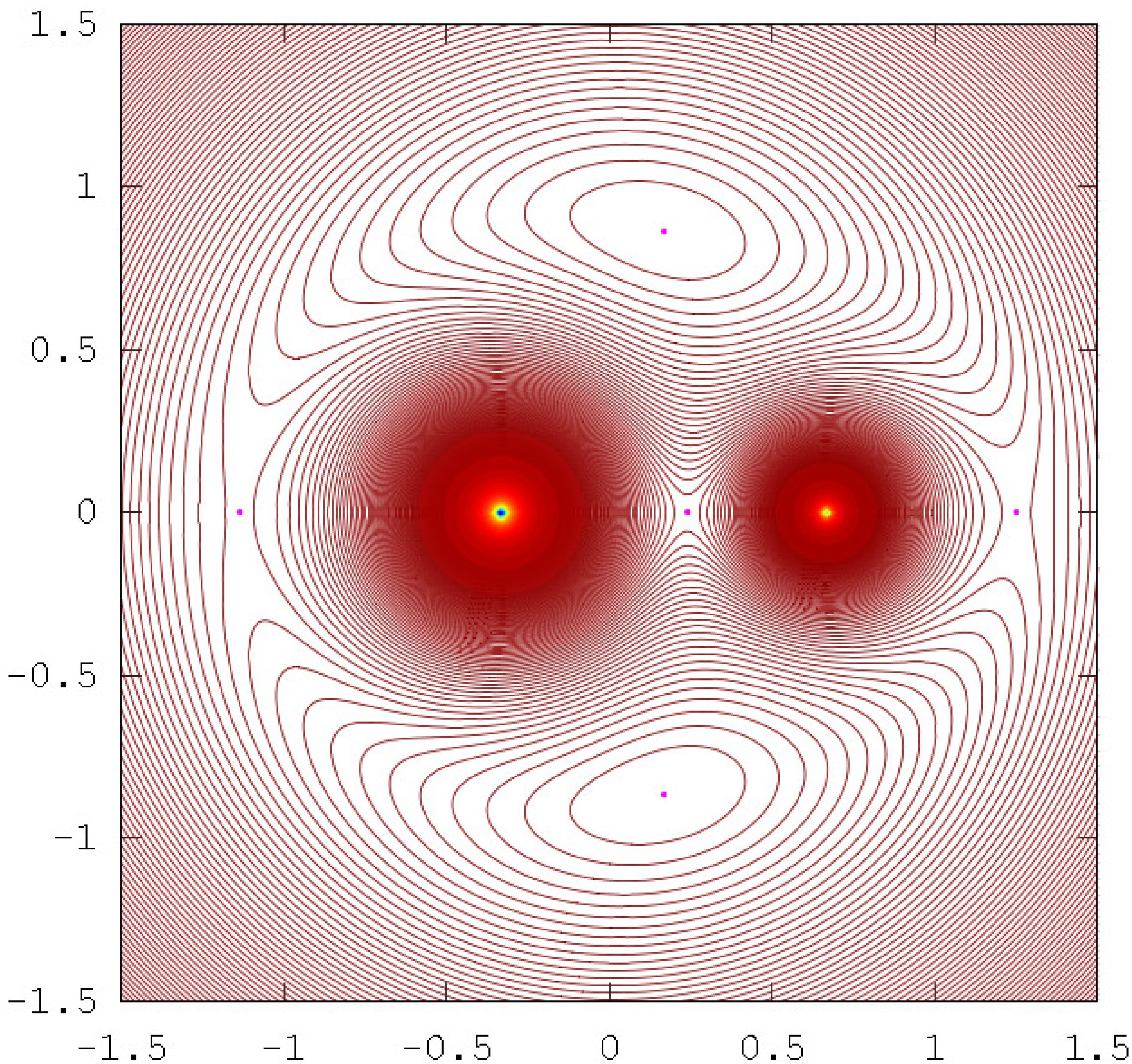
Dr Anandaram has also prepared a number of fascinating drawings in which sample orbits are superimposed, in a second colour, on the equipotential contours. These include tadpole orbits in the vicinity of the equilateral points; "triangular" orbits of the Hilda asteroids, which are in 2 : 3 resonance with Jupiter; the almost "square" orbit of Thule, which is in 3 : 4 resonance with Jupiter; and half of a complete 9940 year libration period of Pluto, which is in 3 : 2 resonance with Neptune. It is proposed to publish these in a separate paper dedicated to Max, the reference to which will in due course be given in these notes.



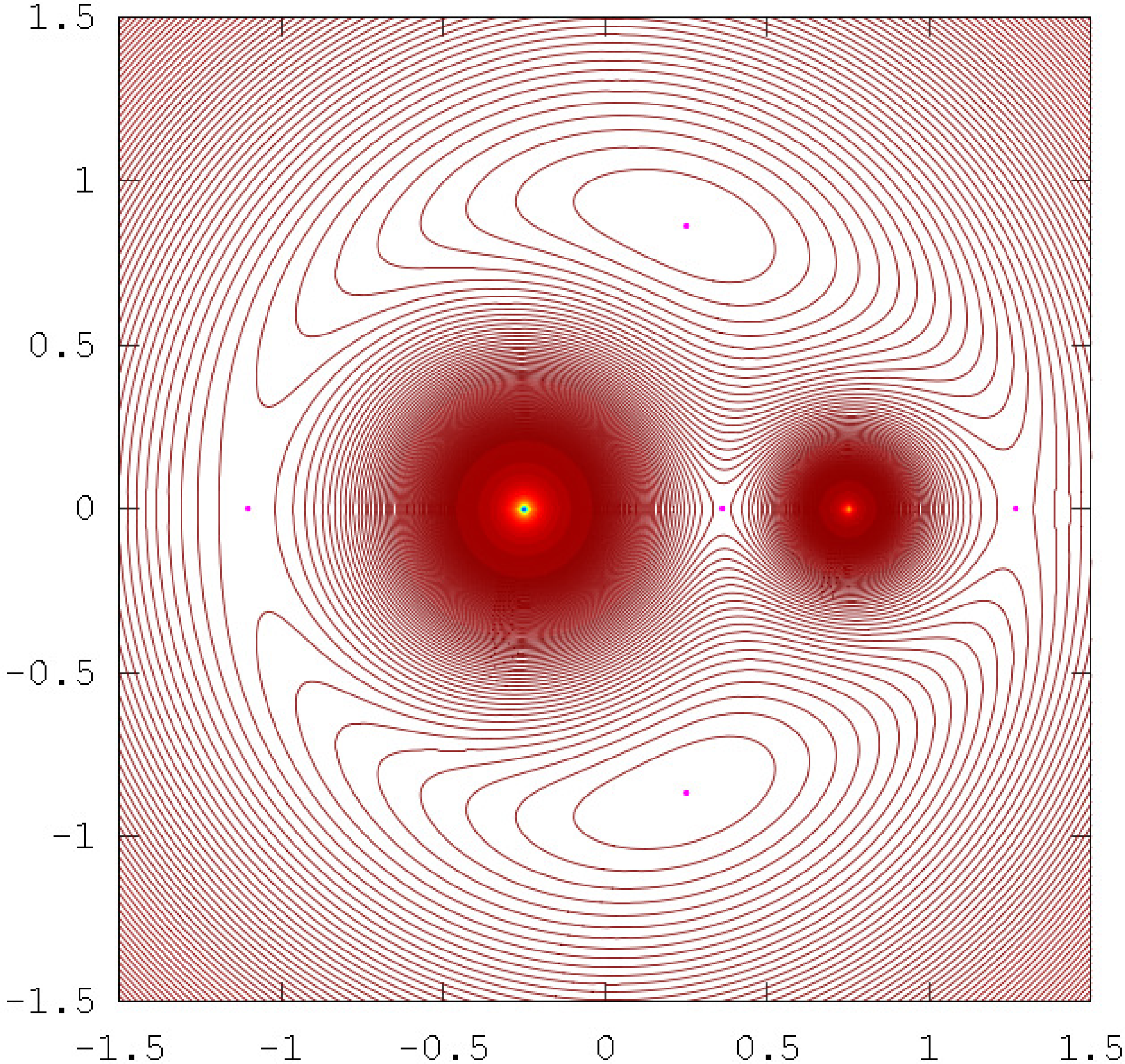
q = M1/M2 = 1 : Equipotential contours and Lagrangian points



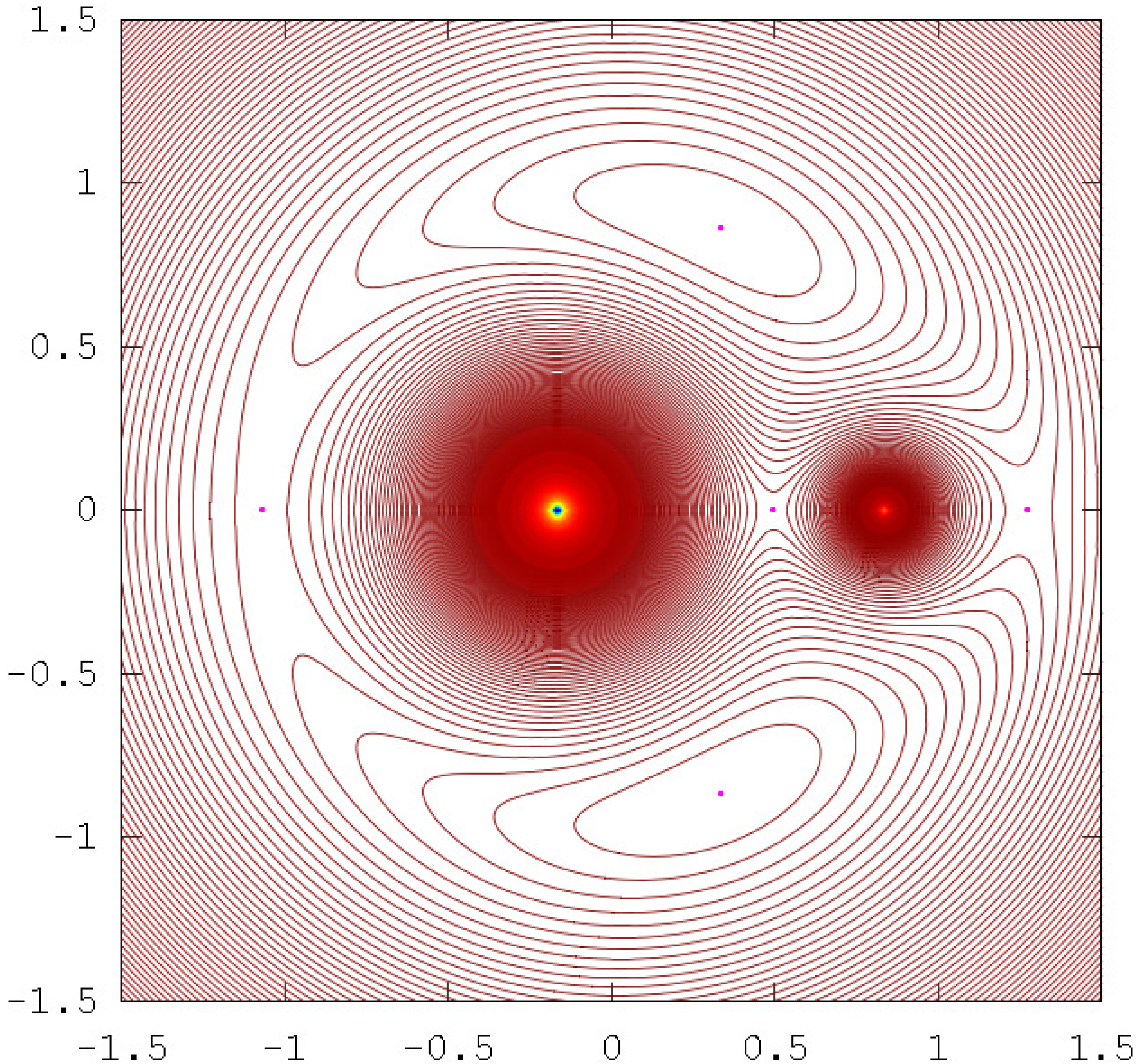
q = M1/M2 = 1.5 : Equipotential contours and Lagrangian points



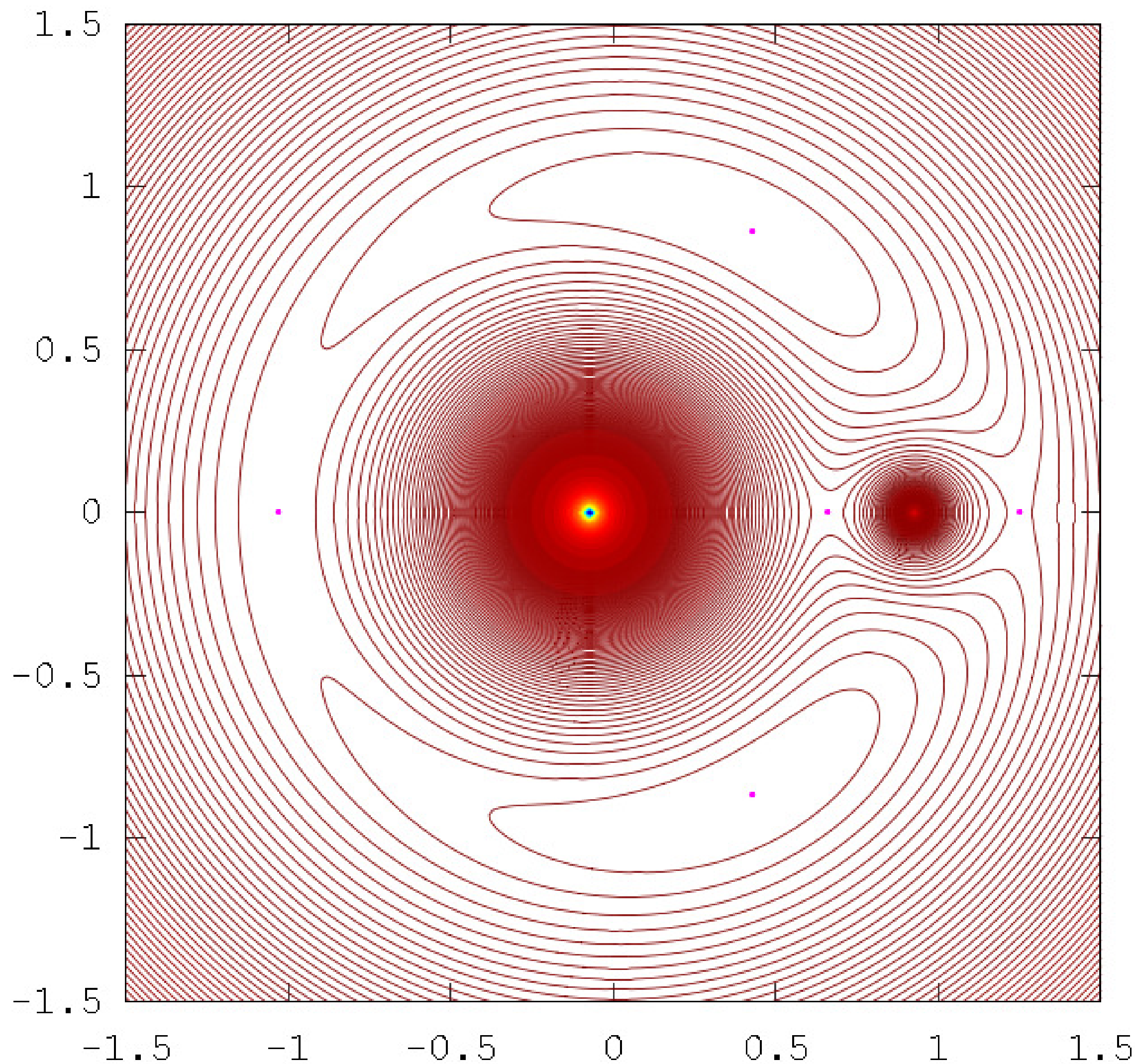
q = M1/M2 = 2 : Equipotential contours and Lagrangian points



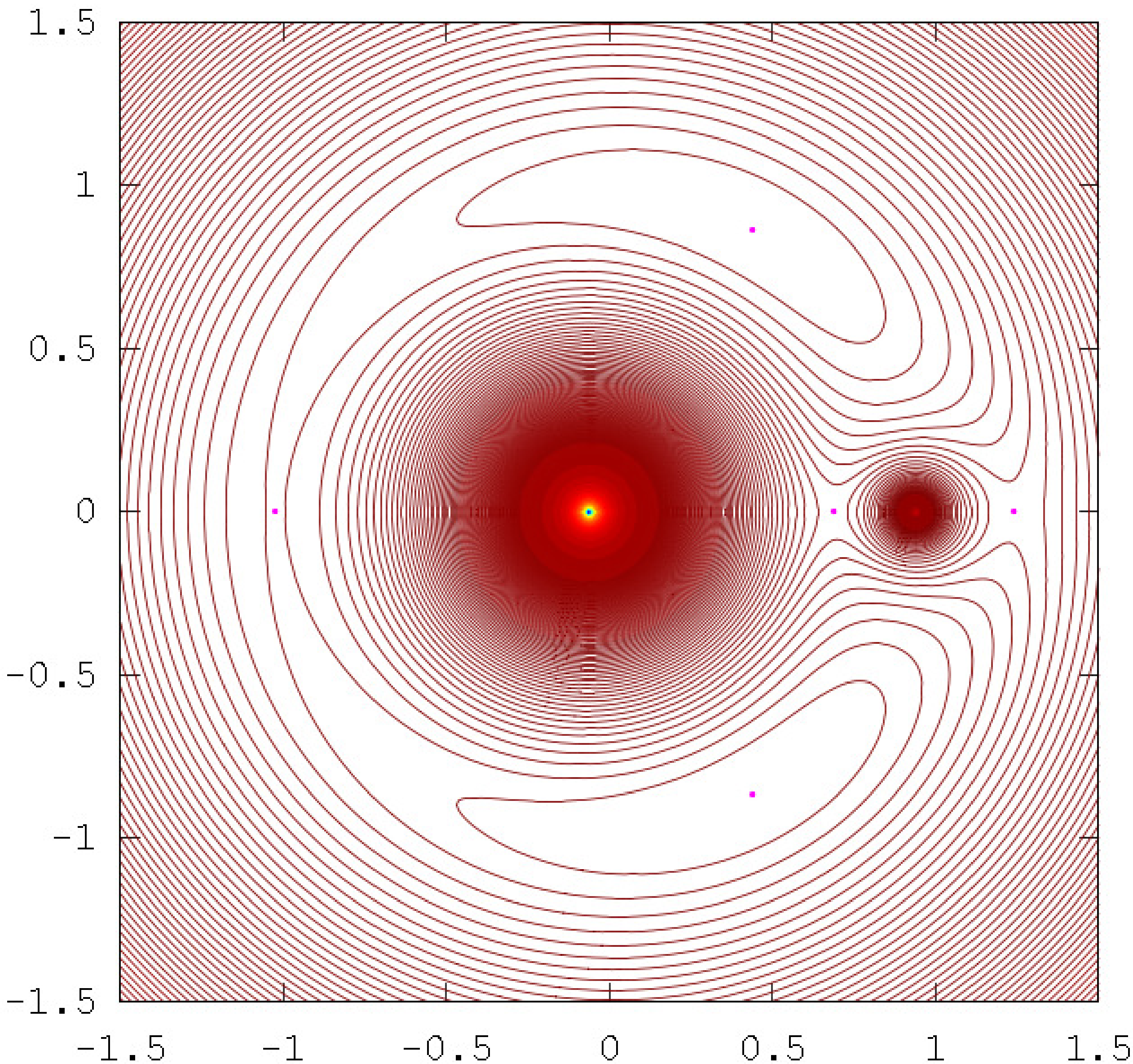
M1/M2 = 3 : Co-rotating Equipotential lines and L1..L5



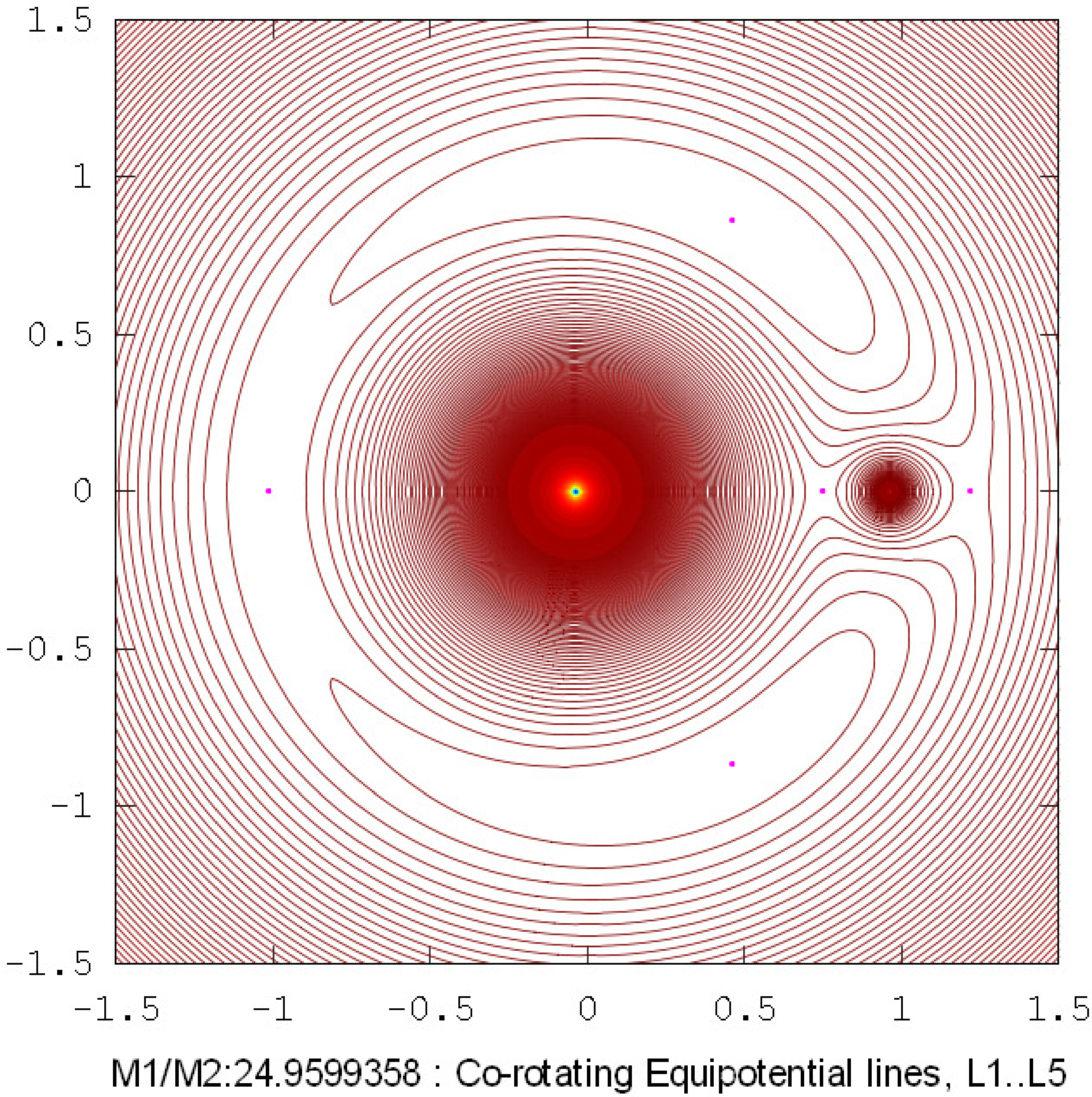
q = M1/M2 = 5 : Equipotential contours and Lagrangian points

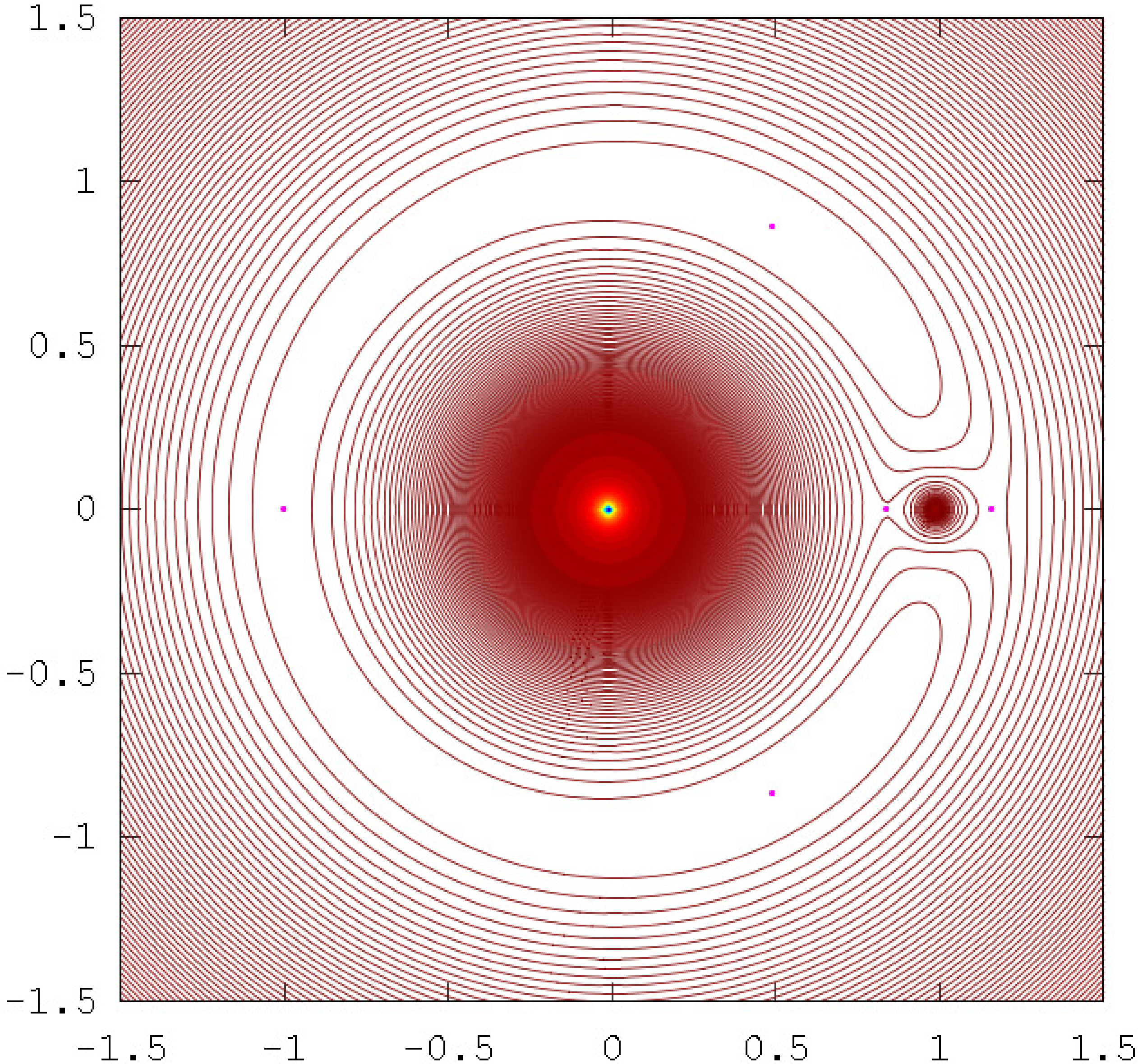


q = M1/M2 = 12.5 : Equipotential contours and Lagrangian points

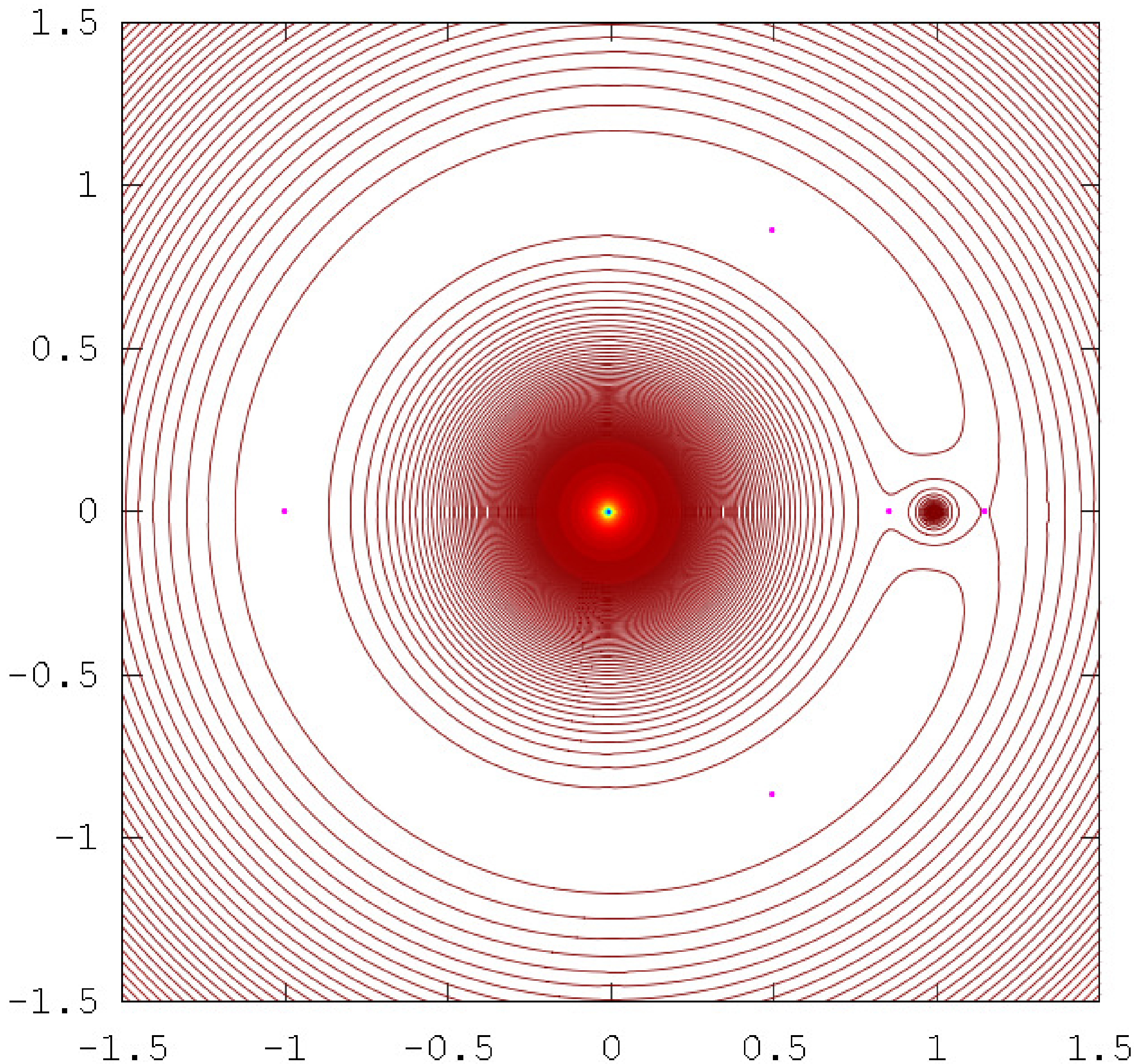


M1/M2 = 15 : Co-rotating Equipotential lines and L1..L5

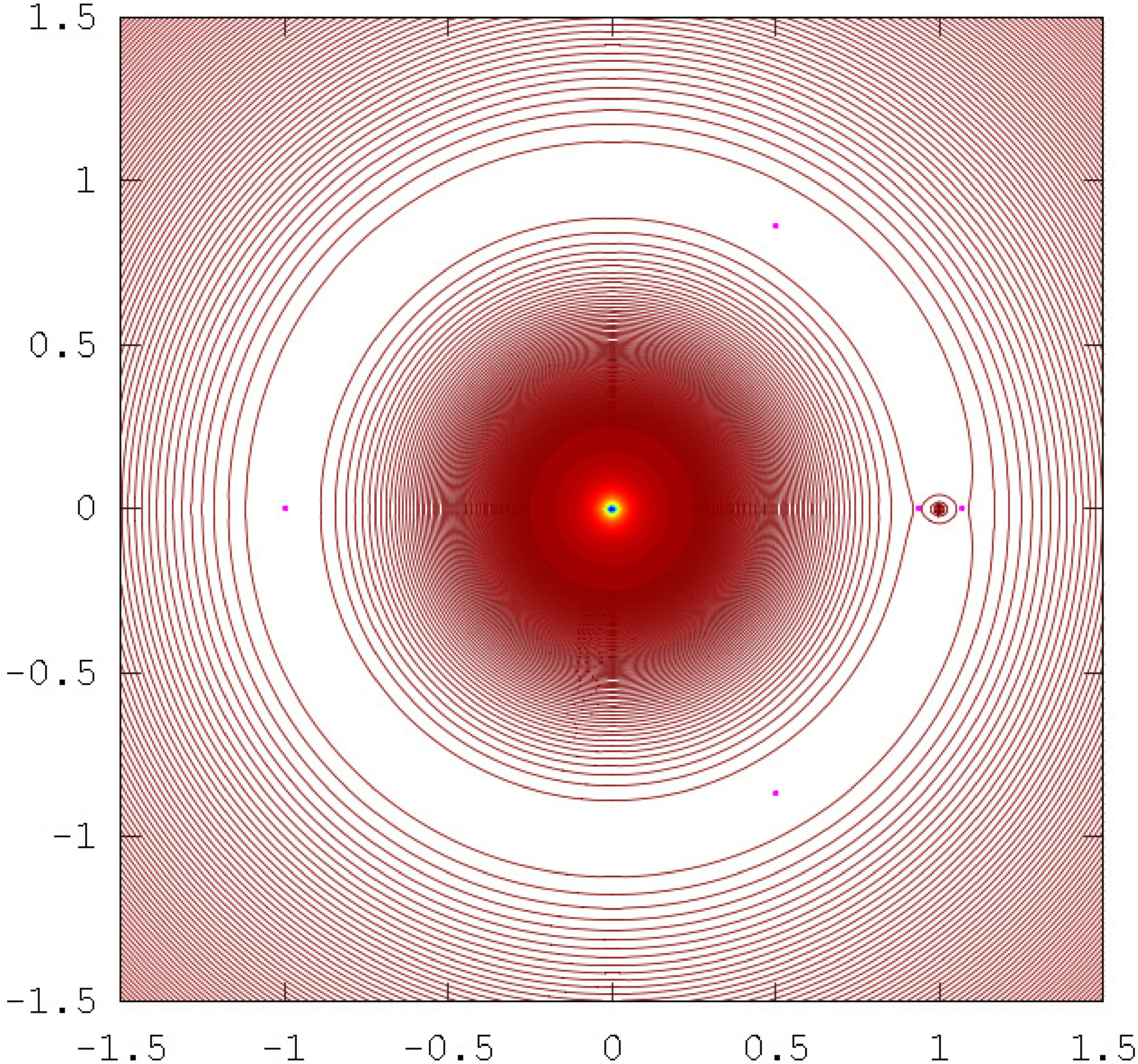

1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5
M1/M2:24.9599358 : Co-rotating Equipotential lines, L1..L5



M1/M2 = 81.3 : Co-rotating Equipotential lines and L1..L5



q = M1/M2 = 100 : Equipotential contours and Lagrangian points



M1/M2 = 1047 : Co-rotating Equipotential lines and L1..L5