CHAPTER 10
COMPUTATION OF AN EPHEMERIS

10.1 *Introduction*

The entire enterprise of determining the orbits of planets, asteroids and comets is quite a large one, involving several stages. New asteroids and comets have to be searched for and discovered. Known bodies have to be found, which may be relatively easy if they have been frequently observed, or rather more difficult if they have not been observed for several years. Once located, images have to be obtained, and these have to be measured and the measurements converted to usable data, namely right ascension and declination. From the available observations, the orbit of the body has to be determined; in particular we have to determine the *orbital elements*, a set of parameters that describe the orbit. For a new body, one determines *preliminary elements* from the initial few observations that have been obtained. As more observations are accumulated, so will the calculated preliminary elements. After all observations (at least for a single opposition) have been obtained and no further observations are expected at that opposition, a *definitive orbit* can be computed. Whether one uses the preliminary orbit or the definitive orbit, one then has to compute an *ephemeris* (plural: *ephemerides*); that is to say a day-to-day prediction of its position (right ascension and declination) in the sky. Calculating an ephemeris from the orbital elements is the subject of this chapter. Determining the orbital elements from the observations is a rather more difficult calculation, and will be the subject of a later chapter.

10.2 *Elements of an Elliptic Orbit*

Six numbers are necessary and sufficient to describe an elliptic orbit in three dimensions. These include the four (a, e, ω and T) that we described in section 9.9 for the two dimensional case. Two additional angles, which will be given the symbols i and Ω, will be needed to complete the description of the orbit in 3-space.

The six elements of an elliptic orbit, then, are as follows.

a the semi major axis, usually expressed in *astronomical units* (AU).
e the eccentricity
i the inclination
Ω the longitude of the ascending node
ω the argument of perihelion
T the time of perihelion passage

The three angles, i, Ω and ω must always be referred to the equinox and equator of a stated epoch. For example, at present they are usually referred to the mean equinox and equator of J2000.0. The meanings of the three angles are explained in figure X.1 and the following paragraphs.

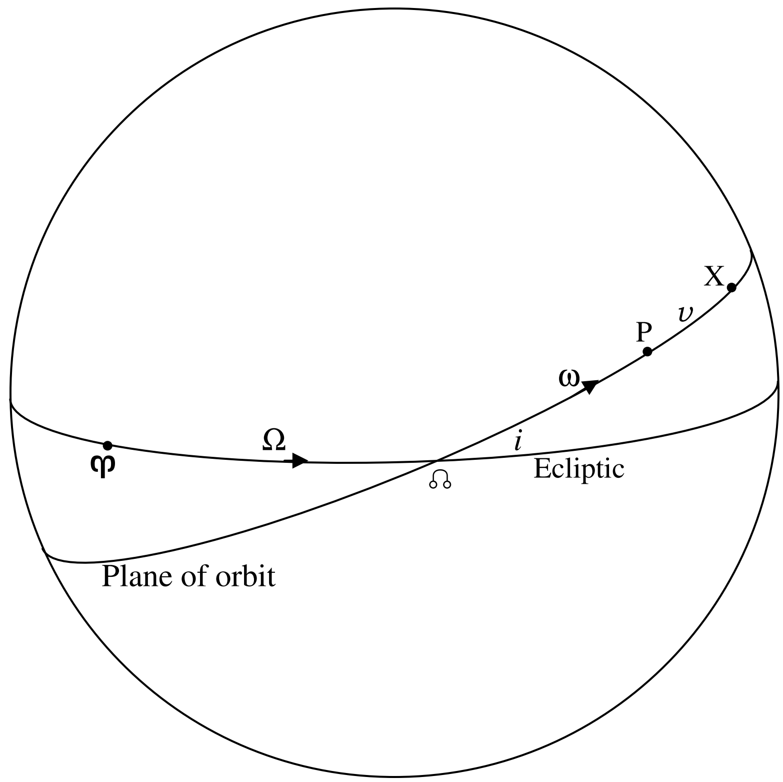


FIGURE X.1

In figure X.1 I have drawn a celestial sphere centred on the Sun. The two great circles are intended to represent the plane of Earth's orbit (i.e. the ecliptic) and the plane of a planet's orbit – (i.e. not the orbit itself, but its projection on to the celestial sphere.) The point P is the projection of the perihelion point of the orbit on to the celestial sphere, and the point X is the projection of the planet on to the celestial sphere at some time. The two points where the plane of the ecliptic and the plane of the planet's orbit intersect are called the *nodes*, and the point marked ☊ is the *ascending node.* The descending node, ☋, not shown in the figure, is on the far side of the sphere. The symbol ♈ is the First point of Aries (now in the constellation Pisces), where the ecliptic crosses the equator. As seen from the Sun, Earth is at ♈ on or near September 22. (For the benefit of personal computer users, I found the symbols ☋, ☊ and ♈ in Bookshelf Symbol 3.)

The *inclination* i is the angle between the plane of the object's orbit and the plane of the ecliptic (i.e. of Earth's orbit). It lies in the range $0^\circ \leq i < 180^\circ$. An inclination greater than 90° implies that the orbit is retrograde – i.e. that it is moving around the Sun in a direction opposite to that of Earth's motion.

The angle Ω, measured eastward from ♈ to ☊, is the *ecliptic longitude of the ascending node.* (The word "ecliptic" is usually omitted as understood.) It goes from 0° to 360°.

The angle ω, measured in the direction of the planet's motion from ☊ to P, is the *argument of perihelion.* It goes from 0° to 360°.

There is no need to add the period P of the orbit to the list of elements, since P in sidereal years is related to a in AU by $P^2 = a^3$.

10.3 *Some Additional Angles*

For a direct orbit, the sum of the two angles Ω and ω is often given the symbol ϖ (a form of the Greek letter pi), and is called (not entirely accurately) the *longitude of perihelion.* It is the sum of two angles measured in different planes. Thus, for a direct orbit, $\varpi = \Omega + \omega$. For a retrograde orbit, however, $\varpi = \Omega - \omega$.

The angle v, measured from perihelion to the planet, is the *true anomaly* of the planet at some time. We imagine, in addition to the true planet, a "mean" planet, which moves at constant angular speed $2\pi/P$, so that the angle from perihelion to the mean planet at time t is $M = \dfrac{2\pi(t-T)}{P}$, which is called the *mean anomaly* at time t. The words "true" and "mean" preceding the word "anomaly" refer to the "true" planet and the "mean" planet.

The angle $\theta = \omega + v$, measured from ☊, is the *argument of latitude* of the planet at time t.

The angle $l = \Omega + \theta = \Omega + \omega + v = \varpi + v$ measured in two planes, is the *true longitude* of the planet. This is a rather curious term, since, being measured in two planes, it is not really the true longitude at all. The word "true" refers to the "true" planet rather than to the longitude.

Likewise the angle $L = \Omega + \omega + M = \varpi + M$ is the *mean longitude* (i.e. the "longitude" of the "mean" planet.).

10.4 *Elements of a Circular or Near-circular Orbit*

For a near-circular orbit (such as the orbits of most of the major planets), the position of perihelion and the time of perihelion passage are ill-defined, and for a perfectly circular orbit they cannot be defined at all. For a near-circular orbit, the argument of perihelion ω (or sometimes the "longitude of perihelion", ϖ) is retained as an element, because there is really no other way of expressing the position of perihelion, though of course the more circular the orbit the less the precision to which ω can be determined. However, rather than specify the time of perihelion passage T, we usually specify some instant of time called the *epoch*, which I denote by t_0, and then we specify either the mean anomaly at the epoch, M_0, or the mean longitude at the epoch, L_0, or the true longitude at the epoch, l_0. For the meanings of mean anomaly, mean longitude and true longitude, refer to section 3, especially for the meanings of "mean" and "true" in this context. Of the three, only l_0 makes no reference whatever to perihelion.

Note that you should not confuse the epoch for which you specify the mean anomaly or mean longitude or true longitude with the equinox and equator to which the angular elements i, Ω and ω are referred. These may be the same, but they need not be (and usually are not). Thus it is often convenient to refer i, Ω and ω to the standard epoch J2000.0, but to give the mean longitude for an epoch during the current year.

10.5 *Elements of a Parabolic Orbit*

The eccentricity, of course, is unity, so only five elements are necessary. In place of the semi major axis, one usually specifies the parabola by the perihelion distance q. Presumably no orbit is ever exactly parabolic, which implies an eccentricity of exactly one. However, many long-distance comets move in large and eccentric orbits, and we see them over such a short arc near to perihelion that it is not possible to calculate accurate elliptic orbits, and we usually then fit a parabolic orbit to the observations.

10.6 *Elements of a Hyperbolic Orbit*

In place of the semi major axis, we have the semi transverse axis, symbol a. This amounts to just a name change, although some authors treat a for a hyperbola as a negative number, because some of the formulas, for example for the speed in an orbit, $V^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$, are then identical for an ellipse and for a hyperbola.

Although there is no fundamental reason why the solar system should not sometime receive a cometary visitor from interstellar space, as yet we know of no comet with an original hyperbolic orbit around the Sun. Some comets, initially in elliptic orbits, are perturbed into hyperbolic orbits by a close passage past Jupiter, and are then lost from the solar system. Such orbits are necessarily highly perturbed and one cannot in general compute a reliable ephemeris by treating it as a simple two-body problem; the instantaneous osculating elements will not predict a reliable ephemeris far in advance.

10.7 *Calculating the Position of a Comet or Asteroid*

We suppose that we are given the orbital elements of an asteroid or comet. Our task is to be able to predict, from these, the right ascension and declination of the object in the sky at some specified future (or past) date. If we can do it for one date, we can do it for many dates - e.g. every day for a year if need be. In other words, we will have constructed an ephemeris. Nowadays, of course, we can obtain ephemeris-generating programs and ephemerides with a few deft clicks on the Web, without knowing so much as the difference between a sine and a cosine; but that way of doing it is not the purpose of this section.

For example, according to the Minor Planet Center, the osculating elements for the minor planet (1) Ceres for the epoch of osculation $t_0 = 2002$ May 6.0 TT are as follows:

$a = 2.766\ 412\ 2$ AU	$\Omega = 80^\circ.486\ 32$
$e = 0.079\ 115\ 8$	$\omega = 73^\circ.984\ 40$
$i = 10^\circ.583\ 47$	$M_0 = 189^\circ.275\ 00$

i, Ω and ω are referred to the equinox and equator of J2000.0.

Calculate the right ascension and declination (referred to J2000.0) at 2002 July 15.0 TT.

We have already learned how to achieve much of our aim from Chapter 9. Thus, from the elements a, e, ω and T for an elliptic orbit (or the corresponding elements for a parabolic or hyperbolic orbit) we can already compute the *true anomaly* v and the heliocentric distance r as a function of time. These are the heliocentric polar coordinates of the body (henceforth "asteroid"). In order to find the right ascension and declination (i.e. geocentric coordinates with the celestial equator as xy-plane) all we have to do is to find the coordinates relative to the ecliptic, rotate the coordinate system from ecliptic to equatorial, and shift the origin of coordinates from Sun to Earth,. We just have to do some straightforward geometry, and no further dynamics.

Let's start by doing what we already know how to do from Chapter 9, namely, we'll calculate the true anomaly and the heliocentric distance.

Mean anomaly at the epoch ($t_0 =$ May 6.0) is $M_0 = 189^\circ.275\ 00$.
Mean anomaly at time t (= July 15. which is 70 days later) is given by

$$M - M_0 = \frac{2\pi}{P}(t - t_0). \qquad 10.7.1$$

The quantity $2\pi/P$ is called the *mean motion* (actually the average orbital angular speed of the planet), usually given the symbol n. We can calculate P in sidereal years from $P^2 = a^3$, and, given that a sidereal year is $365^d.25636$ and that 2π radians is 360 degrees, we can calculate the mean motion in its usual units of degrees per day. We find that n = 0.214 205 degrees per day. In fact the Minor Planet Center, as well as giving the orbital elements, also lists, for our convenience, the mean motion, and they give n = 0.214 204 57 degrees per day. The small discrepancy between the n given by the Minor Planet Center and the value that we have calculated from the published value of a presumably arises because the published values of the elements have been rounded off for publication, and the Minor Planet Center presumably carries all digits in its calculations. I would recommend using the value of n published by the Minor Planet Center, and I do so here. By July 15, then, equation 10.7.1 tells us that the mean anomaly is $M = 204^\circ.269$ 342. (I'm carrying six decimal places, even though M_0 is given only to five, just to be sure that I'm not accumulating rounding-off errors in the intermediate calculations. I'll round off properly when I reach the final result.)

We now have to find the eccentric anomaly from Kepler's equation $M = E - e\sin E$. Easy. (See chapter 9 if you've forgotten how.) We find $E = 202^{\circ}.532\ 2784$ and, from equations 2.3.16 and 17, we obtain the true anomaly $\upsilon = 200^{\circ}.854\ 0289$. The polar equation to an ellipse is $r = \dfrac{a(1-e^2)}{1+e\cos\upsilon}$, so we find that the heliocentric distance is r = 2.968 5716 au. (The Minor Planet Centre gives r, to four significant figures, as 2.969 au.) So much we could already do from Chapter 9. Note also that $\omega + \upsilon$, known as the *argument of latitude* and often given the symbol θ, is $274^{\circ}.838\ 429$.

We are going to have to make use of three heliocentric coordinate systems and one geocentric coordinate system.

1. *Heliocentric plane-of-orbit*. $\odot xyz$ with the $\odot x$ axis directed towards perihelion. The polar coordinates in the plane of the orbit are the heliocentric distance r and the true anomaly υ. The z-component of the asteroid is necessarily zero, and $x = r\cos\upsilon$ and $y = r\sin\upsilon$.

2. *Heliocentric ecliptic.* $\odot XYZ$ with the $\odot X$ axis directed towards the First Point of Aries ♈, where Earth, as seen from the Sun, will be situated on or near September 22. The spherical coordinates in this system are the heliocentric distance r, the ecliptic longitude λ, and the ecliptic latitude β, such that $X = r\cos\beta\cos\lambda$, $Y = r\cos\beta\sin\lambda$ and $Z = r\sin\beta$.

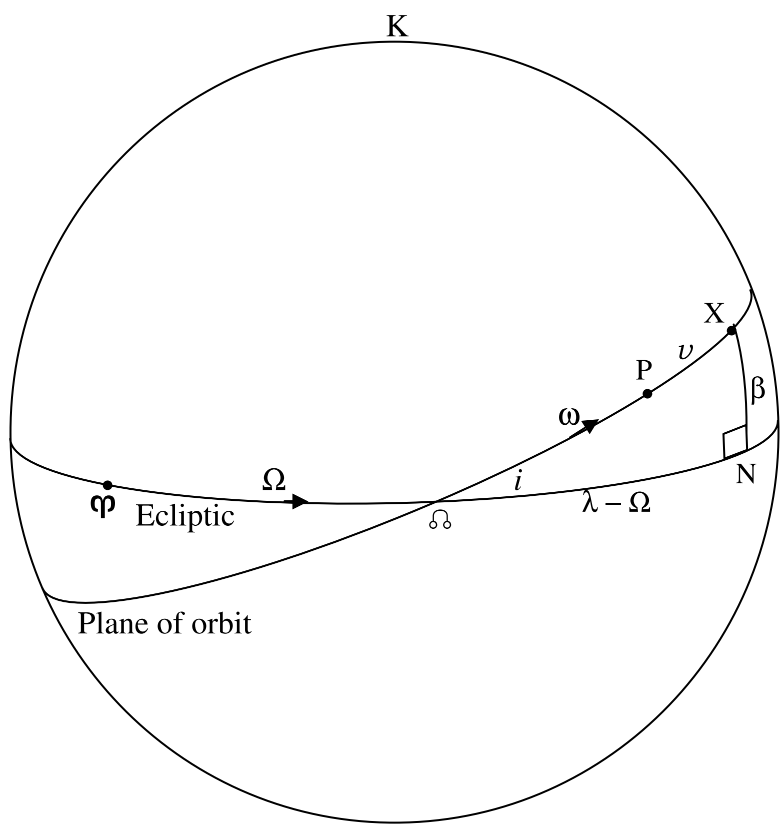


FIGURE X.2

3. *Heliocentric equatorial coordinates.* $\odot\xi\eta\zeta$ with the $\odot\xi$ axis directed towards the First Point of Aries and therefore coincident with the $\odot X$ axis . The angle between the $\odot Z$ axis and the $\odot\zeta$ axis is ε, the obliquity of the ecliptic. This is also the angle between the *XY*-plane (plane of the ecliptic, or of Earth's orbit) and the ξη-plane (plane of Earth's equator). See figure X.4.

4. *Geocentric equatorial coordinates.* $\oplus xyz$ with the $\oplus x$ axis directed towards the First Point of Aries. The spherical coordinates in this system are the geocentric distance Δ, the right ascension α and the declination δ, such that $x = \Delta\cos\delta\cos\alpha$, $y = \Delta\cos\delta\sin\alpha$ and $z = \Delta\sin\delta$.

In figure X.2, the arc ♈N is the heliocentric ecliptic longitude λ of the asteroid, and so ☊N is $\lambda - \Omega$. The arc NX is the heliocentric ecliptic latitude β. By two applications of equation 3.5.5 we find

$$\cos(\lambda-\Omega)\cos i = \sin(\lambda-\Omega)\cot(\omega+\upsilon) - \sin i\cot 90^\circ \qquad 10.7.2$$

and $$\cos(\lambda-\Omega)\cos 90^\circ = \sin(\lambda-\Omega)\cot\beta - \sin 90^\circ\cot i. \qquad 10.7.3$$

These reduce to $$\tan(\lambda-\Omega) = \cos i\tan(\omega+\upsilon) \qquad 10.7.4$$

and $$\tan\beta = \sin(\lambda-\Omega)\tan i. \qquad 10.7.5$$

In our particular example, we obtain (if we are careful to watch the quadrants),

$$\lambda - \Omega = 274^\circ.921\,7550, \quad \lambda = 355^\circ.408\,0750, \quad \beta = -10^\circ.545\,3234$$

Now, we'll take the *X*-axis for the heliocentric ecliptic coordinates through ♈ and the *Y*-axis 90° east of this. Then, by the usual formulas for converting between spherical and rectangular coordinates, that is, $X = r\cos\beta\cos\lambda$, $Y = r\cos\beta\sin\lambda$ and $Z = r\sin\beta$, we obtain

$$X = +2.909\,0661, \quad Y = -0.233\,6453, \quad Z = -0.543\,2880 \quad \text{au.}$$

(Check: $X^2 + Y^2 + Z^2 = r^2$.)

Exercise: Show, by elimination of λ and β, or otherwise, that:

$$X = r(\cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i) \qquad 10.7.6$$

$$Y = r(\sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i) \qquad 10.7.8$$

$$Z = r \sin\theta \sin i. \qquad 10.7.9$$

This will provide a more convenient way of calculating the coordinates. Verify that these give the same numerical result as before. Here are some suggestions for doing it "otherwise".

Refer to figure X.3, in which K is the pole of the ecliptic, and X is the asteroid. The radius of the celestial sphere can be taken as equal to r, the heliocentric distance of the asteroid. The rectangular heliocentric ecliptic coordinates are

$X = r \cos$ ♈⊙X $\qquad Y = r \cos$ R⊙X $\qquad Z = r \cos$ K⊙X

where ⊙ is the Sun (not drawn), at the centre of the sphere. To find expressions for X, Y and Z, solve the triangles ♈☊X , R☊X and K☊X respectively.

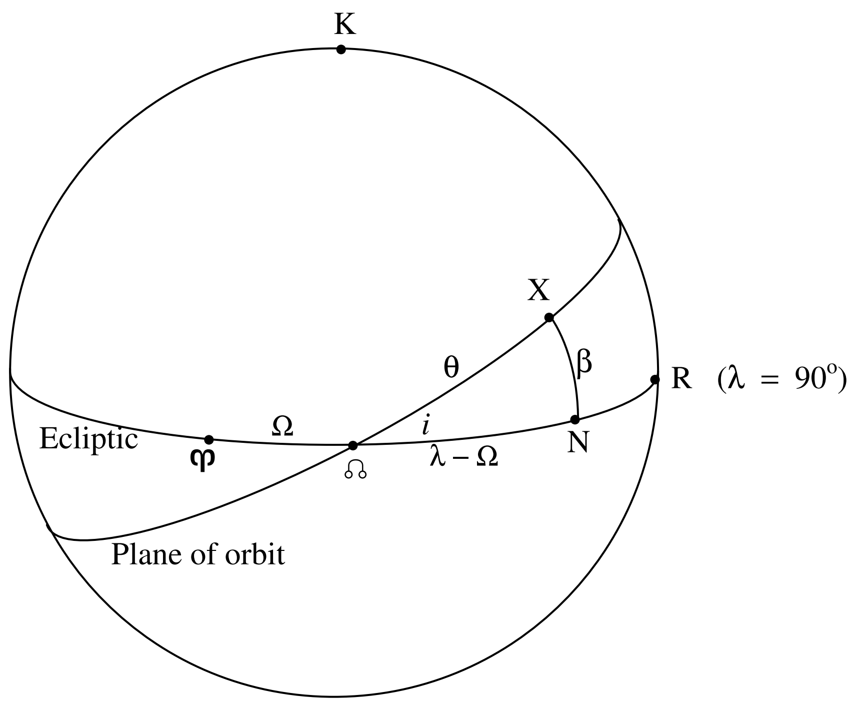


FIGURE X.3

The next step is to find the heliocentric *equatorial* coordinates. The ecliptic is inclined to the equator at an angle ε, the *obliquity of the ecliptic* (see figure X.4) If ⊙XYZ is the heliocentric ecliptic coordinate system and ⊙ξηζ is the heliocentric equatorial coordinate system, the (X, Y, Z) coordinates of an asteroid are related to its (ξ , η, ζ) coordinates by the usual relation for rotation of coordinates:

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} . \qquad 10.7.10$$

Assuming that we are using coordinates referred to the ecliptic and equator of J2000.0, the obliquity of the ecliptic at J2000.0 was 23°.439 291. We thus obtain, for the heliocentric equatorial coordinates of the asteroid,

$$\xi = +2.909\,0661, \quad \eta = +0.001\,7422, \quad \zeta = -0.591\,3957 \quad \text{au.}$$

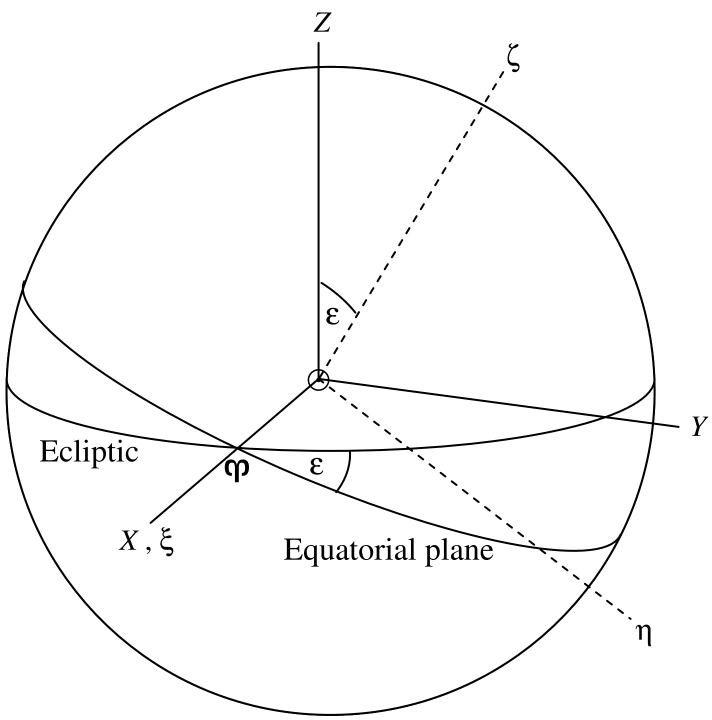


FIGURE X.4

Now we move the origin of coordinates by a translational shift from Sun to Earth. Let ($x_\odot$, $y_\odot$, $z_\odot$) be the *geocentric equatorial coordinates of the Sun*, and (x , y , z) be the geocentric equatorial coordinates of the asteroid (which we want), then

$$x = x_\odot + \xi \qquad 10.7.11$$

$$y = y_\odot + \eta \qquad 10.7.12$$

$$z = z_\odot + \zeta . \qquad 10.7.13$$

This looks like the easiest step of all, but in fact it is the most difficult. How do we calculate the geocentric equatorial coordinates of the Sun?

One answer might be to start from the elements of Earth's orbit around the Sun, and just calculate the heliocentric coordinates of Earth in the same say that we have done for the asteroid. The geocentric coordinates of the Sun are then just minus the heliocentric coordinates of Earth. Unfortunately, for precise ephemerides, this does not work. Earth does not move around the Sun in an ellipse. What moves around the Sun approximately in an ellipse (neglecting planetary perturbations) is the barycentre of the Earth-Moon system. The presence of the Moon makes a lot of difference in calculating the exact position of Earth. What is needed is what is known as *Newcomb's complete theory of the Sun.* I am going to side-step that here. Instead we shall find that the geocentric rectangular equatorial coordinates of the Sun, referred to the mean equator and equinox of J2000.0 (which we want), are published each year, for every day of the year, in *The Astronomical Almanac.* An alternative, then, to running Newcomb's complete theory is to transfer the table of the Sun's coordinates from *The Astronomical Almanac* each year to your own computer. If you do a month's worth at a time, it will not be too tedious – but you will then want to check that you haven't made any mistakes. This can be done by writing a short program to compute the daily increment in the coordinates, and then use a graphics package to plot the increments as a function of date. If you have made any mistakes, these will immediately become obvious. Alternatively (and I haven't tried this) you might want to see if you can find *The Astronomical Almanac* on the Web, and see if you can transfer the table of the Sun's coordinates to your own computer. Either way, you will want to write a nonlinear interpolation program (see Chapter 1, Section 10) to calculate the Sun's $(x_\odot , y_\odot , z_\odot)$ for times between the tabulated values.

In our example, the solar coordinates for 2002 July 15, referred to the mean equator and equinox J2000.0 are

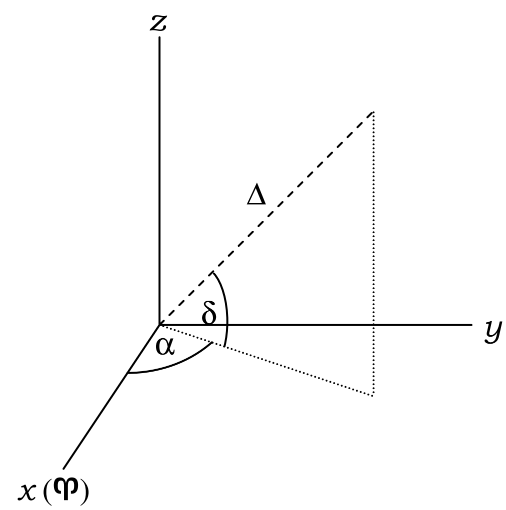
$$x_\odot = -0.386\,1944, \quad y_\odot = +0.862\,6457, \quad z_\odot = +0.374\,9996 \text{ au.}$$

The geocentric equatorial coordinates of the asteroid are therefore

$$x = +2.522\,8717, \quad y = +0.864\,3879, \quad z = -0.216\,3961 \text{ au.}$$

These are the geocentric rectangular coordinates. The geocentric distance Δ, the declination δ and the right ascension α are the corresponding spherical coordinates, and can be calculated in the usual way (see figure X.5).

FIGURE X.5



Thus $$\alpha = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = \sin^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right), \tag{10.7.14}$$

$$\delta = \sin^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} \tag{10.7.15}$$

and $$\Delta = \sqrt{x^2+y^2+z^2}\ . \tag{10.7.16}$$

In our example, $\alpha = 18^\circ.912\,5181 = 01^{\mathrm{h}}\ 15^{\mathrm{m}}.7$

$\delta = -4^\circ.638\,9995 = -4^\circ\ 38'$

$\Delta = 2.676$ A.U.

The Minor Planet Center gives

$\alpha = 01^{\mathrm{h}}\ 15^{\mathrm{m}}.5$

$\delta = -04^\circ\ 40'$

$\Delta = 2.676$ au

It is to be remembered that this result was obtained from the *osculating elements* (see Chapter 9, Section 10) for the *epoch of osculation* 2002 May 6.0 TT. Because of planetary perturbations, the orbits are continuously changing. Generally the orbits are adjusted for perturbations (and new observations, if any) every 200 days, when the orbital elements of asteroids are published for a new epoch of osculation. However, even without planetary perturbations, there are a few small refinements that need to be made in order to calculate a very precise ephemeris. We shall deal with these later in this chapter, and with planetary perturbations in a later chapter.

10.8 *Quadrant Problems*

Any reader who has followed in detail thus far will be aware that there are quadrant problems. That is, problems of the sort: what is $\sin^{-1} 0.5$? Is it 30° or is it 150°? Quadrant problems can be among the most frustrating in celestial mechanics problems, unless one is always aware of them and takes the necessary precautions. I made a quadrant mistake in preparing the first draft of section 10.7, and it took me a frustrating time to find it. I can offer only a few general hints, which are as follows.

i. If you find that the position you have calculated for your asteroid is way, way off, and you have calculated it to be in a quite different part of the sky from where it really is, the most likely cause of the problem is that you have made a quadrant mistake somewhere, and that would be the first place to look.

ii. All inverse trigonometric functions have two solutions between 0° and 360°, so you *always* have to be sure that you select the right one.

iii. To determine the correct quadrant, always check the signs of *two* trigonometric functions. For example, check the signs of $\sin\theta$ and of $\cos\theta$.

iv. The FORTRAN function ATAN2 (DATAN2 in double precision) (there are doubtless similar functions in other computing languages and probably on some hand calculators) is very useful in determining the correct quadrant. Learn how to use it.

v. The little diagram,

S	A
T	C

which you may have learned in high school trigonometry classes, is very useful.

vi. The mean, eccentric and true anomalies need not be in the same quadrant. For example, try $e = 0.95$, $M = 80^\circ$, or $e = 0.5$, $M = 50^\circ$, and work out E and υ in each case. All three, however, at least are either together in the range 0° to 180° or 180° to 360°. Another way of putting it is that all three have the same sign.

10.9 *Computing an Ephemeris*

In section 7, we calculated the position in the sky of an asteroid on a single date, and we showed, step by step, the way that the calculation was done. To construct an ephemeris, we just have to do the same calculation over and over again for as many days as we wish. However, there are efficient and inefficient ways of doing the calculation. For example, there are many terms, such as cos Ω, which you don't want to have to calculate over and over again each day. The important thing is to calculate all the necessary terms that do not depend on the time before you begin the day-to-day ephemeris. In FORTRAN language, make sure that anything that does not depend on the time is outside the DO-loop. I shall describe two methods that are fairly efficient.

Method i

In this method we first calculate certain non-time-dependent functions of the elements and the obliquity, which I refer to as auxiliary constants. These are as follows, in which I have also given the numerical values of these constants for our example of Section 10.7.

$a = \sin^2 \varepsilon$	+0.158 226 66	10.9.1
$b = \sin^2 i$	+0.033 733 85	10.9.2
$c = 1 - b\cos^2 \Omega$	+0.999 078 44	10.9.3
$d = \frac{1}{2}\sin 2\varepsilon \sin 2i \cos\Omega$	+0.021 780 97	10.9.4
$e = \cos\Omega \cos i$	+0.162 471 35	10.9.5
$f = \sqrt{1 - b\sin^2 \Omega}$	+0.983 457 02	10.9.6
$g = a(b-c)$	−0.152 743 26	10.9.7
$h = g - d$	−0.174 524 22	10.9.8
$j = \sqrt{c+h}$	+0.908 049 68	10.9.9
$k = \sqrt{b-h}$	+0.456 353 01	10.9.10
$A = \omega + \mathrm{DATAN2}(\cos\Omega, -\sin\Omega\cos i)$	+4.263 999 28 rad	10.9.11
$B = \omega + \mathrm{DATAN2}(\sin\Omega\cos\varepsilon, e\cos\varepsilon - \sin i \sin\varepsilon)$	+2.778 267 50 rad	10.9.12
$C = \omega + \mathrm{DATAN2}(\sin\Omega\sin\varepsilon, e\sin\varepsilon + \sin i\cos\varepsilon)$	+2.325 865 55 rad	10.9.13

The constants a, b and e are not, of course, the semi major and semi minor axes, and the eccentricity. In particular, the e in equations 10.9.12 and 10.9.13 is the number calculated from equation 10.9.5. That deals with the auxiliary constants. They need not be calculated again.

The only time-dependent quantities are the heliocentric distance (radius vector) r and the true anomaly υ, and the geocentric equatorial coordinates of the Sun, ($x_\odot$, $y_\odot$, $z_\odot$).

These may be calculated as in Chapter 9, Section 9.6, or looked up as in this Chapter, Section 10.7.

In our example of Section 10.7, for July 15, these were

$$r = 2.968\,572 \text{ AU} \quad \text{and} \quad \upsilon = 200^{\circ}.854\,029 = 3.505\,565\,87 \text{ rad}$$

$$x_{\odot} = -0.386\,1944, \quad y_{\odot} = +0.862\,6457, \quad z_{\odot} = +0.374\,9996 \text{ au}$$

We can immediately calculate the rectangular heliocentric equatorial coordinates from

$$\xi = r f \sin(\upsilon + A) = +2.909\,0661 \text{ au} \qquad 10.9.14$$

$$\eta = r j \sin(\upsilon + B) = +0.001\,7422 \text{ au} \qquad 10.9.15$$

$$\zeta = r k \sin(\upsilon + C) = -0.5913957 \text{ au.} \qquad 10.9.16$$

This is exactly the result we obtained in Section 10.7. From this point we calculate (x, y, z) and Δ, α and δ as in that section.

Of course, you'll probably want to know (or you *ought* to), where all of these equations come from. I shan't do it all; I'll start you off, and you can fill in the details yourself.

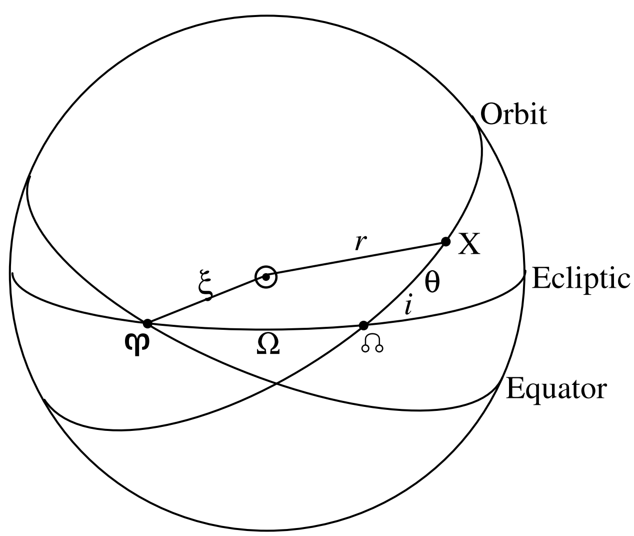


FIGURE X.6

In figure X.6, the cosine of the angle ♈⊙X is ξ/r, and by solution of the triangle ♈☊X it is also $\cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i$. Thus

$$\xi = r(\cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i). \qquad 10.9.17$$

Now introduce two auxiliary constants f and A' defined by

$$\cos\Omega = f\sin A' \qquad 10.9.18$$

$$-\sin\Omega\cos i = f\cos A' \qquad 10.9.19$$

The equation 10.9.17 becomes

$$\xi = r(f\sin A'\cos\theta + f\cos A'\sin\theta) = rf\sin(A' + \theta). \qquad 10.9.20$$

Here, θ is the argument of latitude $\omega + \upsilon$, and if we now let $A = A' + \omega$, this becomes

$$\xi = r\,f\sin(A + \upsilon). \qquad 10.9.21$$

Add $x_\odot$ to each side, and we arrive at the geocentric coordinate x of the asteroid (see equation 10.7.11).

The formulas for η and ζ are a bit more difficult. From equation 10.7.10, we have

$$\eta = Y\cos\varepsilon - Z\sin\varepsilon \qquad 10.9.22$$

and

$$\zeta = Y\sin\varepsilon + Z\cos\varepsilon. \qquad 10.9.23$$

Now, just as we showed, by solving a triangle, that cos ♈⊙X = ξ/r = $\cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i$, you need to show that

$$Y/r = \sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i \qquad 10.9.24$$

and

$$Z/r = \sin\theta\sin i. \qquad 10.9.25$$

Then introduce auxiliary constants j, B', k and C' defined by

$$\sin\Omega\cos\varepsilon = j\sin B', \qquad 10.9.26$$

$$\cos\Omega\cos i\cos\varepsilon - \sin i\sin\varepsilon = j\cos B', \qquad 10.9.27$$

$$\sin\Omega\sin\varepsilon = k\sin C' \qquad 10.9.28$$

and

$$\cos\Omega\cos i\sin\varepsilon + \sin i\cos\varepsilon = k\cos C'. \qquad 10.9.29$$

Proceed from there, slowly and carefully, in the same way as we did for ξ, and you should eventually arrive at equations 10.9.15 and 10.9.16.

Method ii

This method is very useful for an elliptic orbit. It uses auxiliary constants P_x , Q_y ,etc, which are functions of the angular elements and the obliquity and which have a simple and direct geometric interpretation, allow us to calculate the heliocentric equatorial coordinates (ξ, η, ζ) as soon as we have calculated the eccentric anomaly E (without having the calculate the true anomaly v and the attendant quadrant trap) and, best of all, the Minor Planet Center publishes these auxiliary constants at the same time that it publishes the orbital elements.

As in Method i, we discuss four coordinate systems:

Heliocentric plane-of-orbit.	$\odot xyz$
Heliocentric ecliptic.	$\odot XYZ$
Heliocentric equatorial.	$\odot\xi\eta\zeta$
Geocentric equatorial.	$\oplus xyz$

A review of Chapter 3 might be useful before proceeding.

We need to establish the matrix of direction cosines of the three axes $\odot\xi\eta\zeta$ with respect to the system $\odot xyz$, which we can do in two stages. The conversion between $\odot\xi\eta\zeta$ and $\odot XYZ$ is easy, since this involves merely a rotation by ε (the obliquity of the ecliptic) about the mutually coincident $\odot\xi$ and $\odot X$ axes:

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \qquad 10.9.30$$

The other conversion is a bit more lengthy. Obviously one has

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos(X,x) & \cos(X,y) & \cos(X,z) \\ \cos(Y,x) & \cos(Y,y) & \cos(Y,z) \\ \cos(Z,x) & \cos(Z,y) & \cos(Z,z) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad 10.9.31$$

But then one has to find expressions for these direction cosines in terms of the orbital elements.

Refer to figure X.2.

The $\odot X$ axis is directed towards ♈.
The $\odot x$ axis is directed towards P

The angle (X, x) is the angle between ♈ and P.

Solve the triangle ♈☊P:

$$\cos(X, x) = \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i. \qquad 10.9.32$$

For $\cos(X, y)$ we just have to substitute $\omega + 90^\circ$ for ω in equation 10.9.32, to obtain

$$\cos(X, y) = -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i. \qquad 10.9.33$$

I leave it to the reader to identify and to solve the triangles necessary for the remaining cosines. You should get:

$$\cos(X, z) = \sin\Omega\sin i. \qquad 10.9.34$$

$$\cos(Y, x) = \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i. \qquad 10.9.35$$

$$\cos(Y, y) = -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i. \qquad 10.9.36a$$

$$\cos(Y, z) = -\cos\Omega\sin i \qquad 10.9.36b$$

$$\cos(Z, x) = \sin\omega\sin i. \qquad 10.9.37$$

$$\cos(Z, y) = \cos\omega\sin i. \qquad 10.9.38$$

$$\cos(Z, z) = \cos i. \qquad 10.9.39$$

One doesn't really need to identify and solve nine triangles to obtain all nine cosines, for the matrix is orthogonal, and every element is equal to its own cofactor, so only six of the cosines are independent. Work out six of them (and the last of them in particular is quite trivial), and you can work out the remainder by this orthogonal property. However, this does not allow one an opportunity for detecting mistakes. It is better to work out each of the cosines independently, and then check for mistakes by verifying that each element is equal to its cofactor.

Now, by combining equations 10.9.30 and 10.9.31, we obtain

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} P_x & Q_x & R_x \\ P_y & Q_y & R_y \\ P_z & Q_z & R_z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad 10.9.40$$

where P_y, for example is $\cos(\eta, x)$, and the direction cosines are given explicitly by

$$P_x = \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i, \qquad 10.9.41$$

$$Q_x = -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i, \qquad 10.9.42$$

$$P_y = (\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i)\cos\varepsilon - \sin\omega\sin i\sin\varepsilon, \qquad 10.9.43$$

$$Q_y = (-\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i)\cos\varepsilon - \cos\omega\sin i\sin\varepsilon, \qquad 10.9.44$$

$$P_z = (\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i)\sin\varepsilon + \sin\omega\sin i\cos\varepsilon, \qquad 10.9.45$$

$$Q_z = (-\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i)\sin\varepsilon + \cos\omega\sin i\cos\varepsilon. \qquad 10.9.46$$

We don't actually need R_x , R_y or R_z, and in any case they are not independent of the *P*s and *Q*s, for each element is equal to its cofactor – for example, $R_y = P_z Q_x - P_x Q_z$, but for the record,

$$R_x = \sin\Omega\sin i, \qquad 10.9.47$$

$$R_y = -\cos\Omega\,\sin i\cos\varepsilon - \cos i\sin\varepsilon, \qquad 10.9.48$$

and

$$R_z = -\cos\Omega\,\sin i\sin\varepsilon + \cos i\cos\varepsilon. \qquad 10.9.49$$

Let use recall our example of section 10.7:

$$\begin{array}{ll} a = 2.766\,412\,2 \text{ au} & \Omega = 80^\circ.486\,32 \\ e = 0.079\,115\,8 & \omega = 73^\circ.984\,40 \\ i = 10^\circ.583\,47 & M_0 = 189^\circ.275\,00 \end{array}$$

together with the obliquity $\varepsilon = 23^\circ.439\ 291$. We assume that we have carried out the calculation described in Section 10.7 as far as determining We find that the direction cosine matrix is

$$\begin{pmatrix} -0.886\,238\,71 & -0.426\,343\,35 & +0.181\,141\,63 \\ +0.322\,706\,58 & -0.848\,772\,46 & -0.418\,862\,48 \\ +0.332\,327\,26 & -0.312\,756\,55 & +0.889\,798\,81 \end{pmatrix}$$

(You may note that these numbers are given to eight significant figures, in order to avoid rounding-off errors. The matrix has been checked for orthogonality, which is an important check for numerical errors. In practice, you need not calculate the direction cosines, nor need you understand equations 10.9.41-49, for the direction cosines $(P_x, Q_x \text{ etc.})$ are generally published by the MPC in conjunction with the orbital elements.)

Having calculated (or looked up) the time-independent constants, we can now start on the time-dependent part of the calculation.

The z-coordinate of a planet in its orbit is zero (which is why we have no need of R_x, R_y or R_z) so the heliocentric equatorial coordinates are

$$\xi = P_x x + Q_x y, \tag{10.9.50}$$

$$\eta = P_y x + Q_y y, \tag{10.9.51}$$

$$\zeta = P_z x + Q_z y. \tag{10.9.52}$$

Now the plane-of-orbit coordinates (x , y) are related to the radius vector r and true anomaly υ by

$$x = r\cos\upsilon, \tag{10.9.53}$$

and

$$y = r\sin\upsilon, \tag{10.9.54}$$

and from the geometry of the ellipse we have

$$r\cos\upsilon = a(\cos E - e) \tag{10.9.55}$$

and

$$r\sin\upsilon = b\sin E. \tag{10.9.56}$$

(These equations are not given explicitly in section 2.3 of chapter 2, but they may readily be deduced from that section. The symbols a and b are the semi major and semi minor axes of the ellipse.)

Hence we obtain

$$\xi = aP_x(\cos E - e) + bQ_x \sin E, \tag{10.9.57}$$

$$\eta = aP_y(\cos E - e) + bQ_y \sin E \tag{10.9.58}$$

and

$$\zeta = aP_z(\cos E - e) + bQ_z \sin E. \tag{10.9.59}$$

Thus the procedure is first to work out (or look up!) the Ps and Qs, and then work out the eccentric anomalies for the dates required (by solving Kepler's equation). After that we just proceed as from equation 10.7.11, and we are on the home stretch. The reader should try this method using the same data as we used for our numerical example. The method has taken a little while to describe, but, once it has been set up, it is very quick and routine.

In our numerical example, in the paragraph following equation 10.7.1, we had found that the eccentric anomaly at July 15.0 was $E = 202^\circ.532\,2784$. Equations 10.9.57-59 now yield:

$$\xi = +2.909\,0661, \quad \eta = +0.001\,7422, \quad \zeta = -0.591\,3957 \quad \text{au},$$

which agrees exactly with what we obtained in method *i* and in Section 10.7

We can also easily get the equatorial velocity components. Thus

$$\dot{\xi} = [-aP_x \sin E + bQ_x \cos E]\dot{E}, \qquad 10.9.60$$

and similarly for the η and ζ components. But $M = E - e\sin E = 2\pi(t - T)/P$ and so $\dot{M} = (1 - e\cos E)\dot{E} = 2\pi/P = \frac{\sqrt{GM}}{a^{3/2}}$. (Here M is the mean anomaly, and M is the mass of the Sun.)

$$\dot{\xi} = \left(\frac{bQ_x \cos E - aP_x \sin E}{1 - e\cos E}\right)F, \qquad 10.9.61$$

where
$$F = \frac{\sqrt{GM}}{a^{3/2}}. \qquad 10.9.62$$

Here, if a is expressed in au, and if F is expressed in au per mean solar day, $\sqrt{GM}$ has the numerical value 0.017 202 098 95.

The equations 10.9.50-54 are valid for any conic section. Subsequent to these we examined an elliptic orbit. However, we can carry out similar procedures for a parabola and for a hyperbola. Thus for a *parabola* (see section 9.7)

$$r = \frac{2q}{1 + \cos\upsilon}, \quad \cos\upsilon = \frac{1-u^2}{1+u^2}, \quad \sin\upsilon = \frac{2u}{1+u^2} \qquad 10.9.63\text{a,b,c}$$

so that
$$r\cos\upsilon = q(1-u^2), \quad r\sin\upsilon = 2qu. \qquad 10.9.64\text{a,b}$$

From this we obtain

$$\xi = q[P_x(1-u^2) + 2Q_x u], \qquad 10.9.65$$

and similarly for η and ζ. Computation of the geocentric ephemeris then proceeds as for the ellipse. The velocity components can be obtained as follows. We have

$$\dot{\xi} = 2q\dot{u}(-uP_x + Q_x). \qquad 10.9.66$$

But $$u + \tfrac{1}{3}u^3 = \frac{\sqrt{\tfrac{1}{2}G\mathbf{M}}}{q}(t-T), \text{ hence } \dot{u}(1+u^2) = \frac{\sqrt{\tfrac{1}{2}G\mathbf{M}}}{q}. \qquad 10.9.67\text{a,b}$$

From these we obtain

$$\dot{\xi} = \frac{F(-uP_x + Q_x)}{1 + u^2}, \qquad 10.9.68$$

where $$F = k\sqrt{2/q} \qquad 10.9.69$$

and k has the same value as for the ellipse.

For a *hyperbola* (see section 9.8):

$$r = \frac{a(e^2 - 1)}{1+e\cos\upsilon}, \quad \cos\upsilon = \frac{-[u(u - 2e)+1]}{u(eu-2)+e}, \quad \sin\upsilon = (1-\cos^2\upsilon)^{1/2}, \qquad 10.9.70\text{a,b,c}$$

from which we obtain, after a little algebra,

$$r\cos\upsilon = \frac{-a[u(u-2e)+1]}{2u}, \quad r\sin\upsilon = \frac{a(e^2-1)^{1/2}(u^2-1)}{2u}. \qquad 10.9.71\text{a,b}$$

From this we obtain

$$\xi = \frac{-aP_x[u(u-2e)+1)] + bQ_x(u^2-1)}{2u}, \qquad 10.9.72$$

and similarly for η and ζ. Here b is of course the semi transverse axis of the conjugate hyperbola, $a\sqrt{e^2-1}$.

The velocity components can be obtained as follows. We have

$$\dot{\xi} = \frac{[-aP_x(u^2-1) + bQ_x(u^2+1)]\dot{u}}{2u^2}, \qquad 10.9.73$$

or $$\dot{\xi} = (-aP_x\sinh E + bQ_x\cosh E)\dot{E}, \quad \text{where } E = \ln u. \qquad 10.9.74$$

But $\qquad e\sinh E - E = \frac{\sqrt{GM}}{a^{3/2}}(t-T),\ \mathit{hence}\ (e\cosh E - 1)\dot{E} = \frac{\sqrt{GM}}{a^{3/2}}\cdot$ 10.9.75a,b

From these we obtain

$$\dot{\xi} = \left(\frac{bQ_x \cosh E - aP_x \sinh E}{e\cosh E - 1}\right)F, \qquad 10.9.76$$

where

$$F = k/a^{3/2}. \qquad 10.9.77$$

Exercise. While on the subject of velocity components, show that the radial velocity of a planet or comet with respect to the Sun is greatest at the end of a latus rectum.

10.10 *Orbital Elements and Velocity Vector*

In the two-dimensional problem of section 9.9, we saw how the four orbital elements could be obtained from the two positional coordinates and the two components of the velocity vector. Likewise in three dimensions, the six orbital elements can be obtained from the three positional coordinates and the three components of the velocity vector. An orbit is completely determined by the six numbers a, e, i, Ω, ω, T, or by the six numbers P_x , P_y , P_z , Q_x , Q_y , Q_z or by the six components of the position and velocity vectors. If we know the heliocentric equatorial position (ξ, η, ζ) and velocity $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$, we can easily calculate the heliocentric ecliptic position (X, Y, Z) and velocity $(\dot{X}, \dot{Y}, \dot{Z})$ by inversion of equation 10.9.30 (which applies to the velocity components as well as to the coordinates), so we shall take as our task in this section: given the heliocentric position (X, Y, Z) and velocity $(\dot{X}, \dot{Y}, \dot{Z})$, calculate the orbital elements a, e, i, Ω, ω, T.

As in the two-dimensional case (section 9.9), the semi major axis is determined if the heliocentric distance and speed are known, and we merely repeat here equation 9.9.2:

$$a = \frac{r}{2 - rV^2} \quad \text{au.} \qquad 10.10.1$$

Here r is the heliocentric distance in au given by

$$r^2 = X^2 + Y^2 + Z^2 \qquad 10.10.2$$

and V is the speed in units of 29.7846917 km s^{-1} given by

$$V^2 = \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2. \qquad 10.10.3$$

That one was easy. Now for the others.

Let the position and velocity of a planet at time t, in heliocentric ecliptic coordinates, be (X_1, Y_1, Z_1) and $(\dot{X}_1, \dot{Y}_1, \dot{Z}_1)$. The plane of the orbit contains the three points (0, 0, 0), (X_1, Y_1, Z_1) and $(\tau\dot{X}_1, \tau\dot{Y}_1, \tau\dot{Z}_1)$, where τ is an arbitrary constant of dimension T. I shall call these three points O, X and Q respectively. To see that Q is on the orbit, consider that the vector **V** is, of course, confined to the orbital plane. Translate the vector **V** to the origin, i.e to the Sun, and it will be clear that the line of the vector intersects the orbit. The equation to the orbital plane is therefore

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ \dot{X}_1 & \dot{Y}_1 & \dot{Z}_1 \end{vmatrix} = 0 \qquad 10.10.4$$

That is,
$$AX + BY + CZ = 0, \qquad 10.10.5$$

where
$$A = Y_1\dot{Z}_1 - Z_1\dot{Y}_1, \quad B = Z_1\dot{X}_1 - X_1\dot{Z}_1, \quad C = X_1\dot{Y}_1 - Y_1\dot{X}_1 . \qquad 10.10.6a,b,c$$

A, B and C are the direction ratios of the normal to the plane of the orbit. If we divide each by $\sqrt{A^2 + B^2 + C^2}$, we obtain

$$aX + bY + cZ = 0, \qquad 10.10.7$$

where a, b and c are the direction cosines of the normal to the plane, and the <u>inclination</u> is given by

$$\cos i = c, \qquad 10.10.8$$

with no quadrant ambiguity.

.

The next element to yield is the <u>longitude of the ascending node</u>, for the plane intersects the ecliptic at $Z = 0$ in the line $aX + bY = 0$, from which we see that

$$\sin\Omega = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \cos\Omega = -\frac{b}{\sqrt{a^2+b^2}}, \qquad 10.10.9a,b$$

with no quadrant ambiguity.

So far, we have found a, i and Ω. We are going to have to work a little bit for the remaining elements.

Let us first see if we can find the argument of latitude θ at time t. Refer to figure X.2. The arc ☊X is the argument of latitude θ. The arc XN is the ecliptic latitude β, given by

$$\sin\beta = \frac{Z_1}{\sqrt{X_1^2 + Y_1^2 + Z_1^2}} = \frac{Z_1}{r}\cdot \qquad 10.10.10$$

Apply the sine formula to triangle ☊XN:

$$\sin\theta = \frac{\sin\beta}{\sin i}\cdot \qquad 10.10.11$$

This gives the argument of latitude except for a quadrant ambiguity, which must be resolved before we can continue. The arc ♈N is the ecliptic longitude λ, given without quadrant ambiguity by

$$\sin\lambda = \frac{Y_1}{\sqrt{X_1^2 + Y_1^2}} \quad \text{and} \quad \cos\lambda = \frac{X_1}{\sqrt{X_1^2 + Y_1^2}}\cdot \qquad 10.10.12\text{a,b}$$

Apply the cotangent formula to triangle ☊XN:

$$\tan\theta = \frac{\tan(\lambda - \Omega)}{\cos i}\cdot \qquad 10.10.13$$

The argument of latitude of the planet at time t is now determined without quadrant ambiguity by equations 10.10.11 and 10.10.13.

I draw in figure X.7, schematically, the orbit and the position vector **r** and the velocity vector **V.** I have drawn the vector **V** twice – once originating at the planet X, and again translated to the origin O, and you can see the point Q, whose coordinates are $(\tau\dot{X}_1, \tau\dot{Y}_1, \tau\dot{Z}_1)$. The angle ψ that **V** makes with the line of nodes can fairly be called the argument of latitude of the point Q. Let (β' , λ') be the ecliptic latitude and longitude of Q. Then we can calculate ψ by exactly the same procedure by which we calculated θ from equations 10.10.10 to 10.10.13.

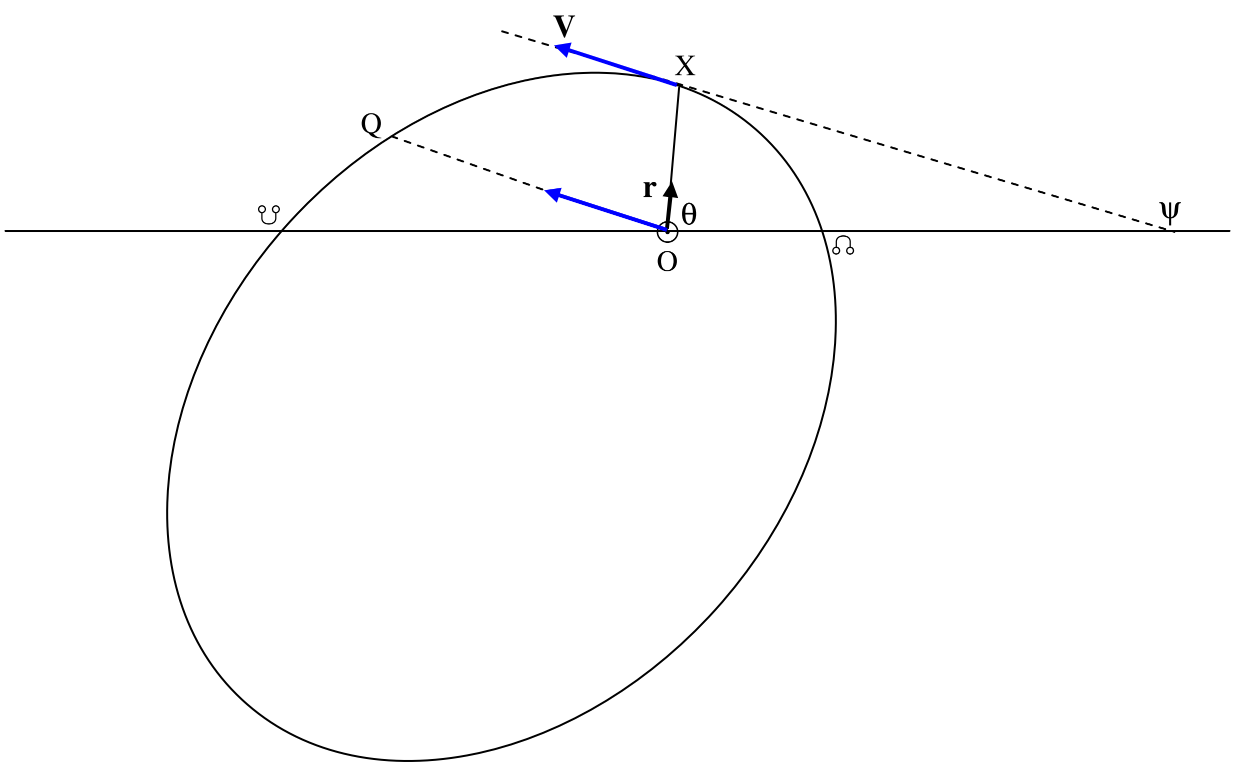


FIGURE X.7

Thus: $$\sin\beta' = \frac{\dot{Z}_1}{\sqrt{\dot{X}_1^2 + \dot{Y}_1^2 + \dot{Z}_1^2}}\cdot \qquad 10.10.14$$

$$\sin\psi = \frac{\sin\beta'}{\sin i}\cdot \qquad 10.10.15$$

$$\sin\lambda' = \frac{\dot{Y}_1}{\sqrt{\dot{X}_1^2 + \dot{Y}_1^2}} \quad \text{and } \cos\lambda' = \frac{\dot{X}_1}{\sqrt{\dot{X}_1^2 + \dot{Y}_1^2}}\cdot \qquad 10.10.16\text{a,b}$$

$$\tan\psi = \frac{\tan(\lambda' - \Omega)}{\cos i}\cdot \qquad 10.10.17$$

The argument of latitude of the point Q at time t is now determined without quadrant ambiguity by equations 10.10.15 and 10.10.17.

We are now going to find the semi latus rectum of the ellipse. From equation 9.5.21 we recall that the angular momentum per unit mass is

$$h = \sqrt{GMl}\ , \qquad 10.10.18$$

and from figure X.7 it is

$$h = rV\sin(\psi - \theta). \qquad 10.10.19$$

(In case you have forgotten your Euclid, the exterior angle of a triangle is equal to the sum of the two opposite interior angles.) From these we obtain, provided that we express distances in au and speeds in units of 29.784 691 7 km s^{-1},

$$l = r^2V^2\sin^2(\psi - \theta). \qquad 10.10.20$$

But $l = a(1 - e^2)$, so we now have the eccentricity from

$$e = \sqrt{1 - l/a}\,. \qquad 10.10.21$$

Two more to go :– ω and T.

The equation to the ellipse is $r = l/(1 + e\cos\upsilon)$, where υ is the true anomaly, equal to $\theta + \omega$. Therefore

$$\cos \upsilon = \frac{1}{e}\left(\frac{l}{r} - 1\right), \qquad 10.10.22$$

and, provided that we are careful with the quadrant in solving equation 10.10.22, we now have the argument of perihelion ω:

$$\omega = \upsilon - \theta . \qquad 10.10.23$$

After that we calculate the eccentric anomaly E, the mean anomaly M and the time of perihelion passage T from equations 2.3.16, 9.6.5 and 9.6.4, and we are finished:

$$\cos E = \frac{e + \cos \upsilon}{1 + e \cos \upsilon}, \qquad 10.10.24$$

$$M = E - e \sin E, \qquad 10.10.25$$

and

$$T = t - \frac{MP}{2\pi}. \qquad 10.10.26$$

Example. Let us suppose that a comet at heliocentric ecliptic coordinates

$$(X_1 \quad Y_1 \quad Z_1) = (1.5 \quad 0.6 \quad 0.2) \text{ au}$$

has a velocity

$$(\dot{X}_1 \quad \dot{Y}_1 \quad \dot{Z}_1) = (20 \quad 10 \quad 4) \text{ km s}^{-1}.$$

From these, we have r = 1.627 882 060 AU and V = 0.762 661 357 in units of 29.784 691 7 km s^{-1}. (I prefer to carry all the significant figures given by my calculator, and to round off only for the final answers, which I shall give to four significant figures or to one arcminute.) From equation 10.10.1, I obtain

$$\underline{a = 1.545\,743\,444 \cong 1.546 \text{ au.}}$$

From equations 10.10.6:

$$A = 0.4, \quad B = -2.0, \quad C = 3.0 \quad \text{au km s}^{-1}.$$

The direction cosines are then

$$a = 0.110\,263\,569\,3, \quad b = -0.551\,317\,846\,4, \quad c = 0.826\,976\,769\,6.$$

From equation 10.10.8 we find

$$\underline{i = 34^\circ.210\ 579\ 85 \cong 34^\circ\ 13'.}$$

There is no quadrant ambiguity, since i always lies between 0 and 180°.

From equation 10.10.9:

$$\sin \Omega = +0.196\ 116\ 135\ 1\ , \quad \cos \Omega = +0.980\ 580\ 675\ 7$$

from which

$$\underline{\Omega = 11^\circ.309\ 932\ 47 \cong 11^\circ\ 19'.}$$

From equations 10.10.10 and 10.10.11:

$$\sin \theta = 0.218\ 518\ 564\ 1$$

and from equations 10.10.12 and 10.10.13:

$$\tan \theta = 0.223\ 930\ 335\ 2,$$

from which $\theta = 12^\circ.622\ 036\ 03.$

This is the argument of latitude of X. Now for the argument of latitude of Q.

From equations 10.10.14 and 10.10.15:

$$\sin \psi = 0.313\ 196\ 153\ 6$$

and from equations 10.10.16 and 10.10.17:

$$\tan \psi = 0.329\ 788\ 311\ 8$$

from which $\psi = 18^\circ.251\ 951\ 53.$

(It may be noted by those who are following the calculation in detail that calculating both $\sin \psi$ and $\tan \psi$ not only eliminates any quadrant ambiguity, but it also serves as a check on the arithmetic.)

Equation 10.10.20: $l = 0.014\ 834\ 389\ 3$ AU.

Equation 10.10.21: $\underline{e = 0.995\ 189\ 967\ 6 \cong 0.9952.}$

Equation 10.10.22 $\cos \upsilon = -0.995\ 676\ 543\ 6\ ,$

$$\therefore \qquad \upsilon \ = \ \pm 174^{\circ}.670\ 214\ 0.$$

With $\theta \ \simeq \ 12^{\circ}$ and $\psi \ \simeq 18^{\circ}$, a quick sketch will convince us that the comet is past perihelion and is becoming more distant from the Sun, and therefore the true anomaly is

$$\upsilon \ = \ +174^{\circ}.670\ 214\ 0.$$

Equation 10.10.23: $\underline{\omega \ = \ 162^{\circ}.048\ 178\ 0 \ \cong \ 162^{\circ}\ 03'.}$

Equation 10.10.24: $\cos E \ = \ -0.053\ 395\ 416\ 1\ ,$

$$E \ = \ \pm 93^{\circ}.060\ 787\ 59.$$

But E and υ must have the same sign, and so

$$E \ = \ +93^{\circ}.060\ 787\ 59.$$

Equation 10.10.25: $M \ = \ +0.630\ 446\ 871\ \text{rad} \ = \ 36^{\circ}.121\ 944\ 93.$

The period is $P \ = \ a^{3/2}$ sidereal years $=$ 1.921 790 844 sidereal years =701.946 328 2 solar days.

Equation 10.10.26: $\underline{T \ = \ t \ - \ 70.432\ 407\ 26 \ \cong \ t \ - \ 70.43 \text{ days.}}$

10.11 *Hamiltonian Formulation of the Equations of Motion*

This section will require some knowledge by the reader of hamiltonian dynamics and the Hamilton-Jacobi theorem. The analysis will result in yet another set of six parameters for describing an orbit, which will be denoted by α_1, α_2, α_3, β_1, β_2, β_3. These will of course be related to the familiar six elements, and an orbit can be described by either one set or another. This section may be slightly more demanding than some previous sections, requiring as it does, knowledge of hamiltonian dynamics, and is not immediately essential. However, results arising will be used in Chapter 14 on general perturbations.

The Hamilton equations of motion (which will be familiar only to those who are acquainted with hamiltonian dynamics) are

$$\frac{\partial H}{\partial p_i} \ = \ \dot{q}_i \qquad \text{and} \qquad \frac{\partial H}{\partial q_i} \ = -\dot{p}_i\,, \tag{10.11.1a,b}$$

where, for a conservative system, $H \ = \ T \ + \ V.$

Now let us suppose that we know the hamiltonian for a system as a function of the generalized coordinates, generalized momenta and the time: $H(q_i , p_i , t)$. We want to find some function of the coordinates and the time, $S(q_i , t)$, which is a solution to the hamiltonian equations of motion. The Hamilton-Jacobi theorem says the following. Let us set up the following equation, in which i goes from 1 to n, n being the number of required generalized coordinates for the system. (In our orbital context, n will be six, since six elements are necessary to describe an orbit).

$$H\left(q_i\ ,\ \frac{\partial S}{\partial q_i}\ ,\ t\right) + \frac{\partial S}{\partial t} = 0. \qquad 10.11.2$$

This is the *Hamilton-Jacobi equation.*

If we can integrate this equation, there will be $n + 1$ constants of integration, which I call α_0 , α_1 , ... α_n. Suppose that $S(q_i , \alpha_i , t) + \alpha_0$ is *any* solution of equation 10.11.2 (not necessarily a solution to Hamilton's equations; it could be quite a simple solution). Then set up n additional equations of the form

$$\frac{\partial S}{\partial \alpha_i} = \beta_i\,, \qquad i = 1 \text{ to } n \qquad 10.11.3$$

where we have introduced n additional constants β_i. If we can solve these equations for S, according to the Hamilton-Jacobi theorem, these solutions are solutions of the hamiltonian equations of motion.

Let us see if we can apply this theorem to the problem of a particle of mass m moving around a body of mass M ($m << M$).

The hamiltonian is

$$H(q_i\,, p_i\,, t) \;=\; \frac{1}{2m}\left(p_r^2 \;+\; \tfrac{1}{r^2}\,p_\theta^2 \;+\; \tfrac{1}{r^2 \sin^2\theta}\,p_\phi^2\right) \;-\; \frac{GMm}{r}\,. \qquad 10.11.4$$

Here I have merely resolved the momentum into its components in spherical coordinates.

We can now set up the Hamilton-Jacobi equation:

$$\tfrac{1}{2m}\left(\left(\tfrac{\partial S}{\partial r}\right)^2 \;+\; \tfrac{1}{r^2}\left(\tfrac{\partial S}{\partial \theta}\right)^2 \;+\; \tfrac{1}{r^2 \sin^2\theta}\left(\tfrac{\partial S}{\partial \phi}\right)^2\right) - \;\tfrac{GMm}{r} \;+\; \tfrac{\partial S}{\partial t} \;=\; 0. \qquad 10.11.5$$

Let us see if we can find *any* solution to this equation, for example a solution of the form $S \;=\; S_r(r) \;+\; S_\theta(\theta) \;+\; S_\phi(\phi) \;+\; S_t(t)$. The equation 10.11.5 is then

$$\frac{dS_t}{dt} \;=\; -\frac{1}{2m}\Big(\big(\frac{dS_r}{dr}\big)^2 \;+\; \frac{1}{r^2}\big(\frac{dS_\theta}{d\theta}\big)^2 \;+\; \frac{1}{r^2\sin^2\theta}\big(\frac{dS_\phi}{d\phi}\big)^2\Big) + \frac{GMm}{r}\,. \qquad 10.11.6$$

The left hand side is a function of t alone, and the right hand side is a function of r , θ and ϕ. This is possible only if each side is a constant independent of r , θ , ϕ and t. Let us call this constant α_1. Then integration of the left hand side gives

$$S_t(t) \;=\; \alpha_1 t \;+\; C_1\,. \qquad 10.11.7$$

In a similar manner we can isolate $\frac{\partial S_\phi}{\partial\phi}$ from equation 10.11.6, so that it is equal to a function of the other variables and therefore it, too, is a constant and independent of the other variables. (This is a seeming contradiction – but "constant" is a special case of a function, and indeed is the only function that will satisfy the condition that a function of one variable equals a function of another.). Therefore, on integrating with respect to ϕ, we obtain

$$S_\phi(\phi) \;=\; \alpha_2\phi \;+\; C_2\,. \qquad 10.11.8$$

We are now left with

$$\alpha_1 \;=\; -\frac{1}{2m}\Big(\big(\frac{dS_r}{dr}\big)^2 \;+\; \frac{1}{r^2}\big(\frac{dS_\theta}{d\theta}\big)^2 \;+\; \frac{1}{r^2\sin^2\theta}\alpha_2^2\Big) + \frac{GMm}{r}\,. \qquad 10.11.9$$

If we multiply through by r^2, we can then separate r and θ. Thus we find that

$\big(\frac{dS_\theta}{d\theta}\big)^2 + \alpha_2^2\csc^2\theta$ is equal to a function of r, and therefore both must be equal to a constant, which we'll call α_3^2 , and so we get

$$S_\theta(\theta) = \int_0^\theta (\alpha_3^2 - \alpha_2^2\csc^2\theta)^{1/2}\,d\theta. \qquad 10.11.10$$

We are now left with

$$\alpha_1 r^2 \;=\; -\frac{1}{2m}\Big(r^2\big(\frac{dS_r}{dr}\big)^2 \;+\; \alpha_3^2\Big) + GMmr\,. \qquad 10.11.11$$

Thus we find that

$$S_r(r) \;=\; \int_{r_1}^{r} \frac{1}{r}(2GMm^2 r - 2m\alpha_1 r^2 - \alpha_3^2)^{1/2}\,dr. \qquad 10.11.12$$

Here r_1 is a lower bound to r (perihelion) being the lower solution of

$$2GMm^2r - 2m\alpha_1 r^2 - \alpha_3^2 = 0. \qquad 10.11.13$$

Thus we have found a solution of the Hamilton-Jacobi equation 10.11.5, which contains four arbitrary constants α_0 , α_1 , α_2 and α_3 , where α_0 incorporates C_1 , C_2 and r_1.

Now, in order to find the solution to the hamiltonian equations of motion, we need to solve the equations $\frac{\partial S}{\partial \alpha_1} = \beta_1$, $\frac{\partial S}{\partial \alpha_2} = \beta_2$ and $\frac{\partial S}{\partial \alpha_3} = \beta_3$.

i. $$\beta_1 = \frac{\partial S}{\partial \alpha_1} = t + \frac{\partial}{\partial \alpha_1}\int_{r_1}^{r} \frac{1}{r}(2GMm^2r - 2m\alpha_1 r^2 - \alpha_3^2)^{1/2}\,dr. \qquad 10.11.14$$

When $r = r_1$, $\beta_1 = t$. Thus we immediately identify β_1 with T, the time of perihelion passage.

Now let us put $r_1 = a(1-e)$ and $r_2 = a(1+e)$, so that $r_1 + r_2 = 2a$ (definition of ellipse!) and $r_1 r_2 = a^2(1-e^2)$. But the sum and product of the roots of the quadratic equation 10.11.13 are GMm/α_1 and $\alpha_3^2/(2m\alpha_1)$ respectively. Thus we can identify α_1 and α_3 with

$$\alpha_1 = \frac{GMm}{2a} \qquad 10.11.15$$

and

$$\alpha_3^2 = GMm^2a(1-e^2). \qquad 10.11.16$$

ii. $$\beta_2 = \frac{\partial S}{\partial \alpha_2} = \phi + \frac{\partial}{\partial \alpha_2}\int_0^{\theta}(\alpha_3^2 - \alpha_2^2\csc^2\theta)^{1/2}\,d\theta. \qquad 10.11.17$$

If we choose the direction of the x-axis such that $\phi = \Omega$ when $\theta = 0$, then we must identify β_2 with Ω and θ with $90^\circ - i$ when the integrand is zero.

Therefore we see that

$$\alpha_2^2 = GMm^2a(1 - e^2)\cos^2 i. \qquad 10.11.18$$

There remains one more integration constant, which is quite arbitrary, and we can choose it such that $\beta_3 = \omega$.

The six parameters α_1 , α_2 , α_3 , β_1 , β_2 , β_3 that can be used to characterise an orbit are related, then, to the more familiar orbital elements by

$$\alpha_1 = \frac{GMm}{2a} \qquad 10.11.19$$

$$\alpha_2^2 = GMm^2a(1 - e^2)\cos^2 i \qquad 10.11.20$$

$$\alpha_3^2 = GMm^2a(1 - e^2) \qquad 10.11.21$$

$$\beta_1 = T \qquad 10.11.22$$

$$\beta_2 = \Omega \qquad 10.11.23$$

$$\beta_3 = \omega. \qquad 10.11.24$$

Conversely:

$$a = \frac{GMm}{2\alpha_1} \qquad 10.11.25$$

$$e = \sqrt{1 - \frac{2\alpha_1\alpha_3^2}{G^2M^2m^3}} \qquad 10.11.26$$

$$i = \cos^{-1}(\alpha_2 / \alpha_3) \qquad 10.11.27$$

$$T = \beta_1 \qquad 10.11.28$$

$$\Omega = \beta_2 \qquad 10.11.29$$

$$\omega = \beta_3 \qquad 10.11.30$$

Thus an orbit can be characterized by stating the values of the α_i and β_i just as easily as by stating the conventional elements a, e, i, Ω, ω, T. At this stage there may seem little point in doing so, though there is no actual difficulty in doing so. In Chapter 14, however, we shall make good use of these parameters.

Likewise, those who are familiar with hamiltonian mechanics are accustomed to describing a system in terms of the canonical variables (q_i , p_i) (generalized coordinates and momenta). But we might also sometimes want to describe the system with some other choice of variables, which we'll call (Q_i , P_i). These of course will be related to the canonical variables (q_i , p_i) by some sort of transformation. A transformation of the form

$$p_i = \frac{\partial S}{\partial q_i}, \qquad P_i = -\frac{\partial S}{\partial Q_i}, \qquad 10.11.31\text{a,b}$$

where

$$S = S(q_i , Q_i , t) \qquad 10.11.32$$

is a *contact transformation.*

Now if the original (q_i , p_i) obey the hamiltonian equations 10.11.1a,b, then it is easy to see that the Q_i , P_i satisfy the equations

$$\frac{\partial K}{\partial P_i} = \dot{Q}_i \qquad \text{and} \qquad \frac{\partial K}{\partial Q_i} = -\dot{P}_i , \qquad 10.11.33\text{a,b}$$

where
$$K = H + \frac{\partial S}{\partial t} \cdot \qquad 10.11.34$$

The Hamilton-Jacobi theorem amounts to making a suitable (contact) transformation – that is, choosing a suitable combination of the elements and the coordinates – such that the hamiltonian is zero. In fact, if we identify Q_i with α_i and P_i with $-\beta_i$, the equation $P_i = -\frac{\partial S}{\partial Q_i}$ is just $\beta_i = \frac{\partial S}{\partial \alpha_i}$.