1 The experimental basis of quantum theory

This introduction to quantum physics covers some very simple yet far reaching ideas:

- the discovery of **new particles** (the electron and the photon) that require
- **new physical laws** which are
- not related to "classical" physics.
- A new **quantum theory** is required.

In 1878 classical physics reached its zenith with the publication of Maxwell's equations describing all electromagnetic phenomena currently known. Consider Coloumb's law (part of Maxwell's equations) describing the force between two charged particles:

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0}\frac{1}{r^2}. \qquad (1)$$

Although Coulomb's law (and the rest of Maxwell's equations) describe the physics of charged objects – Q_1 and Q_2 – there is no limit in the theory on the smallest unit of charge: Q can take any value from some very large negative to positive number. The experimental demonstration that the electron appears to represent some fundamental charge unit with a quantised (i.e. discrete, not continuous) charge was the first indication that Maxwell's electromagnetic theory did not provide a complete description of electromagnetism as encountered in nature.

When manipulated, Maxwell's equations state that accelerated charged particles produce electromagnetic waves that propagate with a wavespeed c (derived from fundamental electromagnetic constants) – this is the fundamental physical nature of light. However, Einstein demonstrated that, in order to understand the photo–electric effect, light must exist in finite packets – photons – with a fixed relationship between energy and frequency. These effects (and others) were unforeseen within Maxwell's equations and a new theory was required – a quantum theory.

Within this topic, we will discuss the experiments (and their interpretation) that indicate – at the smallest physical scales (individual electrons and photons) – that charge and light come in individual packets of fixed properties: quanta.

- Discovery of the electron and the quantum of charge, e.
- Blackbody radiation: $E = hf$ but is this just a mathematical fix?
- The photo-electric effect – a direct link between photons and electrons at the quantum level.
- X-rays and high-energy scattering from electrons: the Compton effect.

1.1 The discovery of the electron: cathode rays and charge quantisation (T–Rex p.85)

Cathode rays were known from various experiments in electromagnetism (EM) as an emission of unknown nature produced from cathode (negative plate) of a pair of metal plates over which a large potential (voltage difference) exists. See T–Rex Figures 3.1 to 3.3. Cathode rays were demonstrated to be electrons (negatively charged particles) with a fixed quantum of charge as a result of two experiments:

- J.J. Thomson – by observing the deflection of cathode-rays under the influence of an electric and a magnetic field, Thomson demonstrated that they behave as a charged particle – **the electron**. In addition he measured the ratio of the electron charge to its mass – q/m.

- R.A. Millikan – by measuring the mass of charged oil droplets suspended in an electric field, Millikan observed that the charge on each drop was not distributed continuously. Instead, charge came in small packets – quanta – that were always an integer multiple of 1.6×10^{-19} Coulombs – thus $n = 1$ corresponds to the charge of the electron.

1.1.1 Thomson's experiment: determining q/m

See T–Rex Figures 3.2 and 3.3. Thomson directed a collimated beam of cathode rays (electrons) through an electric field – created by applying a voltage across a close pair of metal plates. The force on the electron beam as it passes through the electric field is:

$$F_y = ma_y = qE, \tag{2}$$

where m is the electron mass, q is the charge, E is the field strength (aligned along y and a_y is the resulting acceleration. The scattering angle θ is the ratio of the electron velocity components:

$$\tan\theta = \frac{v_y}{v_x} = \frac{a_y t}{v_0} = \frac{qE}{m}\frac{l}{v_0^2}, \tag{3}$$

where l is the length over which the electric field operates and v_0 is the original velocity of the electrons in the beam – **currently unknown**. So we measure θ and control E and l – how do we measure v_0?

A charged object moving through a magnetic field experiences a force according to the formula $\vec{F} = q\vec{v} \times \vec{B}$ (where the arrows denote vectors). Therefore, Thomson orientated the magnetic field to create a force opposing the force created by the electric field, i.e.

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} = 0, \tag{4}$$

or, because $\vec{v}$ and $\vec{B}$ are perpendicular,

$$\vec{E} = -\vec{v} \times \vec{B} \Rightarrow |\vec{E}| = |\vec{v}||\vec{B}|. \tag{5}$$

One then adjusts the strength of the B–field to generate zero deflection and the electron velocity, $v_0 = v_x = E/B$. We can substitute this back into Equation 3 to obtain a relationship for q/m

$$\frac{q}{m} = \frac{E\tan\theta}{B^2 l}. \tag{6}$$

Example: Consider a beam of electrons entering a Thomson apparatus as shown in T-Rex Figure 3.3. The deflection angle is measured to be 0.2 radians when $V = 200$ V, $l = 5$ cm, and $d = 1.5$ cm. If a perpendicular magnetic field of magnitude 5.5×10^{-4} T is applied simultaneously with the electric field then the particle passes through the plates without deflection. Calculate the value of q/m for the electron.

Using the above formula we have

$$\begin{aligned} \frac{q}{m} &= \frac{E\tan\theta}{B^2 l} \\ &= \frac{V\tan\theta}{B^2 d l} \quad \text{using } E = V/d \\ &= \frac{(200\text{ V})(\tan 0.2)}{(5.5\times10^{-4}\text{T})^2(0.015\text{ m})(0.05\text{ m})} \\ &= 1.76\times10^{11}\text{ C kg}^{-1}. \end{aligned} \tag{7}$$

The measured value of q/m for the electron (often referred to as e/m) is 1.76×10^{11} C kg^{-1}. This is about 2000 times larger than the previous record obtained for the Hydrogen atom and it was the first indication that the electron was either much smaller or much more highly charged than anything previously known.

1.1.2 Millikan's oil drop experiment

Having measured e/m for the newly discovered electron, the challenge was next to determine its charge explicitly. The idea behind Millikan's experiment is very simple – see T–Rex Figure 3.4. A charged oil droplet will normally fall under the influence of gravity. However, by applying an electric field a force can be generated to oppose gravity and maintain the charged droplet in a stationary position (much easier to measure), i.e.

$$q\vec{E} = -m\vec{g}. \tag{8}$$

As the magnitude of the electric field is $E = V/d$, where V is the voltage applied over a separation d, once you have a stable oil drop, the charge must be

$$q = \frac{mgd}{V}. \tag{9}$$

So how to measure the mass of the drop? Well, you know the density of the oil, so once you know the radius of the droplet, you can calculate the mass as $m = \frac{4}{3}\pi r^3 \rho$. OK, so how do we measure the radius? When permitted to fall freely under gravity, each droplet soon reaches terminal velocity – when the gravitational force is balanced by frictional forces encountered falling through the air – see T-Rex, Chapter 3, Problem 7 for an introduction to Stokes' law giving relationship between radius and terminal velocity for an oil droplet.

The results from a typical realisation of Millikan's experiment are shown in T–Rex Figure 3.5. Though there is some experimental error associated with the charge measured (some droplets may be almost stationary but with some imperceptible movement), droplets appear to carry set values of charge such that $q = nq_0$, where n is an integer and $q_0 = -1.6 \times 10^{-19}$C.

Why are there a series of peaks? To answer this we have to consider how the droplets become charged. Millikan would atomize the oil in a chamber illuminated by X–rays. The X–rays would ionize air in the chamber, liberating electrons which occasionally become associated with individual oil drops. Negatively charged ions are attracted to the anode in T–Rex Figure 3.4 and some fall through the gap into the observation chamber. The fact that oil drops may carry one or more electrons just reflects the fact that they are bumping around randomly in the ionization chamber.

Therefore, the charge peak at $n = 1$ corresponds to an oil drop carrying one electron of charge equal to $-e = -1.6 \times 10^{-19}$C.

Once the electron charge was known the electron mass could be computed as $m_e = 9.109 \times 10^{-31}$ kg – this is approximately 2,000 times less massive than the proton (known at the time only as H^+). A new, fundamental unit of nature had been discovered.

1.2 Blackbody radiation: Hinting at the existence of the photon (T–Rex p.97)

A Blackbody simply refers to an object that absorbs and emits radiation very effectively. However, why should *all* blackbodies emit radiation according to the same observed laws? Planck demonstrated that to explain this theoretically, the emitted (and absorbed) light had to be organized into "light quanta" – later known as photons: packets of radiation whose energy is strictly related to their frequency. Planck's theoretical spectrum is referred to as Planck's radiation law and introduces the **Planck constant**.

The interaction of light and matter is one of the cornerstones of quantum physics. To really get to grips with the *emission* and *absorption* of light from an object, we have to be able to discount light that is simply *reflected* – i.e. most of what we see around us. For this reason we consider radiation from a **Blackbody** as a good example to study.

A **Blackbody** is an object that absorbs all of the radiation incident upon it. If a Blackbody is in thermal equilibrium with its surroundings then it must also re-emit all of the radiation that it absorbs. So one definition of a Blackbody is that it is a highly efficient absorber and emitter of radiation.

Blackbodies aren't simply an abstract idea – they exist all around us in nature. T–Rex Figure 3.8 gives an example of a radiation cavity as a Blackbody but it is also a pretty good approximation to a blast furnace. The Sun emits radiation like a Blackbody, as do the planets (there is some reflection from the planets but not much). Even a log burning on a fire has the colour it does because it emits radiation as a Blackbody.

All blackbodies emit radiation according to the same experimentally determined laws. Spectral intensity is defined as the total power (energy per unit time) radiated per unit area per unit wavelength. The spectral intensity of a Blackbody is a function of wavelength and temperature *only* – $l(\lambda, T)$. Examples of the spectral intensity as a function of wavelength of blackbodies of different temperatures are shown in T–Rex Figure 3.9. The two main features of the spectral intensity of a Blackbody emitter are as follows:

- The wavelength of peak emission shifts to shorter (bluer) wavelengths as the temperature of the Blackbody increases. This is referred to as **Wien's displacement law** and takes the form:

$$\lambda_{max} T = 2.898 \times 10^{-3} \text{ mK}. \tag{10}$$

  Even on its own this is a very useful law: One can observe the spectral intensity of the Sun and infer the temperature of the photosphere (T-Rex Example 3.5); by observing the spectrum of the Cosmic Microwave Background we can measure the "temperature" of the universe; you could even observe the emission from an iron bar and work out whether it was safe to pick up!

- The total power (over all wavelengths) increases as the temperature increases. This was determined experimentally by Stefan and derived theoretically by Boltzmann. The **Stefan–Boltzmann law** takes the form:

$$R(T) = \int_0^\infty l(\lambda, T) \mathrm{d}\lambda = \epsilon \sigma \mathrm{T}^4, \tag{11}$$

  where ϵ is the emissivity of the object ($\epsilon = 1$ for a perfect Blackbody) and σ is measured to be 5.6705×10^{-8} $\mathrm{Wm^{-2}K^{-4}}$.

T–Rex Example 3.5

The sun may be assumed to be a Blackbody. Its wavelength of peak emission is located at approximately 500nm. Calculate (1) the surface temperature of the sun and (2) the total power emitted by the sun.

1. We can write Wien's displacement law in terms of nanometres

$$(500\text{nm}) \times T_{sun} = 2.898 \times 10^{-3}\ \text{mK} \times \frac{10^9\,\text{nm}}{\text{m}}$$

$$T_{sun} = \frac{2.898 \times 10^6}{500}\ \text{K} = 5800\ \text{K} \tag{12}$$

2. The total emitted power per unit area is

$$R(T) = \sigma T^4 = 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\text{K}^4}\,(5800\ \text{K})^4$$

$$= 6.42 \times 10^7\,\text{W/m}^2. \tag{13}$$

   The radius of the sun is $r = 6.96 \times 10^5$km and the surface area is $4\pi r^2$. The total emitted power from the sun is therefore

$$P_{sun} = 4\pi r^2 \times R(T)$$

$$= 4\pi (6.96 \times 10^8)^2\,\text{m}^2 \times 6.42 \times 10^7\,\text{W/m}^2$$

$$= 3.91 \times 10^{26}\,\text{W}. \tag{14}$$

1.2.1 Rayleigh–Jeans and the ultra–violet catastrophe!

OK, so if we know so much about Blackbody radiation, why haven't I shown you an equation describing the spectral intensity $l(\lambda, T)$? That's the problem – getting an equation that works.

To understand the problem we have to go back to some basic thermodynamics (heat sharing) and statistical mechanics (the statistics of large numbers of particles in a system). In a large system of interacting particles, the total energy contained in the system will tend to average out among the particles. So if there are n particles and a total energy E, the energy per particle will be E/n – this is what is meant by the **equipartition of energy**. Of course there will be fluctuations from this mean and these fluctuations follow the Maxwell distribution (i.e. the differential distribution of particles of given velocity or energy for a fixed temperature).

Similarities between the Blackbody spectrum and the Maxwell distribution led Rayleigh and Jeans to independently investigate whether energy equipartition and statistical mechanics could solve the mystery of the Blackbody spectrum. Their result is called the Rayleigh–Jeans formula and takes the form (T–Rex Figure 3.10):

$$l(\lambda, T) = \frac{2\pi ckT}{\lambda^4}, \tag{15}$$

where c is the speed of light and k is Boltzmann's constant. The only problem is that it doesn't work at all wavelengths.

Their theory saw the total energy emitted by a Blackbody shared over all the "vibrational modes" permitted within the previously mentioned radiation cavity. The vibrational modes are the result of thermal oscillations of atoms forming the walls of the cavity. By permitted vibrational modes we mean wavelengths of radiation, correctly thought to be electromagnetic waves, that will fit between the cavity – see Gamow Figure 3. The longest permitted wave is one which will just fit in the cavity and all shorter wavelengths are permitted. As you can arrange more shorter wavelengths inside the cavity than larger ones, the number of permitted vibrational modes tends to infinity at short wavelengths. If the energy is shared over all modes, the total emitted energy must be dominated by light at short wavelengths. Put another way the spectral intensity of the Rayleigh-Jeans Blackbody spectrum diverges at short wavelengths – **the ultra-violet catastrophe**.

1.2.2 The Planck radiation law and the quantum of light

What happens if we modify the assumptions used by Rayleigh and Jeans that led to the UV catastrophe? In their model light emitted with a given frequency f can take any energy value – just like a vibrating string can (within reason) have any amplitude. The total energy of the Blackbody is shared out over all possible frequencies (referred to above as vibrational modes) and, as the frequency distribution is dominated by the boundless distribution of short wavelengths, we get the UV catastrophe.

Planck suggested that light of a given frequency **cannot** be emitted with any given energy value. Light instead can only be emitted from a Blackbody in little energy packets – quanta – of energy $E = hf$, where h is some constant[1]. Therefore at any given frequency the total energy associated with all of the n emitters oscillating in that mode is nhf. If we apply energy equipartition to this version of the Blackbody emitter energy is still shared out equally over all frequencies. However, at high frequency (short wavelength), the number n of light packets that can each take their share of the energy hf gets smaller and smaller and tends to zero – **see Figure 1**. The Planck model of the Blackbody spectrum tends to zero spectral intensity at short wavelengths and averts the UV catastrophe.

The Planck radiation law takes the form:

$$l(\lambda, T)\,\mathrm{d}\lambda = \frac{2\pi c^2 h}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}\,\mathrm{d}\lambda, \tag{16}$$

[1]There is an important subtlety here: Planck dealt strictly with the case of light emitted from a Blackbody. Though he stated that light is only emitted or absorbed in finite packets, he did not attempt to extend this approach to light (electromagnetic radiation) in general. For that we have to wait for Einstein's explanation of the photo–electric effect (Section 1.3.3) – it was Einstein who generalised Planck's approach to state that the *light itself* (alternatively the EM field) and not the emitting "vibrational mode" was quantised.

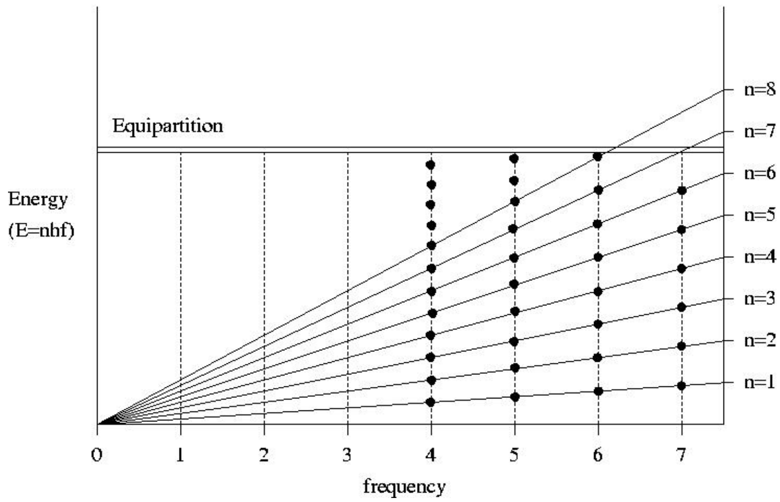

Figure 1: A cartoon description of how the quantisation of light into packets of energy $E = hf$ prevents the ultra–violet catastrophe.

where the intensity l describes the energy emitted per unit surface area of emitter per unit wavelength. The symbol h is called the **Planck constant**. If you set $h = 6.6261 \times 10^{-34}$ Js one obtains an *exact fit* to all observed Blackbody spectra. The quantum of light – the photon – had arrived!

Example: Express the Planck radiation law in terms of frequency, i.e. $l(f,T)\,\mathrm{d}f$, where wavelength and frequency are related by $c = f\lambda$.

One must insert values for both λ and $\mathrm{d}\lambda$ into the Planck radiation law, i.e.

$$\begin{aligned} \lambda &= \frac{c}{f} \\ \frac{\mathrm{d}\lambda}{\mathrm{d}f} &= -\frac{c}{f^2} \\ \mathrm{d}\lambda &= -\frac{c}{f^2}\,\mathrm{d}f \end{aligned} \tag{17}$$

Note that the minus sign just means that as $\mathrm{d}\lambda$ increases, $\mathrm{d}f$ decreases. Inserting these values into the Planck radiation law yields

$$l(f,T) = 2\pi c^2 h \left(\frac{f}{c}\right)^5 \frac{1}{e^{hf/kT} - 1} \left(\frac{c}{f^2}\right) \mathrm{d}f \tag{18}$$

$$= \frac{2\pi h f^3}{c^2} \frac{1}{e^{hf/kT} - 1}\,\mathrm{d}f. \tag{19}$$

1.2.3 From the Planck law to the Stefan–Boltzmann law

As we saw earlier, the Stefan–Boltzmann law for an ideal Blackbody states that the total emitted power $R(T) = \sigma T^4$. If Planck's radiation law is to stand up to further scrutiny, we should be able to derive the Stefan–Boltzmann law using Planck's law. We start with the definition of total emitted power

$$\begin{aligned} R(T) &= \int_0^\infty l(\lambda, T)\,\mathrm{d}\lambda \\ &= 2\pi c^2 h \int_0^\infty \frac{1}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda kT} - 1}\,\mathrm{d}\lambda. \end{aligned} \tag{20}$$

We make the following substitutions,

$$\begin{aligned} x &= \frac{hc}{\lambda kT}, \\ \mathrm{d}x &= -\frac{hc}{kT}\frac{\mathrm{d}\lambda}{\lambda^2}, \end{aligned} \tag{21}$$

and rearrange Equation 20 to obtain

$$\begin{aligned} R(T) &= -2\pi c^2 h \int_\infty^0 \left(\frac{kT}{hc}\right)^5 x^5 \frac{1}{e^x - 1}\frac{1}{x^2}\left(\frac{hc}{kT}\right)\mathrm{d}x \\ &= +2\pi c^2 h \left(\frac{kT}{hc}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}\,\mathrm{d}x. \end{aligned} \tag{22}$$

The integral terms has the value $\pi^4/15$ (T–Rex Appendix 7), generating

$$R(T) = \frac{2\pi^5 k^4}{15 h^3 c^2} T^4 = \sigma T^4, \tag{23}$$

where $\sigma = 5.67 \times 10^{-8}$ is confirmed by insertion of the numerical values for k, h and c. **Read T–Rex Examples 3.6 and 3.8 for further examples investigating the Planck law.**

1.3 The photo–electric effect: confirming the existence of the photon (T–Rex p.103)

In Sections 1.1 and 1.2 we have seen that both the electron and the photon (at least emission and absorption) must be described as quantum units with an associated quantum of charge and energy respectively. With the photo–electric effect we now investigate the interaction of the electrons and photons (albeit at relatively low energies). We will discuss the photo–electric effect from several perspectives:

- Quick reminder – how to liberate electrons from metal conductors. By the early 1900s it was known that valence electrons in metals move relatively freely within the conducting material. However, they are still bound, albeit loosely, and further techniques are required to liberate them from the metal surface:
  1. Thermionic emission – heating the metal allows electrons to escape (the method used by Thomson to produce electrons).
  2. Secondary emission – if another high–speed (high KE) particle strikes the metal, an electron can be ejected.
  3. Field emission – application of a sufficiently strong electric field will literally tear the electrons off the metal surface.
  4. Photo–electric effect – incident light will transfer energy to the electrons, permitting them to escape. **Classical EM theory cannot explain the photo–electric effect – only a quantum explanation works.**
- Experimental evidence.
- Problems with classical theory.
- Einstein's theory.

1.3.1 Experimental results of the photo–electric effect

The typical apparatus used to investigate the photo–electric effect is shown in T–Rex Figure 3.11: light illuminates the cathode plate of a pair of metal plates (the emitter) and liberated photo–electrons strike the anode plate (the collector). The photo–current I thus generated is registered on an ammeter and a voltage V – either positive (accelerating) or negative (retarding) – can be applied across the plates to regulate the passage of electrons. The main observed results are:

1. The kinetic energy (KE) of liberated photo–electrons is *independent* of the light intensity (T–Rex Figure 3.12). The measured photo–current drops to zero when a voltage $-V_0$ is applied across the metal plates.

2. The maximum KE of the photo–electrons (measured by determining $-V_0$ is determined *solely* by the frequency of incident light (T–Rex Figure 3.13).

3. The *smaller* the work function – the binding energy of the electron to the material (T–Rex Table 3.3) – of the conducting metal used for the cathode, the *smaller* is the threshold frequency required to liberate photo–electrons (T–Rex Figure 3.14). No electrons are liberated when the frequency of incident light is smaller than the threshold frequency.

4. When the frequency of incident light exceeds the threshold frequency and the applied voltage remains constant, the number of photo–electrons produced (measured by the photo–current) is proportional to the intensity of incident light (T–Rex Figure 3.15).

5. Photo–electron emission occurs almost instantly ($\leq 3 \times 10^{-9}$s) after the cathode is illuminated, regardless of the light intensity.

1.3.2 Problems with classical theory

Classical electromagnetic theory allows electromagnetic radiation to liberate photo–electrons from conducting metals. However a number of problems arise when classical theory is applied to the photo–electric effect.

- The KE of a classical EM wave is proportional to its intensity. However, experimental evidence shows that the KE of liberated electrons is *independent* of the intensity of incident radiation and instead depends upon the frequency of incident light (Point 1). The same is true for the maximum KE of liberated photo–electrons (Point 2), classical theory predicts this to be a function of light intensity and not (as observed) frequency.

- The experimental relationship between work function and threshold frequency (Point 3) is completely inexplicable within classical EM theory. However, Point 4 – more intensity equals more photo–electrons – is in agreement with classical theory.

- Classical theory allows energy from electromagnetic radiation to "trickle into" valence electrons in conductors (see T–Rex Example 3.10). When enough energy is deposited, the electron will be ejected. Therefore, light of very small frequency and intensity should eventually deposit sufficient energy to eject a photo–electron. This is not observed: when even exceptionally small intensities of light (of the threshold frequency or greater) illuminate a cathode plate, photo–electrons are almost instantly detected.

1.3.3 Einstein's quantum explanation of the photo–electric effect

Einstein went one step beyond Planck's view that light observed from a Blackbody was the result of a quantized absorption and emission mechanism: Einstein stated that *all light* exists as quantized packets of energy. His original statement provides an excellent summary:

The energy of a light ray spreading out from a point source is not distributed continuously over an increasing space but consists of a finite number of energy quanta which are localized at points in space, which move without dividing, and which can only be produced and absorbed as complete units.

What Einstein called "light quanta" are now termed **photons**. Each photon is associated with an energy

$$E = hf, \tag{24}$$

where f is the frequency of the EM wave and h is Planck's constant. (Recall that wavelength and frequency are related by $c = f\lambda$.)

Einstein was suggesting that light – in addition to behaving like a wave – also behaved like a discrete particle. As we shall see in Lecture 3, resolving the problem of wave–particle duality led to the development of quantum wave mechanics (e.g. Schroedinger's Equation). However, we must first investigate how Einstein's suggestion provided a consistent explanation for the photo–electric effect.

Einstein suggested that when a photon strikes the metal conductor, it transfers its **total energy** hf to a **single electron**. A photon cannot give up some fraction of its energy and remain the same photon. If the electron is to escape the conductor and be detected it must overcome the binding energy ϕ of the material – energy which is lost from the KE of the electron. The escaping electron may interact with other electrons on its route out of the conductor, losing more energy in the process. However, in the special case where no additional interactions take place we can restrict ourselves to using the maximum KE of the electron. Conservation of energy requires:

$$\begin{aligned} \text{Energy before (photon)} &= \text{Energy after (electron)} \qquad (25) \\ hf &= \phi + \frac{1}{2}\, mv_{max}^2 \end{aligned}$$

In this picture, the point at which the retarding potential V_0 just stops the liberated photo–electrons from reaching the anode must indicate the maximum KE of the electrons, i.e.

$$eV_0 = \frac{1}{2}\, mv_{max}^2. \tag{26}$$

So how does Einstein's theory explain the observed properties of the photo–electric effect?

- Points 1 and 2 state that the measured KE of liberated photo–electrons are independent of the light intensity but are a clear function of the frequency of incident light for a given material work function (ϕ). This is exactly what we see in Equations 25 and 26.
- If we re–write Equation 25 as

$$eV_0 = \frac{1}{2}\, mv_{max}^2 = hf - hf_0, \tag{27}$$

we can associate directly the threshold frequency f_0 with the work function of the material ϕ – thus understanding Point 3.

- In addition, if we interpret light as a collection of energy packets then intensity has units of number of photons falling per unit area per unit time. Therefore, once the threshold frequency has been crossed, more incident photons equals more photo–electrons which equals more measured current – Point 4.
- Finally, we can re–write Equation 27 to obtain

$$eV_0 = h(f - f_0), \tag{28}$$

which states that the "stopping voltage" (alternatively $\mathrm{KE}_{\mathrm{max}}$) should be a linear function of frequency f above the threshold frequency f_0. This relationship was finally confirmed experimentally by Millikan in 1916 with a linearity constant h almost exactly that used by Planck to explain the Blackbody radiation spectrum (see T–Rex Figure 3.16) – the quantum theory of light had been confirmed.

T–Rex Example 3.11

Light of wavelength 400 nm is incident upon lithium ($\phi = 2.93$eV). Calculate 1) the photon energy and 2) the stopping potential V_0.

1. It is fairly straightforward to rewrite the photon energy in terms of wavelength.

$$\begin{aligned} E &= hf = \frac{hc}{\lambda} \\ &= \frac{(6.626 \times 10^{-34}\mathrm{Js})(2.998 \times 10^{8}\mathrm{ms}^{-1})}{\lambda(1.602 \times 10^{-19}\mathrm{J\ eV}^{-1})(10^{-9}\mathrm{m\ nm}^{-1})} \\ E &= \frac{1.240 \times 10^{3}\mathrm{eV\ nm}}{\lambda} \\ E &= \frac{1.240 \times 10^{3}\mathrm{eV\ nm}}{400\ \mathrm{nm}} = 3.10\ \mathrm{eV}. \end{aligned}$$

2. For the stopping potential we can write

$$eV_0 = hf - \phi = E - \phi = 3.10\ \mathrm{eV} - 2.93\ \mathrm{eV} = 0.17\ \mathrm{eV}, \tag{29}$$

therefore

$$V_0 = 0.17\ \mathrm{V}. \tag{30}$$

1.3.4 X–ray production and the inverse photo–electric effect

Physical processes are often reversible. The inverse of the photo–electric effect sees a moving electron emit a photon (or a series of photons) as it is decelerated in the electric field of an atom. Such "breaking radiation" – Bremsstrahlung – is a common method for producing X–rays.

The typical apparatus for producing bremsstrahlung X–ray emission is shown in T–Rex Figure 3.18. High velocity electrons are directed towards an anode target (e.g. tungsten, molybdenum or chromium – a high melting point is the key) where the electrons lose kinetic energy via Coulomb interactions with the atomic nuclei and liberating the energy as photons.

Photons must be created or destroyed as complete, individual units. However, an electron can lose some fraction of its KE and still exist as the same electron. Its energy is liberated via a series of emitted photons as it decelerates around the nucleus. In the extreme case where the electron comes to a stop around the nucleus, all of its KE can be used to create a single, high-energy photon.

Typical X-ray emission spectra are shown in T–Rex Figure 3.19. In each case the continuum emission is due to bremsstrahlung. Narrow line features are due to the incident electron ejecting a tightly bound atomic electron – the line is created as an electron re–occupies the vacant position, emitting the binding energy (negative) as an X–ray photon.

The minimum wavelength (λ_{min}) of the bremsstrahlung spectrum corresponds to the case mentioned above where the incident electron comes to a complete stop about the atomic nucleus. In this case the total KE of the electron is converted into a single photon:

$$\begin{aligned} \mathrm{KE}_{\mathrm{electron}} &= eV_0 = hf_{max} = \frac{hc}{\lambda_{min}} \quad \text{or} \\ \lambda_{min} &= \frac{hc}{e}\frac{1}{V_0}, \end{aligned} \tag{31}$$

which is independent of the target material used. This relationship (first known experimentally) is known as the **Duane–Hunt limit**. We will come back to X-ray emission in Lecture 3.

1.4 A summary of photon–electron interactions

Photons interact with matter in different ways:

- The photo–electric effect: An incident photon can transfer all its energy to an electron – disappearing in the process – and possibly liberating the excited electron from its parent material.
- Thomson scattering was understood using classical EM theory – an incoming EM wave causes an oscillating motion in the target electron which in turn emits a scattered EM wave of the same frequency at some new angle.
- Compton scattering: In addition to Thomson scattering, energetic photons (X–rays) will scatter relativistically from target electrons. The Compton scattered photon is shifted to a longer wavelength than the incident photon. Compton scattering can only be explained by taking light as discrete particle of energy $E = hf$ and using the relativistic formula for the electron energy–momentum – thus confirming both the photon as a quantum unit and the theory of Special Relativity.
- Pair production and annihilation (T–Rex Section 3.9).
- One could also include spectral line emission and absorption in this list – see Lecture 2. However, at this point the physics of the interaction was not understood as an electron–photon interaction.

1.5 Compton scattering

Compton correctly interpreted the wavelength shift and extreme scattering angles viewed in his experiments as the result of a relativistic collision between a particle of light – the photon – and an electron. The typical geometry of a Compton scattering event is shown in T–Rex Figure 3.20. By viewing the event as an elastic collision, the momentum and energy of both the incident photon plus target electron and the scattered photon plus recoil electron must be conserved. We use the following relativistic expressions for momentum and energy:

$$\text{Photon momentum: } p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda} \tag{32}$$

$$\text{Total electron energy: } E_e^2 = \left(mc^2\right)^2 + p_e^2 c^2 \tag{33}$$

The initial and final energy and momentum values for the photon and electron are summarised in Table 1 below:

Energy or momentum	Initial	Final
Photon energy	hf	hf'
Photon momentum in x–direction	$\frac{h}{\lambda}$	$\frac{h}{\lambda'} \cos\theta$
Photon momentum in y–direction	0	$\frac{h}{\lambda'} \sin\theta$
Electron energy	mc^2	$E_e^2 = (mc^2)^2 + p_e^2c^2$
Electron momentum in x–direction	0	$p_e \cos\phi$
Electron momentum in y–direction	0	$p_e \sin\phi$

Table 1: Results of Compton scattering.

Inserting these values into the conservation laws we can write:

$$\text{Energy} \quad hf + mc^2 = hf' + E_e \tag{34}$$

$$p_x \quad \frac{h}{\lambda} = \frac{h}{\lambda'}\cos\theta + p_e\cos\phi \tag{35}$$

$$p_y \quad \frac{h}{\lambda'}\sin\theta = p_e\sin\phi \tag{36}$$

The aim of the analysis is to relate wavelength change $\Delta\lambda = \lambda' - \lambda$ to the photon scattering angle θ. By squaring and adding Equations 35 and 36 we obtain

$$p_e^2 = \left(\frac{h}{\lambda}\right)^2 + \left(\frac{h}{\lambda'}\right)^2 - 2\left(\frac{h}{\lambda}\right)\left(\frac{h}{\lambda'}\right)\cos\theta. \tag{37}$$

We next substitute expressions for E_e (Equation 34) and p_e (Equation 37) into Equation 33

$$\left[hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) + mc^2\right]^2 = (mc^2)^2 + \left(\frac{hc}{\lambda}\right)^2 + \left(\frac{hc}{\lambda'}\right)^2 - 2\left(\frac{hc}{\lambda}\right)\left(\frac{hc}{\lambda'}\right)\cos\theta. \tag{38}$$

Expanding the l.h.s. of the above equation and canceling terms leaves

$$mc^2\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = \frac{hc}{\lambda\lambda'}(1 - \cos\theta). \tag{39}$$

Finally, rearranging the above terms to express the wavelength shift associated with the Compton scattered photon leaves

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{mc}(1 - \cos\theta). \tag{40}$$

The above formula correctly predicts the wavelength shift of Compton scattered photons and the associated analysis of the recoil electron successfully describes the electron's energy and momentum. The combination of physical constants $\lambda_c = h/mc = 2.426 \times 10^{-3}$ nm is referred to as the **Compton**

wavelength of the electron. Therefore the wavelength shift caused by Compton scattering will only correspond to a significant fraction of the incident photon wavelength for photons with $\lambda \leq \lambda_c$, i.e. X–rays and γ–rays. We are now in a position to describe the physical processes behind Compton and Thomson scattering:

- An incident high energy photon has an energy equal to many times the binding energy of all but the most tightly bound inner electrons. It scatters elastically from the essentially free electron with a modified wavelength – Compton scattering (what is the fate of the electron?).

- A much lower energy photon, or a photon that scatters off an inner electron (higher binding energy), effectively scatters from the electron plus nucleus. The Compton wavelength of the atom is thousands of times larger than the electron alone and the resulting scattering occurs with $\Delta\lambda \approx 0$ – Thomson scattering.

T–Rex Example 3.16

An X–ray photon of wavelength 0.05nm scatters from a gold target. (1) Can the X–ray photon be Compton scattered from an inner electron bound by 62 keV? (2) When scattering off a loosely bound outer electron, what is the largest wavelength of the scattered photon that can be observed? (3) What is the kinetic energy of the most energetic recoil electron and at what angle does it occur?

1. The energy of the incident X–ray photon is

$$E_X = \frac{1.240 \times 10^3 \,\mathrm{eV\,nm}}{0.05\,\mathrm{nm}} = 24.8\,\mathrm{keV}. \tag{41}$$

   Therefore, if the scattering occurs from a tightly bound inner shell electron, we must use the mass of the electron plus nucleus. The expected wavelength shift is small and the scattering is equivalent to Thomson scattering.

2. However, the X–ray photon may Compton scatter from an outer electron whose binding energy is small compared to the X-ray photon energy. In this case the scattered photon wavelength is maximised when $\Delta\lambda$ is maximised, i.e. $\cos\theta = 180°$. In this case we have

$$\begin{aligned} \lambda' &= \lambda + \frac{h}{m_e c}(1 - \cos 180°) = \lambda + \frac{2h}{m_e c} \\ &= 0.05\,\mathrm{nm} + 2 \times 0.00243\,\mathrm{nm} = 0.055\,\mathrm{nm}. \end{aligned} \tag{42}$$

3. The kinetic energy of the recoil electron equals the energy difference between the incident and scattered photon. This value is maximised for the case $\theta = 180°$, i.e. $\phi = 0°$. We may write

$$\mathrm{KE_{electron}} = E_X - E'_X$$

$$= 24.8\,\text{keV} - \frac{1.240 \times 10^3\,\text{eV nm}}{0.055\,\text{nm}}$$

$$= 24.8\,\text{keV} - 22.5\,\text{keV} = 2.3\,\text{keV}. \quad (43)$$

1.6 Pair production and annihilation

We have seen examples of how photons can be converted into electron kinetic energy via the photo–electric effect and Compton scattering – and how electrons can convert some fraction of their kinetic energy into light via Bremsstrahlung and inverse Compton scattering. However, we know from Special Relativity that energy and mass are equivalent. Therefore, we expect reactions where highly energetic photons are converted directly into matter and vice versa.

The rest mass–energy of an electron is 511 keV/c^2. However, a single photon cannot convert its energy into a single electron – charge would not be conserved.

In 1932 the positron (e^+) was observed in cosmic ray experiments[2]. Positrons are also observed when high energy gamma ray beams are directed at matter.

The conversion of a photon into an electron–positron pair is referred to as **pair production**, i.e.

$$\gamma \rightarrow e^+ + e^-. \quad (44)$$

Charge is conserved in this process but what about energy and momentum (see T–Rex Figure 3.22)? If we represent electron and positron values by subscripts "−" and "+" respectively we may write down the following conservation laws for pair production in free space as

$$\text{Energy} \quad hf = E_+ + E_- \quad (45)$$

$$\text{Momentum}, p_x \quad \frac{hf}{c} = p_- \cos\theta_- + p_+ \cos\theta_+ \quad (46)$$

$$\text{Momentum}, p_y \quad 0 = p_- \sin\theta_- + p_+ \sin\theta_+ \quad (47)$$

We can re–write the x–momentum as

$$hf = (p_-c)\cos\theta_- + (p_+c)\cos\theta_+, \quad (48)$$

which has a maximum value of

$$hf = p_+c + p_-c. \quad (49)$$

[2]Cosmic rays are highly energetic particles created via unknown processes in the universe. When they interact with particles in the atmosphere (or cloud chambers) they create cascades of exotic particles.

However, we know that $E^2 = \left(mc^2\right)^2 + p^2c^2$, substituting this into Equation 45, we obtain

$$hf = \sqrt{p_+^2c^2 + m^2c^4} + \sqrt{p_-^2c^2 + m^2c^4}, \text{ or} \quad (50)$$

$$hf > p_+c + p_-c. \quad (51)$$

Apparently energy and momentum do not add up. The momenta of the electron–positron pair is less than the incident photon momentum. The key is that pair production cannot take place in "free space". Pair creation can only take place in the vicinity of an additional mass – typically an atomic nucleus (see T–Rex Figure 3.22). The minimum energy condition for pair production is

$$hf > 2m_ec^2 = 1.022\,\text{MeV}. \quad (52)$$

The opposite reaction – **annihilation** – can take place in free space

$$e^+ + e^- \rightarrow \gamma + \gamma. \quad (53)$$

Momentum is conserved as two photons are produced. Energy conservation sets the mimimum photon energy produced as

$$2m_ec^2 = 2hf \Rightarrow hf = 0.511\,\text{MeV}. \quad (54)$$

1.7 Summary of Lecture 1

- The electron is tiny particle that carries an apparently fundamental unit of charge $-e$. This quantum charge unit $e = 1.6^{-19}$C.
- Electrons can be liberated from metal atoms (and other elements) via a number of processes. However, the fact that atoms could be divided into electrons and positive ions challenged the view that atoms were indivisible units of nature.
- The logical extension of this point is that, if neutral atoms contain electrons – apparently the number of electrons and the atomic number featuring on the Periodic Table are equal – where does the equal and opposite quantity of positive charge reside? This point is continued in the next lecture.
- In classical EM theory, light is a wave phenomenon. This is confirmed experimentally via diffraction and interference experiments.
- However, Blackbody radiation can only be explained if light is emitted and observed in quantum packets of energy $E = h\nu$.

- Einstein went further and stated that light itself is not a purely wave phenomenon – in the photo–electric effect, light acts as a particle of energy $E = h\nu$ – the photon. Millikan eventually confirmed experimentally all aspects of Einstein's equations describing the photo–electric effect.

- Compton scattering provides further evidence that light acts as a particle not just with an energy $E = h\nu$ but with an associated momentum $p = E/c = h\nu/c$. This can be understood within Special Relativity only if photons have zero rest mass.