8 Lambda

The cosmological constant, parameterised as Λ, has a history as long as relativistic cosmology itself. The presence (or otherwise) of Λ within cosmological models has always been motivated by observations. As we shall see by the end of this lecture, current discussion of the nature of the Λ term has generalized to determining the properties of universal "dark energy" contribution.

8.1 Back to basics: Λ within a Friedmann cosmology

We have seen in Lecture 1 how GR can be applied to describe the dynamical evolution of the universe via the Friedmann equation. Though Friedmann published his equations in 1922, Einstein was considering the problem of applying GR to the universe from 1915 onwards. At this time the "universe" consisted of the matter density contributed by stars within the Milky way (whose average radial velocity was zero) and unidentified faint nebulae, together with the effective mass density contributed by starlight. This led Einstein to the reasonable conclusion that the universe was dominated by non-relativistic, pressureless matter. Could such a universe be static? The answer turns out to be no.

Consider a Newtonian analogue: the mass density is related to the potential via Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho. \tag{1}$$

The gravitational acceleration at any point in space is equal to the gradient of the potential

$$\vec{a} = -\vec{\nabla}\Phi. \tag{2}$$

In a static universe, the acceleration is zero and therefore, the potential must be constant in space. However, if Φ is constant we obtain

$$\rho = \frac{1}{4\pi G}\nabla^2\Phi = 0 \tag{3}$$

The only permissible static universe is empty. Einstein tackled this problem by introducing a cosmological constant Λ to counteract gravity. In Newtonian terms he modified the Poisson equation as follows

$$\nabla^2\Phi + \Lambda = 4\pi G\rho. \tag{4}$$

The units of Λ are $(\text{time})^{-2}$ and setting $\Lambda = 4\pi G\rho$ generates a static universe. In GR terms Einstein added Λ to the Field Equation – the relativistic analogue of the Poisson equation.

If the Friedmann equation is derived with Λ included in the EFQ one obtains

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}. \tag{5}$$

The form of the Fluid equation is unchanged but the Acceleration equation takes the form

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3} \tag{6}$$

One can see an immediate problem with setting a global $\Lambda = 4\pi G\rho$ where ρ represents the average mass density of the universe. Though one obtains a global $\ddot{a} = 0$, the presence of matter fluctuations will create locally unstable regions of the universe and collapse is inevitable.

It is useful to re–write the Friedmann equation in terms of the normalized redshift evolution of the Hubble parameter, i.e.

$$\frac{H(z)}{H_0} \equiv E(z) = \sqrt{\Omega_M(1+z)^3 + \Omega_R(1+z)^4 + \Omega_K(1+z)^2 + \Omega_\Lambda}, \tag{7}$$

where

$$\Omega_M = \left(\frac{3H_0^2}{8\pi G}\right)^{-1}\rho_M,\ \Omega_R = \left(\frac{3H_0^2}{8\pi G}\right)^{-1}\rho_R,\ \Omega_K = \frac{-kc^2}{H_0^2 a_0^2},\ \text{and}\ \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}. \tag{8}$$

Note that all density values are considered at the current epoch. One may further express the total "density" contribution of all sources of energy in the universe as

$$\Omega_M + \Omega_R + \Omega_K + \Omega_\Lambda = 1. \tag{9}$$

Finally, as it will be useful later, we remind ourselves of the construction of the luminosity distance of a source at a redshift z, in terms of the $E(z)$ function,

$$d_L(z) = \frac{c}{H_0}(1+z)\int_0^z \frac{dz'}{E(z')},\ \text{for}\ \Omega_K = 0. \tag{10}$$

8.2 A toy model of Λ

In Lecture 1 we derived the Friedmann equation employing a Newtonian analogue that, in simple terms, can be thought of as a ballistic model of the universe. Introducing Λ to this ballistic model permits a simple investigation of the role of Λ in the expansion of the universe. Consider the equation of motion of a cannon ball fired from a cannon:

$$s_y = u_y t + \frac{1}{2}\,gt^2 \tag{11}$$

$$s_x = u_x t. \tag{12}$$

To this simple model we can add λ as a constant repulsive term per unit length (as we are only considering forces in the vertical direction):

$$s_y = u_y t + \frac{1}{2}\,(g + s_y\lambda)t^2, \tag{13}$$

where s_x is unchanged. The effect of introducing Λ to this ballistic model is displayed in Figure 1.

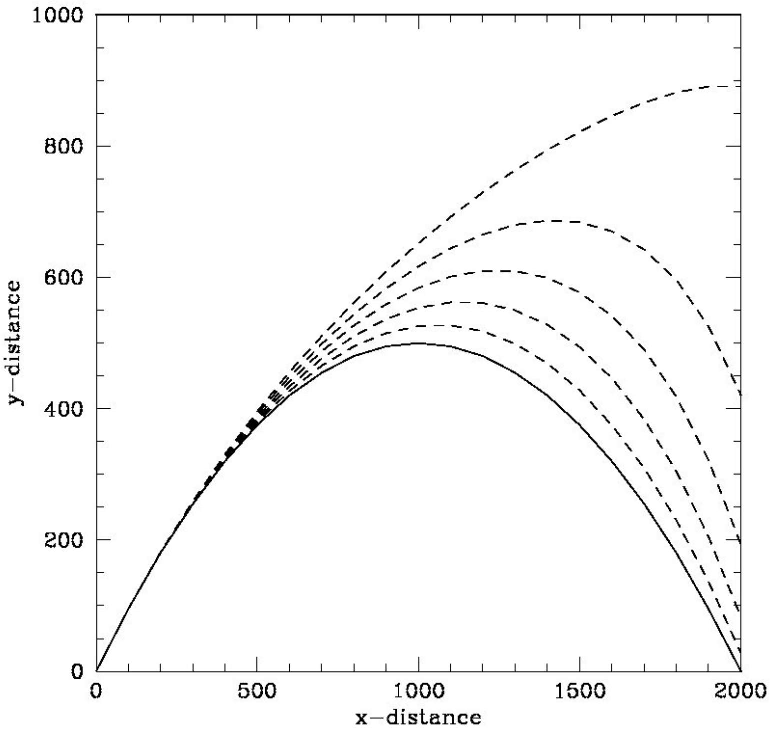

Figure 1: The effect of Λon ballistic models. The solid line shows the case $g = -10$ and $u_x = u_y$. Each dashed line displays the case $\lambda = 0.001, 0.002, \ldots, 0.005$.

8.3 Observations constraining Λ

Studies of type Ia supernovae (i.e. explosion of a carbon–oxygen white dwarf star resulting from matter accreting from a binary companion) have emerged as the principal class of observations that constrain the contribution of Λ to the geometry of the universe.

Type Ia supernovae have evolved with time as the preferred standard candle with which to measure the luminosity distance–redshift relation. Efficient supernovae searches require large field CCD cameras in order to sample a sufficient volume of the universe (i.e. to get enough supernovae) and sufficient depth (i.e. to get enough distant supernovae). Furthermore, 8m–class telescopes are required to obtain redshifts for distant SNe Ia, displaying apparent magnitudes ~ 25 at redshifts $z \sim 1$, where the expected difference between competing world models is expected to be greatest. These technological considerations limited the effectiveness of SNe Ia searches until advances in the available technology in the mid–1990s (wide field cameras plus Keck) permitted new, deeper and wider, studies to begin.

During the 1990s, it was recognized that SNe Ia were not as standard a candle as first believed. It had been thought that SNe Ia displayed mean apparent magnitudes, $M_B \approx -19.5$, with a scatter of $\sigma \approx 0.4 - 0.6$ mag. However, Phillips (1993) reported a variation of a factor 3 in the peak brightness of SNe Ia but that the peak luminosity correlated strongly with the decrease in B luminosity 15 rest–frame days after peak, $\Delta m_{15}(B)$. Studies such as this were improved upon by Hamuy et al. (1993) who compiled a database of SNe Ia with well sampled light curves for a range of peak brightness and colour variation. The results of such studies succeeded in reducing the "observed" scatter associated with SNe Ia to $\sigma \approx 0.15$ mag. As we shall discuss in the following paragraphs, reducing the observed scatter in SNe Ia properties, greatly reduced the size of SNe Ia survey required to constrain cosmological parameters to a given precision. Importantly, the study of Hamuy et al. also provided a low redshift sample of SNe Ia (34 at $z < 0.15$) to compare to more distant samples.

Steps to compiling the SNe Ia Hubble Diagram:

1. Obtain imaging data: ground–based 4m–class imaging covers about 4 square degrees to 23rd magnitude; HST imaging covers smaller fields yet to much greater depths.

2. Difference image time–series data to identify variable objects. Identify SNe Ia on the basis of colour and light curve behaviour.

3. Obtain SNe Ia redshifts: direct confirmation of the SNe Ia spectrum is best but if one cannot get the redshift in time, then obtaining the redshift of a well–identified host galaxy is acceptable.

4. Reduce sources of uncertainty: systematic (photo zero points, selection effects, evolution[upper limits only], evolution of the extinction law, gravitational lensing) and statistical (individual zero points, shot noise, K–corrections, extinction, σ of SNe Ia).

5. Determine the "best–fitting" cosmological model given the SNe Ia magnitudes, i.e. form an equation of the form

$$\chi^2 (H_0, \Omega_M, \Omega_\Lambda) = \sum_i \frac{\left[\mu_{p,i} (z_i | H_0, \Omega_M, \Omega_\Lambda) - \mu_{0,i}\right]^2}{\sigma^2_{\mu 0,i} - \sigma^2_v}, \tag{14}$$

The SNe Ia Hubble Diagram has changed considerably over the past 6 years; from the presentation of the first "high–redshift" SNe Ia at $z \sim 0.5$ (Schmidt et al. 1998), through to the extending of the $z < 1$ sample (Riess et al. 1998) which placed the first "interesting" constraints upon Λ, to the discovery of numerous $z > 1$ SNe Ia with HST that appears to illustrate the changeover from a decelerating to an accelerating universe.

8.4 Investigating "dark energy"

The behaviour of the luminosity distance versus redshift relation given by SNe Ia observations appears conclusively consistent with the effects of a universal repulsion counteracting gravity. Alternative explanations for the observed "turnover" in the luminosity distance all invoke considerable cosmic conspiracies. The cosmological constant as it appears in the expression for $E(z)$ may be generalized to a universal "dark energy" described by an equation of state of the form, $w = P/\rho c^2$. The full expression for $E(z)$ then takes the form

$$E(z) = \sqrt{\Omega_M(1+z)^3 + \Omega_R(1+z)^4 + \Omega_K(1+z)^2 + \Omega_\Lambda(1+z)^{3(1+w)}}, \tag{15}$$

where $w = -1$ represents the former case of a cosmological constant. New models describing the behaviour of Λ via the equation of state are becoming more and more numerous. However, the best observations to date remain completely consistent with the case $w = -1$.

8.5 Consequences of Λ

Confirmation of the presence of a cosmological constant forces us to reconsider previous statements regarding the evolution of the universe and its contents. Namely:

1. Observations of the CMB by WMAP indicate that the universe is spatially flat. Observations of SNe Ia indicate that the universe is Λ–dominated. An Einstein de–Sitter universe ($\Omega_M = 1$) expands effectively forever – reaching a finite yet large value of the scale factor at infinite time. Inclusion of Λ means that the flat universe expands forever at an accelerating rate.

2. Hang on – at an accelerating rate? If we return to Equation 13. we see that, in the case of a flat, Λ–dominated universe at large time, $E(z) \to \sqrt{\Omega_\Lambda}$. This indicates that the universe will reach a constant expansion rate (Hubble parameter) at large time, i.e.

$$\begin{aligned} \frac{H(t)}{H_0} &\to \sqrt{\Omega_\Lambda} = H_\Lambda \\ \frac{\mathrm{d}a}{a} &= H_\Lambda \mathrm{d}t \\ \ln a &= H_\Lambda t + C \\ a &\propto e^{H_\Lambda t}. \end{aligned} \tag{16}$$

   Another way of thinking about this is via the horizon distance travelled by a photon, $r_h = \int_0^{t_0} c\,dt/a(t)$. If $a(t)$ were to increase at an exponential rate, it would eventually expand faster than $c\,dt$ and the light horizon would begin to decrease, i.e. sources that are not currently bound to our local matter distribution will expand with the Hubble Flow at an increasingly large rate until the amount of time required for the universe to double in size is shorter than the light travel time measured between the two expanding sources.

3. If it turns out that the dark energy behaves exactly like a cosmological constant, one challenge will be to explain *why* takes the value $\Omega_\Lambda \approx 0.7$. Universal energies similar to the cosmological constant appear within certain particle physics models as the zero point energy of the vacuum. However, naive calculations lead one to expect $\Omega_\Lambda \approx 10^{60}$. If Λ really is a vacuum energy, why is its present value so small? Is it the decaying remnant of some primordial field, possibly associated with the epoch of inflation? If so, is that not inconsistent with present estimates of $w = -1$?

4. Structure forms in a flat Λ–dominated universe at a slower rate than both a flat matter–dominated universe or a low density universe without Λ. Structure formation is driven by gravity and suppressed by universal expansion. Less matter equals less gravity, more Λ equals more expansion. This relationship was exploited within structure formation models even before SNe Ia observations provided direct observational evidence for Λ. Structure forms too quickly within flat or open CDM dominated universes, resulting in highly clustered structure in the present day universe. The addition of Λ reduces both the early–time growth of small–scale structure – mainly via the absence of matter – and the late–time growth of large–scale structure – via the increased expansion rate of the universe compared to non–Λ models.