2 A Universe of galaxies

Stars exist in galaxies but galaxies are distributed throughout the Universe. Therefore, to understand questions such as the physical origins of galaxies, distances to galaxies and the ages of stars in galaxies one must consider the fundamental properties of the Universe in which they exist.

2.1 Geometry

General relativity describes the Universe as a four dimensional spacetime.

The relationship between matter/energy and geometry is provided by the Einstein field equation and the metric tensor.

$$R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = -8\pi\, G\, T_{\mu\nu}, \tag{1}$$

where $T_{\mu\nu}$ is the stress–energy tensor (i.e. matter and energy) and G is Newton's constant in relativistic ($c = 1$) units. We ignore the possible effect of Λ at present.

The most basic geometrical operation is to measure the distance between nearby points via the line element, e.g. Pythagoras's theorem. Within the spacetime defined by the Riemann tensor, the line element, ds, describing the infinitesimal distance between two nearby points is

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu. \tag{2}$$

Depending upon the exact definition of $g_{\mu\nu}$, the diagonal elements are $g_{00}, g_{11}, g_{22}, g_{33} = +1, -1, -1, -1$ for Euclidean or flat space (Index 1 refers to time and indices 2, 3 and 4 refer to space). Non–diagonal elements are zero in this case.

The Universe is isotropic on the largest detectable scales. Evidence for isotropy comes from the cosmic microwave background (CMB), they sky distribution of distant sources such as gamma ray bursts (GRBs) and, in a statistical sense, from the large scale distribution of galaxies.

The Copernican principle states that we occupy no special place in the solar system. The cosmological principle extends this to state that we occupy no special place in the Universe.

The implication of observed isotropy and the cosmological principle is that we exist in a homogeneous Universe.

The simplest metric describing a homogeneous and isotropic spacetime is the Robertson-Walker metric

$$\mathrm{d}s^2 = c^2\ \mathrm{d}t^2 - a(t)^2 \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\ (\mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\phi^2) \right], \tag{3}$$

where $a(t)$ is a time varying scale factor and k is a constant describing curvature.

2.2 The Friedmann equation

The Freidmann equation describes the time variation of the scale factor $a(t)$[1]. It can be derived by substituting the metric terms described by the RW line element into the EFQ. However, we will investigate a version of the Friedmann equation using a Newtonian analogue and Birkhoff's theorem. Consider a universe with coordinate distances defined by the RW line element. The universe is populated with galaxies with a space density ρ and pressure $P = 0$ (i.e. the galaxies do not interact with each other).

Consider a spherical volume of the universe of radius l and mass M. We further consider the dynamical behaviour of a test particle (a single galaxy if you like) of mass m located on the surface of this spherical shell. Birkhoff's theorem states that the mass within the sphere will act upon the test particle as if the entire mass M were concentrated at the centre of the sphere. From the Newtonian equation of motion of the test particle we therefore obtain

$$m \, \frac{\mathrm{d}^2 l}{\mathrm{d}t^2} = -\frac{GMm}{l^2}. \tag{4}$$

Multiplying the equation by $\dot{l}$ generates

$$\frac{\mathrm{d}}{\mathrm{d}t} \, \frac{\dot{l}^2}{2} = \frac{\mathrm{d}}{\mathrm{d}t} \, \frac{GM}{l}. \tag{5}$$

Integrating yields

$$\frac{\dot{l}^2}{2} - \frac{GM}{l} = E, \tag{6}$$

where the integration constant has units of energy. Note: this equation may be interpreted as a basic energy relation of the form, *Kinetic* + *Potential* = *Total*. The RW line element indicates that $l(t) = l_0 \, a(t)$, where l_0 is independent of time and $a(t)$ is the time–varying, universal scale factor. Therefore, we may write

$$\begin{aligned}
\frac{\dot{l}^2}{2} - \left(\frac{G}{l}\right)\left(\frac{4\pi l^3 \rho(t)}{3}\right) &= E \\
\frac{\dot{a}^2}{2} - \frac{4\pi G \rho(t) a^2(t)}{3} &= \frac{E}{l_0^2}, \\
\dot{a}^2 - \frac{8\pi G}{3} \, \rho(t) \, a^2(t) &= \frac{2E}{l_0^2}.
\end{aligned} \tag{7}$$

The result is one form of Friedmann's equation and it has three general solutions:

[1]To simplify the discussion of the scale factor and its history we (and others) set $a(t_0) = 1$ with $a(t)$ taking a value relative to the current epoch.

E < 0 : The potential term is proportional to $1/a(t)$ and always dominates (see Equation 6). Hence $a(t)$ cannot increase without limit, instead it must reach some maximum value (at which point $\ddot{a} < 0$) and decrease.

E $= 0$: $a(t)$ increases throughout time, tending toward (but never reaching) an asymptotic maximum scale as $t \to \infty$. The critical density value corresponding to this case is

$$\rho_c(t) = \frac{3}{8\pi G}\left(\frac{\dot{a}}{a}\right)^2 = \frac{3H^2(t)}{8\pi G}. \tag{8}$$

The current value of the Hubble parameter is $H_0 = H(t_0) = 70 \pm 7$ kms^{-1} Mpc^{-1} and the corresponding value of the critical density is $\rho_{c,0} = (9.2 \pm 1.8) \times 10^{-27}$ kg m^{-3} $= (1.4 \pm 0.3) \times 10^{11}$ M$_\odot$ Mpc^{-3}. Current estimates of the space density of galaxies are of order 1.4×10^{-2} Mpc^{-3}. If one assumes that the mass of a typical galaxy is 10^{11}M$_\odot$, then one concludes that visible galaxies contribute about 1% of the value of $\rho_{c,0}$.

E > 0 : $a(t)$ increases monotonically for all time. The universe expands forever.

At this point we should justify our statement that the Hubble parameter $H(t) = \dot{a}/a$. Recalling the Hubble law, $v = H_0 l$ and $l(t) = a(t) \times l_0$, we can write that $\mathrm{d}l/\mathrm{d}t = H_0 l(t)$ which is the same as $\mathrm{d}a/\mathrm{d}t = H_0 a(t)$ and that $H_0 = \dot{a}/a$ evaluated at $t = t_0$. In general, $H(t) = \dot{a}/a$.

Returning to the form of Friedmann's equation, one may redefine the constant of integration to be

$$\frac{2E}{l_0^2} = -kc^2. \tag{9}$$

Re-writing the Friedmann equation yields

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G}{3}\rho(t) = -\frac{kc^2}{a^2}. \tag{10}$$

Note that this form of the equation was that obtained by Friedmann (and re–discovered by Lemaitre) by inserting the corresponding elements of the metric tensor applicable to the RW line element into the EFQ. The parameter k takes one of three values

$$\begin{aligned} E < 0 &\Rightarrow k = +1 \\ E = 0 &\Rightarrow k = 0 \\ E > 0 &\Rightarrow k = -1. \end{aligned} \tag{11}$$

We have now obtained the complete link between the time dependence of the universal scale factor $a(t)$ and the curvature constant k first encountered in the RW line element. Proceeding further, one may redefine the Friedmann equation in terms of a new dimensionless density parameter

$$\Omega(t) = \frac{\rho(t)}{\rho_c(t)}. \tag{12}$$

If we divide the Friedmann equation (Equation 10) by $H(t)^2$ and use the above identity, we obtain

$$1 - \Omega(t) = -\frac{kc^2}{H^2a^2}, \quad \text{or}$$
$$\frac{kc^2}{H^2a^2} = \Omega(t) - 1. \quad (13)$$

This expression has the following consequences

$$\begin{aligned} k = +1 &\Rightarrow \Omega(t) > 1 \\ k = 0 &\Rightarrow \Omega(t) = 1 \quad \text{for all times,} \\ k = -1 &\Rightarrow \Omega(t) < 1 \end{aligned} \quad (14)$$

where

$$\begin{aligned} \Omega(t) > 1 &\Rightarrow \text{a closed universe,} \\ \Omega(t) = 1 &\Rightarrow \text{a spatially flat universe,} \\ \Omega(t) < 1 &\Rightarrow \text{an open or hyperbolic universe.} \end{aligned} \quad (15)$$

Therefore, the total matter/energy content of the universe determines the overall spatial geometry, the time variation of $a(t)$ and the ultimate fate of the universe[2] Importantly, these equations indicate that the universe has a well–defined characteristic matter density that marks the limit of each of the above cases. Therefore, the determination of the total matter content of the universe became an immediate challenge for early observational cosmologists. However, to determine this critical density, one requires $H(t)$ or H_0 – the present day value of the Hubble parameter.

2.3 H_0 and the age of the universe

In an expanding universe with low deceleration/acceleration, the quantity H^{-1} approximately defines the time taken for the distance between any two galaxies to double[3]. A more definite relationship between the value of the scale factor and time may be obtained from the Friedmann equation. **Note:** in the following discussion, $H_0 = H(t_0)$, where t_0 indicates the current epoch.

The Einstein–de Sitter universe (EdS): In 1932 Einstein and de Sitter postulated that, in the absence of secure observations to the contrary, the simplest assumptions governing the behaviour

[2]Given the time variation of $\Omega(t)$, one question that vexed cosmologists in the past was, "If Ω_0 is not equal to 1, why should it be anywhere close to 1 at the present epoch?". If $\Omega \neq 1$, why do we appear to be living in a special epoch?

[3]Following the Hubble law, let $v = H_0 \times l$ and let us observe the universe from this moment for an additional time $t = 1/H_0$ into the future. Any galaxy at a distance l from the observer will appear to travel an additional distance $v \times t = l$ in this time and will now lie at a distance $2 \times l$ from the observer.

of the Friedmann equation should be adopted, i.e. $\Lambda = k = P = 0$. Note that these assumptions refer specifically to a universe containing a non-interacting gas of galaxies. At the time when the EdS universe was suggested a radiation dominated universe had not been considered.

Mass conservation implies that $\rho \propto a^{-3}$, i.e.

$$\begin{aligned} \text{Mass} &= \rho \times V \\ \rho_0 a_0^3 l_0^3 &= \rho(t) a^3(t) l_0^3 \\ \rho(t) &= \rho_0 (a_0/a(t))^3 \\ \text{in general} \;\; \rho(t) &\propto a^{-3} \end{aligned}$$

One may then re-write the Friedmann equation in the form

$$\begin{aligned} a\,\dot{a}^2 &= \frac{8}{3}\pi G\rho\, a^3 \\ a\,\dot{a}^2 &= \text{constant} \\ a^{1/2}\,\mathrm{d}a &= A\,\mathrm{d}t \\ \frac{2}{3}\,a^{3/2} &= A\,t + C \quad (C = 0 \;\text{ as }\; \mathrm{a} \rightarrow 0 \;\text{ as }\; \mathrm{t} \rightarrow 0) \\ \text{or} \quad a &\propto t^{2/3}. \end{aligned} \tag{16}$$

Note that $H = \dot{a}/a$, therefore

$$H_0 = \dot{a}/a = (2/3t^{-1/3})/t^{2/3}, \tag{17}$$

which can be rearranged as

$$t_0 = \frac{2}{3}\frac{1}{H_0}. \tag{18}$$

A further case considers an open universe ($\rho \ll \rho_c$), $\Lambda = 0$ scenario, i.e.

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= 0 - \frac{kc^2}{a^2} \\ \dot{a}^2 &= A \end{aligned} \tag{19}$$

which leads to $a = \sqrt{A}\ t$ or, in terms of H one may write

$$
\begin{aligned}
H &= \dot{a}/a = 1/t \\
\\
t_0 &\simeq 1/H_0. \qquad (20)
\end{aligned}
$$

This is the **Hubble time** – the maximum time elapsed since $a = 0$ for a $\Lambda = 0$ universe. Note that the actual age of the universe in this case is a little bit less than $1/H_0$. The degree of inaccuracy in using a tangent to the current expansion curve will lie in the extent of the deviation of ρ from zero. Therefore, consideration of the Friedmann and Fluid equations for several basic models generates a characteristic age of the universe in terms of H_0.

2.4 The nature of redshift

Redshift was originally defined in purely observational terms. However, cosmological redshift arises naturally from a consideration of the RW line element. Consider the light path of a photon travelling through the universe along a radial path (radial null geodesic), i.e. $\mathrm{d}s^2 = \mathrm{d}\theta = \mathrm{d}\phi = 0$,

$$c^2\ \mathrm{d}t^2 = a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2}\right]. \qquad (21)$$

For a light pulse emitted at t_e and observed at t_o by an observer at a distance r_e, one may write

$$\int_{t_e}^{t_o} \frac{c\ \mathrm{d}t}{a(t)} = \int_0^{r_e} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} \equiv f(r_e). \qquad (22)$$

Note, if one has a good enough understanding of $a(t)$ and k once can determine the relation between distance and time in the universe. However, one may consider a second pulse of light emitted and observed at $t_e + \Delta t_e$ and $t_o + \Delta t_o$ respectively. The time between the two light pulses is sufficiently short that $a(t_e) \simeq a(t_e + \Delta t_e)$ etc. and

$$
\begin{aligned}
\int_{t_e+\Delta t_e}^{t_o+\Delta t_o} \frac{c\ \mathrm{d}t}{a(t)} - \int_{t_e}^{t_o} \frac{c\ \mathrm{d}t}{a(t)} &= 0 \\
\frac{c\ \Delta t_o}{a(t_o)} - \frac{c\ \Delta t_e}{a(t_e)} &= 0 \\
\frac{c\ \Delta t_o}{c\ \Delta t_e} &= \frac{a(t_o)}{a(t_e)} \qquad (23)
\end{aligned}
$$

However, $c\ \Delta t = \lambda$, and the above analysis is related to the (observationally defined) redshift by

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{\lambda_o}{\lambda_e} - 1 = \frac{a(t_o)}{a(t_e)} - 1. \qquad (24)$$

Therefore the cosmological redshift describes the relative expansion of the universal scale factor between the epochs of emission and observation.

2.5 The cosmological horizon

The cosmological horizon may be defined as the maximum distance a photon could travel within the lifetime of the universe. It is a convenient definition of the largest region of the universe that could exist in causal contact at any particular epoch. The horizon is defined by considering a radial null geodesic within the RW line element, i.e.

$$\begin{aligned} c\ \mathrm{d}t &= \frac{a(t)\ \mathrm{d}r}{\sqrt{1-kr^2}} \\ \int_0^{t_0} \frac{c\ \mathrm{d}t}{a(t)} &= \int_0^{r_H} \frac{\mathrm{d}r}{\sqrt{1-kr^2}}, \end{aligned} \tag{25}$$

Where t_0 indicates the current age of the universe and r_H is the horizon distance. For an EdS universe with $k = 0$ we may write

$$\begin{aligned} r_H &= \int_0^{t_0} \frac{c\ \mathrm{d}t}{a(t)} \\ &= \int_0^{t_0} \frac{c\ \mathrm{d}t}{a_0} \frac{t_0^{2/3}}{t^{2/3}} \\ &= 3\ c\ t_0. \end{aligned} \tag{26}$$

Consideration of the cosmological horizon at the current epoch provides one method to answer **Olber's paradox**, or "why is the night sky dark?". The universe may well be infinite in space, but it is finite in time. The combination of a finite age of the universe with a finite speed of light ensures that we can only observe photons from a finite region of the universe.

However, computation of the cosmological horizon gives rise to the **horizon problem**. When we look at the universe at the cosmological horizon, it is isotropic, i.e. large scale structure and CMB temperatures are statistically identical even though separated by 180° on the sky. However, these two regions, though in casual contact with us, are themselves causally isolated. How could the CMB and LSS have developed in exactly the same manner? The answer to the horizon problem is provided by **cosmic inflation** that postulates that the universe underwent a rapid phase of expansion at early times. Regions in the early universe were originally in causal contact and were subsequently inflated to scales much larger than the horizon during the inflationary epoch.

2.6 Distance volume and time in an expanding Universe

Distances, volumes and times in the universe cannot be observed "directly". There is no universal ruler or clock with which to measure such quantities. Therefore, quantities such as distance, etc. are defined in terms of their relationship to observed quantities.

Distances and volumes are defined by considering the path of a photon as it travels through the expanding universe from a distant source to the observer. Photons trajectories in spacetime are referred to as null geodesics which satisfy $\mathrm{d}s^2 = 0$ by definition. We further consider a photon travelling along a purely radial trajectory, i.e. $\mathrm{d}\theta = \mathrm{d}\phi = 0$. Returning to the RW line element we may therefore write

$$\begin{aligned} c\ \mathrm{d}t &= \frac{a(t)\ \mathrm{d}r}{\sqrt{1-kr^2}} \\ \frac{c\ \mathrm{d}t}{a(t)} &= \frac{\mathrm{d}r}{\sqrt{1-kr^2}}. \end{aligned} \tag{27}$$

In considering a photon emitted at coordinates $(t = t_1, r = 0)$ and received at $(t = t_0, r = r)$ we generate the following integral

$$\int_{t_1}^{t_0} \frac{c\ \mathrm{d}t}{a(t)} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1-kr'^2}} \tag{28}$$

The coordinate radius describing the co-moving distance between the source and observer is defined as

$$S(r) = \int_0^r \frac{\mathrm{d}r}{\sqrt{1-kr^2}}. \tag{29}$$

Evaluating $S(r)$, one obtains

$$\begin{aligned} &= \arcsin r \quad \text{for} \quad k = +1, \\ S(r) &= r \quad \text{for} \quad k = 0, \\ &= \operatorname{arcsinh} r \quad \text{for} \quad k = -1. \end{aligned} \tag{30}$$

The physical radius giving the source-observer separation at a particular epoch, t, is $a(t)\ S(r)$. Therefore, as we shall see below, when we consider the distance between objects in an expanding universe, we must also consider the time evolution of the scale factor. This leads us to evaluate the integral presented in Equation 28.

2.6.1 Angular diameter distance

Angular diameter distance, d_A, is defined as the ratio of an object's physical transverse size to its apparent angular size (in radians). For an object of fixed (not expanding with the Hubble flow) transverse size, y, and apparent angular size, θ, the angular diameter distance to the object is defined as

$$d_A = y/\theta. \tag{31}$$

For two photons emitted at a time, t_1, from either end of the object, the light paths are defined by two close, radial null geodesics. In this case, θ is simply the transverse distance, y, divided by the radial distance to the observer as calculated **at the time of emission**, i.e.

$$\theta = y/a_1 S(r). \tag{32}$$

However, recalling $a_0/a_1 = 1 + z$, one obtains

$$d_A = a_0 S(r)/1 + z. \tag{33}$$

The angular diameter distance is therefore obtained from the solution to Equation 28. Note that the solution to this general integral will be discussed after all of the distance/volume measures have been defined.

2.6.2 Luminosity distance

Luminosity distance is defined such that the ratio between bolometric luminosity and the observed bolometric flux of a given object is

$$d_L = \sqrt{\frac{L_{bol}}{4\pi f_{bol}}}. \tag{34}$$

The radiation from this object reaches us having travelled a radial distance $a_0 S(r)$ and is distributed over a pseudo–spherical surface (of constant time) of surface area $4\pi(a_0 S(r))^2$. However, two additional factors are required.

1. The energy of each photon decreases proportionally with the redshift factor,

$$E_{\gamma,0} = h\nu_0 = \frac{h\nu_e}{1+z}. \tag{35}$$

2. The rate of reception of photons (remember we are dealing with a flux here) decreases by a further factor $1 + z$ if one considers that $\Delta t_0/\Delta t_e = a_0/a_e = 1 + z$.

Therefore, the original flux versus luminosity relation may be re–written as

$$f_{bol} = \frac{L_{bol}}{4\pi(a_0 S(r))^2(1+z)^2}, \tag{36}$$

i.e. $d_L = (a_0 S(r)) \cdot (1 + z)$. Note that $d_L = (1 + z)^2\ d_A$. We have so far restricted the discussion of luminosity distance to bolometric quantities. The k–correction must be considered when dealing with **spectral** flux and luminosity and will be discussed later.

2.6.3 Co–moving volume

The co–moving volume defines a region of the universe in which the number density of non–evolving objects expanding with the Hubble flow is constant. The co–moving volume computed over the full sky out to a radial coordinate r is therefore,

$$V_c = 4\pi a_0^3 \int_0^r \frac{r'^2 \mathrm{d}r'}{\sqrt{1 - kr'^2}}. \tag{37}$$

2.6.4 Solving for distance versus redshift

General solutions to the distance versus redshift relations are best given by Hogg (astro–ph/9905116). However, the analytic solutions for an EdS universe with $k = 0$ are given here. Recall that the solution for the term $a_0 S(r)$, involved in the computation of d_L, d_A, and V_c, is based upon the following relation obtained from the RW line element

$$\int_{t_e}^{t_0} \frac{c\ \mathrm{d}t}{a(t)} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1 - kr'^2}}. \tag{38}$$

For $k = 0$ the RHS of this equation reduces to r. Taking the LHS one obtains

$$\begin{aligned}
\int_{t_e}^{t_0} \frac{c\ \mathrm{d}t}{a(t)} &= \frac{c\ t_0^{2/3}}{a_0} \int_{t_e}^{t_0} \frac{\mathrm{d}t}{t^{2/3}} \longleftarrow \left[\frac{a(t)}{a_0} = \left(\frac{t}{t_0}\right)^{2/3}\right] \\
&= \frac{3\ c\ t_0^{2/3}}{a_0} \left[t^{1/3}\right]_{t_e}^{t_0} \\
&= \frac{3\ c\ t_0^{2/3}}{a_0} \left[t_0^{1/3} - t_e^{1/3}\right] \\
&= \frac{3\ c\ t_0^{2/3}}{a_0} \left[t_0^{1/3} - \frac{t_0^{1/3}}{\sqrt{1+z}}\right] \\
&= \frac{3\ c\ t_0}{a_0} \left[1 - \frac{1}{\sqrt{1+z}}\right] \\
a_0 r &= \frac{2c}{H_0} \left[1 - \frac{1}{\sqrt{1+z}}\right] \longleftarrow \left[t_0 = \frac{2}{3} H_0^{-1}\right].
\end{aligned} \tag{39}$$

Returning to the previous geometric definitions, we can now write explicit distance and volume formulae for the EdS case, i.e.

Angular diameter distance:

$$d_A(z) = \frac{2c}{H_0}\,\frac{1}{(1+z)}\,\left[1-\frac{1}{\sqrt{1+z}}\right]. \tag{40}$$

Luminosity distance:

$$d_L(z) = \frac{2c}{H_0}\,(1+z)\,\left[1-\frac{1}{\sqrt{1+z}}\right]. \tag{41}$$

Co–moving volume element:

$$V_c = 4\pi a_0^3 \int_0^r \frac{r'^2\,\mathrm{d}r'}{\sqrt{1-kr'^2}}. \tag{42}$$

For $k=0$, $V_c = 4\pi/3\cdot(a_0 r)^3$, therefore

$$V_c = \frac{32\pi}{3}\left(\frac{c}{H_0}\right)^3\left(1-\frac{1}{\sqrt{1+z}}\right)^3. \tag{43}$$

Note that this volume is computed over the full sky, i.e. 41,253 $\deg^2$ or 4π Sr.

2.6.5 Look back time

The look back time is the difference in the age of the universe between the epochs of photon emission and observation, i.e. $t_L = t_0 - t_1$. For specific cases one may solve this equation directly. For example, for the EdS case, i.e. $k=0$, $a \propto t^{2/3}$, we can write

$$\begin{aligned} t_L &= t_0 - \left(\frac{a_1}{a_0}\right)^{3/2} t_0 \\ &= t_0\left(1-\frac{1}{(1+z)^{3/2}}\right) \\ &= \frac{2}{3}H_0^{-1}\left(1-\frac{1}{(1+z)^{3/2}}\right), \end{aligned} \tag{44}$$

recalling that $t_0 \simeq 2/3H_0^{-1}$.

2.6.6 Characteristic scales

The above equations introduce a characteristic universal distance and volume in addition to the characteristic time discussed in Lecture 1:

$$\begin{aligned} \text{Hubble distance} &\equiv \frac{c}{H_0} = 3000\ h^{-1}\text{Mpc} \\ \text{Hubble volume} &\equiv \left(\frac{c}{H_0}\right)^3 = 2.7 \times 10^{10}\ h^{-3}\text{Mpc}^3 \\ \text{Hubble time} &\equiv \frac{1}{H_0} = 9.78 \times 10^9\ h^{-1} yr, \end{aligned} \tag{45}$$

where $H_0 = 100\ h$ kms^{-1} Mpc^{-1} introduces a useful "cosmology free" unit.

2.7 Which type of universe do we live in?

The most compelling evidence that we live in a spatially flat, accelerating Universe comes from the combination of data from the CMB and type Ia supernovae (SNe Ia).

The characteristic physical scale of hot and cold spots in the CMB temperature map can be computed from basic physics. The measurement of the characteristic angular scale of such features determines the ADD at the redshift of CMB creation. This effectively determines the total mass-energy density of the Universe, i.e. $\Omega_M + \Omega_\Lambda = 1$.

SNe Ia are well-calibrated stellar nuclear explosions. They occur when sufficient mass from a giant star accumulates on its white dwarf companion. When the mass of the white dwarf exceeds the Chandrasehkar mass it experiences a catastrophic collapse under gravity resulting in runaway nuclear burning (detonation) of its carbon-oxygen core.

SNe Ia exhibit a narrow range of absolute magnitudes about $M_B = -19$. Therefore by comparing the apparent magnitudes of a sample of SNe Ia observed at a range of redshifts to a reference population one can determine the variation of luminosity distance with redshift. As this depends upon the cosmological model in a unique manner (i.e. non-degenerate with Ω_M and Ω_Λ) it is an important test of the cosmological model.

SNe Ia indicate that the recent expansion of the Universe is accelerating and thus support the existence of a repulsive dark energy denoted Ω_Λ.

2.8 A short cosmic history

The energy density in the early universe is dominated by radiation. Therefore,we may write

$$a(t) \propto t^{1/2}; \quad T(t) \propto a(t)^{-1}; \quad T(t) \propto t^{-1/2}. \tag{46}$$

Ryden uses an analysis of the conditions at the Planck time to obtain the constant of proportionality. Using radiation-matter equality also provides a reasonable answer, i.e.

$$T(t) \simeq 10^{10}\ \mathrm{K} \left(\frac{t}{1\mathrm{s}}\right)^{-1/2} \tag{47}$$

$$kT(t) \simeq 1\ \mathrm{MeV} \left(\frac{t}{1\mathrm{s}}\right)^{-1/2} \tag{48}$$

$$E_{mean} = 2.7kT(t) \simeq 3\ \mathrm{MeV} \left(\frac{t}{1\mathrm{s}}\right)^{-1/2} \tag{49}$$

These formulae provide us with the energy scale as a function of epoch in the early universe. A comparison to the energy associated with particle interactions provides an understanding the physical conditions associated with each epoch.

1. **The hadron era:** $t_{min} < t \lesssim 10^{-5}$ sec. This marks the earliest Universal epoch where experimental physics can be applied with any confidence. Almost all matter, including electrons, protons, neutrons, neutrinos and their associated anti–particles are in thermal equilibrium with the photon radiation field. The disparity between particles and anti–particles is thought to be 1 part in $\gtrsim 10^7$ and is ultimately responsible for all matter in the present day universe. The exact physics (e.g. equation of state) of this epoch is not known. Thus the dependence of the scale factor $a(t)$ and the temperature $T(t)$ upon cosmic time is not known.

2. **The lepton era:** 10^{-5} sec $\lesssim t \lesssim 10$ sec. The temperature decreases such that kT is significantly lower than the rest mass energy of the proton ($m_p = 938$ MeV). Proton–antiproton pairs, in addition to other hadrons present, annihilate. The lepton era begins with photons in thermal equilibrium with electrons and positrons, muons, neutrinos and antineutrinos. The energy released by hadron annihilation is thus shared between all of these particle families (see Rees for an interesting discussion of later epochs where the energy partition is not equal). Each of the relativistic particles (photons, neutrinos and electrons – plus antiparticles) contributes an energy density $\propto T^4$. The lepton era ends when the radiation temperature drops significantly below $T \simeq 5 \times 10^9$ K (i.e. $kT \simeq m_e c^2 = 511$ keV). Electron–positron pairs annihilate and temperatures begin to decrease to levels where a protons fall out of equilibrium with neutrons.

3. **The plasma era:** 10 sec $\lesssim t \lesssim 10^{13}$ sec. The universe consists of photons, neutrinos, electrons, protons and neutrons (the discussion assumes that at this stage Dark Matter particles do not

interact). The early stages of the plasma era remain sufficiently hot and dense to produce light nuclear elements from hydrogen. Matter and radiation are coupled to form a photon–baryon fluid discussed in Lecture 3: photons are coupled to electrons via Thompson scattering, and electrons are coupled to protons via Coulomb interactions. Matter and radiation remain in thermal equilibrium until the photon temperature drops below $T \simeq 3 \times 10^3$ K where electrons combine with protons to form atomic hydrogen. The radiation field continues unimpeded to the present day where it is observed as the Cosmic Microwave Background.

4. **The post–recombination era:** $t \gtrsim t_{rec} \simeq 10^{13}$ sec. Various astrophysical processes combine to produce the present day universe.

Understanding the cosmic history preceeding the formation of galaxies provides insight into two important questions.

1. Why do galaxies contain so much helium? Gaseous nebulae in galaxies (HI and HII regions) contain some 25-27% of ^{4}He by mass. Simple calculation indicates that this abundance could not have been created by stellar nucleosynthesis. The helium is primordial in that it was created in the earliest (first few minutes) of the Universe's history.

2. What were the first baryonic structures to form after recombination? the Jeans relation describes baryonic structures whose self–gravity is supported by internal pressure. Structures that are Jeans stable do not collapse and structure is suppressed on scales smaller than the Jeans scale. Consider a virialised cloud of particles at the stability limit, i.e.

$$3NkT = \frac{GM_J^2}{R_J}, \tag{50}$$

where subscript J indicates the Jeans limit. Employing the relations $N = M_J/\mu m_p$ and $M_J = 4\pi\rho R_J^3/3$ one obtains the Jeans length

$$R_J = \left(\frac{3kT}{\mu m_p} \, . \, (G\rho)^{-1} \, . \, \frac{3}{4\pi}\right)^{1/2} = c_s\sqrt{\frac{\pi}{G\rho}}. \tag{51}$$

The Jeans length is therefore proportional to the sound speed of baryons at each epoch. Prior to decoupling the baryonic sound speed was determined by the properties of the relativistic photon baryon fluid, i.e. $c_s = c/\sqrt{3} = 0.58c$. The corresponding value of the Jeans length may be computed as

$$R_J = 0.58c \times \sqrt{\frac{\pi}{6.67 \times 10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}} \times 5 \times 10^{-19}\ \mathrm{kg\,m^{-3}}}} = 5.3 \times 10^{22}\ \mathrm{m} \approx 1.6\ \mathrm{Mpc}, \tag{52}$$

where we have taken the physical baryon density at the epoch of decoupling. The associated Jeans mass is of order 10^{20} $M_\odot$ which is several orders of magnitude larger than the most massive galaxy cluster known. Immediately after decoupling the sound speed is given by the expression

$$c_s \approx \sqrt{\frac{kT}{\mu m_p}} = \sqrt{\frac{kT}{\mu m_p c^2}}c = \sqrt{\frac{0.26 \text{ eV}}{563 \text{ MeV}}}c = 2 \times 10^{-5}c. \tag{53}$$

The corresponding Jeans length and mass are respectively $R_J = 1.84 \times 10^{18}$m ≈ 50 pc and $M_J \approx 4 \times 10^6$ $M_\odot$ which lies between the mass of a typical globular cluster and a dwarf galaxy. Therefore, the epoch of decoupling marks an important transition in the development of baryonic structure. Prior to decoupling all fluctuation scales up to the horizon scale are effectively suppressed. After decoupling the Jeans scale is dramatically reduced and baryonic structures can begin to grow.

Remember that dark matter haloes exist on all scales down to some mass/length scale determined by the free streaming scale of the dark matter particle itself.

If the mass of the dark matter particle is of order 1 MeV the mass associated with the free streaming scale is of order $10^6 M_\odot$.

Therefore there appears to be a key mass scale of order $10^6 M_\odot$ imprinted on baryonic structures in the post-recombination Universe. This limit is close to the lower mass limits of globular clusters and dwarf galaxies.

In the following lectures we will explore which of these early overdensities ends up in present day globular clusters, dwarf galaxies and giant galaxies.